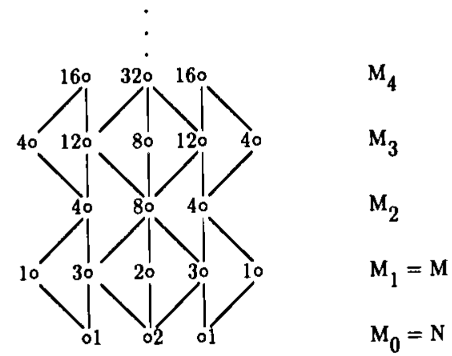
It follows at once by induction that all the algebras M_k in the tower generated by $N \subset M$ are also multi-matrix, and furthermore the inclusion matrix $\Lambda_{M_k}^{M_{k+1}}$ is Λ_N^M for k even and $(\Lambda_N^M)^t$ for k odd. Thus in the infinite Bratteli diagram for the tower, the $(k+1)^{st}$ story (for $M_{k+1} \subset M_{k+2}$) is the reflection of the k^{th} story (for $M_k \subset M_{k+1}$).

We illustrate this with $N = \mathbb{C}[\mathfrak{S}_3]$ and $M = \mathbb{C}[\mathfrak{S}_4]$, as in Example 2.3.8. We have

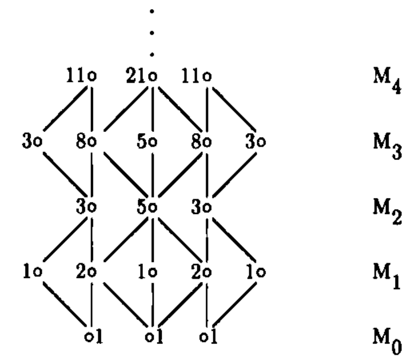
$$\Lambda = \Lambda_N^M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{\nu} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

These data determine the Bratteli diagram of the tower:



In particular $M_2 = \mathrm{End}_N^r(M) \cong \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_8(\mathbb{C}) \oplus \mathrm{Mat}_4(\mathbb{C})$.

An accidental feature of this example is that $\vec{\nu}$ is an eigenvector of $\Lambda^t\Lambda$ with eigenvalue 4. Consequently the vector of dimensions on the $k+2^{nd}$ floor is always 4 times the vector on the k^{th} floor. To get a better idea of the general situation let us also consider the example with the same inclusion matrix Λ but with $N = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, i.e. $\vec{\nu} = (1,1,1)^t$. Then the Bratelli diagram is



The reader might wish to continue the diagram for a few more levels and observe the *convergence of the ratio of dimensions* on the even floors to [1:2:1] and on the odd floors to [1:3:2:3:1], eigenvectors for $\Lambda^t\Lambda$ and $\Lambda\Lambda^t$ respectively. This observation is the key to the next proposition.

Proposition 2.4.2. *Let* $N \subset M$ *be a pair of multi-matrix algebras with inclusion matrix* Λ, *and let* $(M_k)_{k \geq 0}$ *be the associated tower. Then*

$$\lim_{k\to\infty} \{\dim_{\mathbb{K}} M_k\}^{1/k} = \|\Lambda\|^2.$$

Proof. It suffices to prove this in the case that $Z_M \cap Z_N = \mathbb{K}$. Then since $\Lambda^t\Lambda$ and $\Lambda\Lambda^t$ are irreducible and aperiodic (2.3.1.f and 1.3.2) and also positive semi-definite, it follows from Perron–Frobenius theory that

$$\lim_{k\to\infty} \|(\Lambda^t\Lambda)^k\xi\|^{1/k} = \lim_{k\to\infty} \|(\Lambda\Lambda^t)^k\xi\|^{1/k} = \|\Lambda\|^2,$$

for any non-zero $\xi \in \mathbb{R}^n$ with non-negative coordinates. (Set A equal to $\Lambda^t\Lambda$ or $\Lambda\Lambda^t$. Then $A = \|A\|E_0 + \sum \mu_i E_i$, where E_0 is the rank one orthogonal projection onto the span of the Perron–Frobenius eigenvector, E_i are the remaining spectral projections of A, and μ_i satisfy $0 \leq \mu_i < \|A\|$. Hence

$$\frac{A^k\xi}{\|A\|^k} = E_0\xi + \sum \frac{\mu_i^k}{\|A\|^k} E_i\xi,$$

which converges to $E_0\xi$. If z is the unique positive normalized ($\|z\| = 1$) Perron–Frobenius eigenvector, then $E_0\xi = (\sum x_i z_i)z$, which is non-zero because $\xi_i \geq 0$

and $z_i > 0$ for all i, and $\xi_j > 0$ for some j. It follows that

$$\lim \frac{\|\Lambda^k \xi\|}{\|\Lambda\|}^{1/k} = \lim \|E_0 \xi\|^{1/k} = 1.)$$

The inclusion matrix for $M_0 \subset M_{2k}$ is $(\Lambda^t\Lambda)^k$, and that for $M_0 \subset M_{2k+1}$ is $(\Lambda\Lambda^t)^k\Lambda$. Thus $\dim_{\mathbb{K}} M_{2k} = \|(\Lambda^t\Lambda)^k \vec{\nu}\|^2$ and $\dim_{\mathbb{K}} M_{2k+1} = \|(\Lambda\Lambda^t)^k \Lambda \vec{\nu}\|^2$. Therefore

$$\lim_k (\dim_{\mathbb{K}} M_{2k})^{1/2k} = \lim_k (\dim_{\mathbb{K}} M_{2k+1})^{1/2k+1} = \|\Lambda\|^2. \quad \#$$

<u>Lemma 2.4.3</u>. *Let* $N \subset M$ *be a pair of finite dimensional algebras over a field* $\mathbb{K}$, *and let* $(M_k)_{k\geq 0}$ *be the associated tower. Then*

$$[M:N] = \lim_{k\to\infty} \sup \{\dim_{\mathbb{K}} M_k\}^{1/k}.$$

<u>Proof</u>. As M_0 and M_k are finite dimensional $\mathbb{K}$-algebras, one has

$$\frac{\dim_{\mathbb{K}}(M_k)}{\dim_{\mathbb{K}}(M_0)} \leq \mathrm{rk}(M_k | M_0) \leq \dim_{\mathbb{K}}(M_k),$$

and therefore

$$[M:N] = \lim_{k\to\infty} \sup \{\mathrm{rk}(M_k|M_0)\}^{1/k} = \lim_{k\to\infty} \sup \{\dim_{\mathbb{K}} M_k\}^{1/k}. \qquad \#$$

<u>Proposition 2.4.4</u>. Let $\mathbf{1} \in N \subset M$ be a pair of finite dimensional algebras over a field $\mathbb{K}$, let $\mathbb{E}$ be any extension field of $\mathbb{K}$ and set

$$M^{\mathbb{E}} = M \otimes_{\mathbb{K}} \mathbb{E} \quad \text{and} \quad N^{\mathbb{E}} = N \otimes_{\mathbb{K}} \mathbb{E}.$$

Then

(a) $\mathrm{End}_N^r(M) \otimes_{\mathbb{K}} \mathbb{E} \cong \mathrm{End}^r_{N^{\mathbb{E}}}(M^{\mathbb{E}})$.

(b) $[M:N] = [M^{\mathbb{E}} : N^{\mathbb{E}}]$.

<u>Proof</u>. (a) This is an example of a theorem on "change of rings in Hom"; see for example [R], p.24. We give a simple proof appropriate to the special case at hand.

Define

$$\alpha : \mathrm{End}_{\mathbb{K}}(M) \otimes_{\mathbb{K}} \mathbb{E} \to \mathrm{End}_{\mathbb{E}}(M^{\mathbb{E}})$$

by

$$\alpha(\varphi \otimes a)(x \otimes b) = \varphi(x) \otimes ba \quad (\varphi \in \mathrm{End}_{\mathbb{K}}(M),\ a,b \in \mathbb{E},\ x \in M).$$

Define also

$$\beta : \mathrm{End}_{\mathbb{E}}(M^{\mathbb{E}}) \to \mathrm{End}_{\mathbb{K}}(M) \otimes_{\mathbb{K}} \mathbb{E}$$

as follows. Let $\{a_i\}$ be a basis of $\mathbb{E}$ over $\mathbb{K}$. For each $\Phi \in \mathrm{End}_{\mathbb{E}}(M^{\mathbb{E}})$ and each i, there is a unique $\varphi_i \in \mathrm{End}_{\mathbb{K}}(M)$ such that

$$\Phi(x \otimes \mathbf{1}) = \sum \varphi_i(x) \otimes a_i \qquad (x \in M).$$

Only finitely many $\varphi_i(x)$ are non-zero for any particular x, and since M is finite dimensional over $\mathbb{K}$, only finitely many φ_i are non-zero altogether. Then β can be defined by

$$\beta(\Phi) = \sum \varphi_i \otimes a_i.$$

It is easy to check that α and β are isomorphisms of $\mathbb{E}$-algebras which are inverse to each other.

Next observe that

(2.4.4.1) $\qquad \alpha(\lambda(m)\otimes a) = \lambda(m\otimes a)$, and
$\qquad \alpha(\rho(n)\otimes a) = \rho(n\otimes a) \quad (m \in M,\ n \in N,\ a \in \mathbb{E})$.

It follows from this that

$$\alpha(\mathrm{End}_N^r(M)\otimes\mathbb{E}) = \mathrm{End}^r_{N^{\mathbb{E}}}(M^{\mathbb{E}}).$$

(b) Let $(M_k)_{k\geq 0}$ be the tower of extensions generated by $N \subset M$ and let $(A_k)_{k\geq 0}$ be the tower generated by $N^{\mathbb{E}} \subset M^{\mathbb{E}}$. We produce a sequence of isomorphisms $\alpha_k : M_k \otimes_{\mathbb{K}} \mathbb{E} \to A_k$ such that $\alpha_{k+1}\big|_{M_k \otimes_{\mathbb{K}} \mathbb{E}} = \alpha_k$ for all k. Take α_0, α_1 to be the identity and α_2 to be the isomorphism defined in part (a); we have $\alpha_2\big|_M = \alpha_1$ by (2.4.4.1). Suppose $\alpha_1, \cdots, \alpha_k$ have been defined. Let

$$\delta_{k+1} : M_{k+1} \otimes \mathbb{E} = \mathrm{End}^{r}_{M_{k-1}}(M_k) \otimes \mathbb{E} \to \mathrm{End}^{r}_{M^{\mathbb{E}}_{k-1}}(M^{\mathbb{E}}_k)$$

be the isomorphism defined as in part (a), and let

$$\gamma_{k+1} : \mathrm{End}^{r}_{M^{\mathbb{E}}_{k-1}}(M^{\mathbb{E}}_k) \to A_{k+1} = \mathrm{End}^{r}_{A_{k-1}}(A_k)$$

be induced by the pair of isomorphisms

$$\begin{array}{ccc} M^{\mathbb{E}}_k & \xrightarrow{\alpha_k} & A_k \\ \cup & & \cup \\ M^{\mathbb{E}}_{k-1} & \xrightarrow{\alpha_{k-1}} & A_{k-1} \end{array}$$

Set $\alpha_{k+1} = \gamma_{k+1} \circ \delta_{k+1}$; this extends α_k because δ_{k+1} extends the identity on $M^{\mathbb{E}}_k$, and γ_{k+1} extends α_k.

Consequently, we have $\dim_{\mathbb{K}}(M_k) = \dim_{\mathbb{E}}(M^{\mathbb{E}}_k) = \dim_{\mathbb{E}}(A_k)$ for all k, and the equality $[M : N] = [M^{\mathbb{E}} : N^{\mathbb{E}}]$ follows from this and Lemma 2.4.3. #

Proof of Theorem 2.1.1 and Corollary 2.1.2. Because of 2.4.4 and the definition of Λ^M_N for arbitrary semi-simple algebras (given in the chapter introduction), it suffices to consider the case that $\mathbb{K}$ is algebraically closed, so M and N are multi-matrix algebras. But then

$$[M : N] = \lim_k \{\dim_{\mathbb{K}} M_k\}^{1/k} = \|\Lambda^M_N\|^2,$$

by 2.4.2 and 2.4.3. The corollary follows from Kronecker's Theorem 1.1.1. #

Remark. The norm of a product of two matrices is not, in general, the product of their norms. It follows that, given a nested sequence $\mathbf{1} \in L \subset P \subset M$ of semi-simple algebras, the inequality

$$[M : L] \le [M : P][P : L]$$

is in general strict. However, even this inequality fails to hold for algebras with radicals, as we now show.

Example 2.4.5. Consider two integers $m', m'' \ge 1$ and set $m = m' + m''$. Let M b[e] the factor $\mathrm{Mat}_m(\mathbb{C})$, let P be its "parabolic" subalgebra

$$P = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in M : A \in \mathrm{Mat}_{m'}(\mathbb{C}),\ B \in \mathrm{Mat}_{m',m''}(\mathbb{C}),\ C \in \mathrm{Mat}_{m''}(\mathbb{C}) \right\}$$

and let L be the "Levi" subalgebra $\left\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \right\}$ of P. Then L and M are semi-simple and $[M : L] = 2$ as above, but P is of course not semi-simple.

We claim that $[M : P] = 1$. Indeed, from left multiplication $\begin{bmatrix} X & Y \\ Z & T \end{bmatrix} \mapsto \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}$ one has the inclusion

$$\left\{ \begin{array}{ccc} P & \to & \mathrm{End}_{\mathbb{C}}(M) \approx M \otimes M^{\mathrm{opp}} \\ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} & \mapsto & \begin{bmatrix} \lambda_A & 0 & \lambda_B & 0 \\ 0 & \lambda_A & 0 & \lambda_B \\ 0 & 0 & \lambda_C & 0 \\ 0 & 0 & 0 & \lambda_C \end{bmatrix} \end{array} \right.$$

where λ_A is left-multiplication by A (and ρ_A below is right multiplication). As th[e] commutant of P in M is reduced to the center $\mathbb{C}$ of M, the commutant of $\lambda(P)$ i[n] $\mathrm{End}_{\mathbb{C}}(M)$ is isomorphic to M; moreover the natural morphism from M t[o] $C_{\mathrm{End}(M)}(\lambda(P))$ is an isomorphism. Consequently the tower generated by $P \subset M$ i[s] $P \subset M \subset M \subset \cdots$ and the index is 1.

We also claim that $[P : L] = 1$. From left multiplication $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ one has the inclusion

$$\left\{ \begin{array}{ccc} L & \to & \mathrm{End}\,(P) \\ (A, B) & \mapsto & \begin{bmatrix} \lambda_A & 0 & 0 \\ 0 & \lambda_A & 0 \\ 0 & 0 & \lambda_B \end{bmatrix}. \end{array} \right.$$

Thus $C_{\mathrm{End}(P)}(L)$ is the subalgebra

$$N = \left\{ \begin{bmatrix} \rho_R & \rho_S & 0 \\ \rho_T & \rho_U & 0 \\ 0 & 0 & \rho_V \end{bmatrix} : \begin{bmatrix} R & S \\ T & U \end{bmatrix} \in \mathrm{Mat}_{2m'}(\mathbb{C}) \text{ and } V \in \mathrm{Mat}_{m''}(\mathbb{C}) \right\}$$

of $\mathrm{End}_{\mathbb{C}}(P)$, isomorphic to $(\mathrm{Mat}_{m'}(\mathbb{C}) \otimes \mathrm{Mat}_2(\mathbb{C})) \oplus \mathrm{Mat}_{m''}(\mathbb{C})$. As right multiplication $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \mapsto \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is represented in $\mathrm{End}_{\mathbb{C}}(P)$ by the matrix

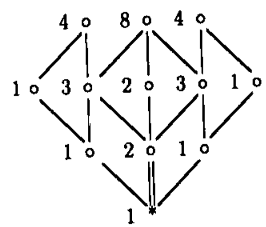
$$\begin{bmatrix} \rho_A & 0 & 0 \\ \rho_B & \rho_C & 0 \\ 0 & 0 & \rho_C \end{bmatrix}$$

the canonical morphism $P \longrightarrow N$ is given by

$$\begin{cases} P & \to N = (\mathrm{Mat}_{m'}(\mathbb{C}) \otimes \mathrm{Mat}_2(\mathbb{C})) \oplus \mathrm{Mat}_{m'}(\mathbb{C}) \\ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} & \mapsto \left[\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}, C\right] \end{cases}$$

The argument used to show $[M:P]=1$ shows also that the canonical construction applied to $P \subset N$ gives an algebra isomorphic to N. Finally, the tower generated by $L \subset P$ is $L \subset P \subset N \subset N \subset \cdots$ and the index is also 1.

2.4.6. <u>A reprise of Proposition 2.4.1.</u> Let $\mathbf{1} \in N \subset M$ be a pair of multi-matrix algebras with inclusion matrix Λ. Write $\{q_j : 1 \le j \le n\}$ and $\{p_i : 1 \le i \le m\}$ for the minimal central idempotents of N and M respectively. Let B be the two-story Bratteli diagram whose 0^{th} story is $B(N \subset M)$ and whose 1^{st} story is the reflection of $B(N \subset M)$; that is $\Lambda^{(0)} = \Lambda$ and $\Lambda^{(1)} = \Lambda^t$. Let $\tilde{B}$ be the augmented diagram, as in 2.3.11. For example for $\mathbb{C}\mathfrak{S}_3 \subset \mathbb{C}\mathfrak{S}_4$, $\tilde{B}$ is



We identify the pair $N \subset M$ with the pair $A_0 \subset A_1$, of path algebras associated with $\tilde{B}$. (See 2.3.11.) Write $\{\tilde{q}_j : 1 \le j \le n\}$ for the minimal central idempotents of the path algebra A_2. According to 2.4.1 and 2.3.9, there is an isomorphism of $\mathrm{End}^r_N(M)$ onto A_2 which takes $\lambda(M)$ onto M. Our purpose here is to use the path model to provide an explicit isomorphism. Except as noted above, our notation is as in 2.3.11.

An edge on $\tilde{B}$ is specified by the data $\eta = (k;i,j,\ell)$, where k is the story on which η lies, v^k_j and v^{k+1}_i are the two vertices of η, and the index ℓ distinguishes among the $\lambda^k_{i,j}$ edges joining v^k_j and v^{k+1}_i. Define an involution $*$ of $\Omega_{[0,1]} \cup \Omega_{[1,2]}$ by

$$(k;i,j,\ell)^* = (1-k;j,i,\ell).$$

Thus $*$ is the reflection through the first floor. (Nevertheless we regard the reflection of an upward oriented edge to be upward oriented.)

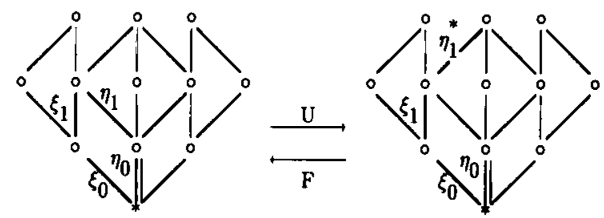
Let $V_j = \mathbb{K}\Omega^j_{2]} \otimes \mathbb{K}\Omega^j_{0]}$ and $V = \bigoplus_{j=1}^{n} V_j$. Define a linear map U from M to V by requiring

$$U(T_{\xi,\eta}) = (\xi_0,\xi_1,\eta_1^*) \otimes \eta_0 \qquad ((\xi,\eta) \in R_1).$$

U is a linear isomorphism, its inverse F being determined by

$$F(\xi' \otimes \eta_0) = T_{(\xi_0',\xi_1'),(\eta_0,\xi_2'^*)}$$
$$(\xi' \in \Omega^j_{2]} \text{ and } \eta_0 \in \Omega^j_{0]};\ 1 \le j \le n).$$

Note that U breaks and unfolds the round trip path $\xi \circ \eta^{-1}$, while F folds and joins the pair (ξ', η_0). For example:



V carries both a right action of N and a left action of A_2, arising from the right action of N on $\mathbb{K}\Omega_{0]}$ and the left action of A_2 on $\mathbb{K}\Omega_{2]}$:

$$\rho(n)(\xi \otimes \eta) = \xi \otimes \rho(n)\eta$$
$$x(\xi \otimes \eta) \ \ = x\xi \otimes \eta.$$

It is easy to check that A_2 is in fact the commutant of $\rho(N)$ in $\mathrm{End}_{\mathbb{K}}(V)$, and that U intertwines the right actions of N on M and V. Hence

$$\alpha : \varphi \mapsto U \circ \varphi \circ F$$

is an isomorphism from $\mathrm{End}^r_N(M) = C_{\mathrm{End}_{\mathbb{K}}(M)}(\rho(N))$ to $C_{\mathrm{End}_{\mathbb{K}}(V)}(\rho(N)) = A_2$.

Let $(\xi,\eta) \in R_2$ and $(\sigma,\tau) \in R_1$ (so $T_{\xi,\eta} \in A_2$ and $T_{\sigma,\tau} \in M$). One checks that

$$\alpha^{-1}(T_{\xi,\eta})(T_{\sigma,\tau}) = \delta(\eta,\sigma\circ\tau_1^*)T_{(\xi_0,\xi_1),(\tau_0,\xi_2^*)}.$$

It follows that $\alpha^{-1}(x) = \lambda(x)$ for $x \in M \subset A_2$. Also

$$\alpha^{-1}(\tilde{q}_j) = \alpha^{-1}\Big(\sum_{\xi\in\Omega^j_{2]}} T_{\xi,\xi}\Big) = \rho(q_j),$$

as required by Proposition 2.4.1.

Remark. Later we will want to modify the definition of U somewhat. If $c : \Omega_{[0,1]} \to \mathbb{K}^*$ is any function and we instead define U by

$$U(T_{\xi,\eta}) = c(\eta_1)(\xi_0,\xi_1,\eta_1^*) \circledast \eta_0,$$

then $\varphi \mapsto U\circ\varphi\circ U^{-1}$ is another isomorphism of $\mathrm{End}^r_N(M)$ onto A_2.

2.5. Traces.

A $\mathbb{K}$-linear map φ from $\mathbb{K}$-algebra M to a $\mathbb{K}$-vector space V is said to be faithful if the corresponding bilinear map

$$(x,y) \mapsto \varphi(xy)$$

is non-degenerate; that is for each non-zero $x \in M$ there is a $y \in M$ such that $\varphi(xy) \neq 0$. This is a one-sided notion, but if M is finite dimensional and $\varphi : M \to \mathbb{K}$ is linear, then φ is faithful on one side if and only if it is faithful on the other. Furthermore, in this case, for each linear $\psi : M \to \mathbb{K}$, there is an $a \in M$ such that $\psi(x) = \varphi(xa)$ for all $x \in M$.

A trace on M is a linear map $\mathrm{tr} : M \to \mathbb{K}$ such that $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x,y \in M$. On a factor, any non-zero trace is faithful, and any two traces are proportional. In fact a trace on $\mathrm{Mat}_d(\mathbb{K})$ satisfies $\mathrm{tr}(e_{i,j}) = \delta^j_i\,\mathrm{tr}(e_{1,1})$, where $\{e_{i,j}\}$ are the standard matrix units.

Let M be a multi-matrix algebra over $\mathbb{K}$, written as before as $M = \bigoplus_{i=1}^m p_iM$, with $p_iM \cong \mathrm{Mat}_{\mu_i}(\mathbb{K})$. We associate to a trace tr on M the row-vector

$$\overleftarrow{s} = (\mathrm{tr}(e_1),\cdots,\mathrm{tr}(e_m)) \in \mathbb{K}^m,$$

where e_i is a minimal idempotent in Mp_i. For example, the trace $\sigma^{(i)}$ defined by $\sigma^{(i)}(e_i) = 1$ and $\sigma^{(i)}|_{p_kM} = 0$ for $k \neq i$ corresponds to the i^{th} vector of the canonical basis of $\mathbb{K}^m$. Any row vector $\overleftarrow{s} \in \mathbb{K}^m$ determines a unique trace

$$\mathrm{tr}_s = \sum_{i=1}^m s_i\sigma^{(i)} : M \to \mathbb{K}$$

with associated vector $\overleftarrow{s}$.

A trace tr on M is faithful if and only if the associated vector $\overleftarrow{s}$ has no zero entries. When the characteristic of $\mathbb{K}$ is zero, we say that tr is positive if $s_i \geq 0$ for all i. (There is an ambiguity here; if $\mathbb{K}$ is given as an extension of the reals, the meaning of $s_i \geq 0$ is clear. Otherwise we take $s_i \geq 0$ to mean that there is an imbedding of $\mathbb{Q}(s_1,\cdots,s_m)$ in $\mathbb{C}$ such that $s_i \geq 0$ for all i.) A positive trace is faithful if $s_i > 0$ for all i.

Proposition 2.5.1. *Let $\mathbf{1} \in N \subset M$ be a pair of multi-matrix algebras with*

$$N = \bigoplus_{j=1}^n q_jN, \quad M = \bigoplus_{i=1}^m p_iM$$

and with inclusion matrix Λ^M_N.

(a) *Let σ be a trace on M corresponding to $\overleftarrow{s} \in \mathbb{K}^m$ and let τ be a trace on N corresponding to $\overleftarrow{t} \in \mathbb{K}^n$. Then σ extends τ if and only if $\overleftarrow{t} = \overleftarrow{s}\Lambda^M_N$.*

(b) *If $\mathrm{char}(\mathbb{K}) = 0$, then there exists a faithful trace on M with faithful restriction to N. If $\mathrm{char}(\mathbb{K}) = p > 0$, then a sufficient condition for the existence of a faithful trace on M with faithful restriction to N is that for all j, the sum $\sum_i \lambda_{i,j}$ is not divisible by p.*

Proof. (a) If f_j is a minimal idempotent in q_jN, then f_jp_i is the sum of $\lambda_{i,j}$ minimal idempotents in p_iM. Hence the restriction of σ to N is described by the vector $\overleftarrow{t}'$ with components

$$t'_j = \sigma(f_j) = \sum_{i=1}^m \sigma(f_jp_i) = \sum_{i=1}^m s_i\lambda_{i,j} = (\overleftarrow{s}\Lambda^M_N)_j.$$

(b) Define the trace on M with weights $s_i = 1$ for all i. The restriction to N has weights $t_j = \sum_i \lambda_{i,j}$ (mod char($\mathbb{K}$)). If char($\mathbb{K}$) = 0, or if the column sums of Λ_N^M are not divisible by characteristic, then the weights t_j are non–zero, and the restricted trace is faithful. #

Remarks

(1) With the notation of the proposition, one has, when σ extends τ,

$$(\vec{s}, \vec{\mu}) = \sum_{i=1}^{m} s_i \mu_i = \sigma(1) = \tau(1) = (\vec{t}, \vec{\nu}).$$

By Propositions 2.3.1.b and 2.5.1.a, this implies

$$(\vec{s}, \Lambda_N^M \vec{\nu}) = (\vec{s}\Lambda_N^M, \vec{\nu}),$$

which is, of course, obvious!

(2) A faithful trace on M may have zero restriction to N. Consider for example

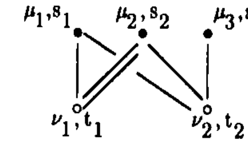
$$N = \mathbb{C}[\mathfrak{S}_2] \subset M = \mathbb{C}[\mathfrak{S}_3] \cong \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$$

as in Example 2.3.7, and the trace on M associated to the vector $(1,-1,1) \in \mathbb{C}^3$. Or consider the two element field $\mathbb{F}_2$ and the pair $\mathbb{F}_2 \subset Mat_2(\mathbb{F}_2)$ (with inclusion matrix [2]); any trace on $Mat_2(\mathbb{F}_2)$ has zero restriction to the center $\mathbb{F}_2$! One may thus say about traces on multi–matrix algebras, that positivity is hereditary, but faithfulness is not.

(3) The assignment of a vector $\vec{s} \in \mathbb{K}^m$ to a trace $tr : M \to \mathbb{K}$ has been defined above via the values of tr on (classes of) minimal idempotents of M. In Chapter 3, we shall consider a new situation, where M is a finite direct sum of continuous (type II_1) von Neumann factors; since no minimal idempotents are present in this situation, we shall describe a trace tr by the vector $\tilde{s} = (tr(p_1),\cdots,tr(p_m))$ of values of tr on minimal *central* projections of M. In principle, the description of tr via $\tilde{s}$ is also possible for multi–matrix algebras, but this causes complications which would be out of place in the present chapter.

(4) Given a Bratteli diagram representing a sequence of inclusions of multi–matrix algebras, $A_k \subset A_{k+1}$, and a trace on $\bigcup_k A_k$, it is sometimes convenient to record all the data on the diagram by marking each vertex with the weight of the trace on the corresponding factor, that is the value of the trace on a minimal idempotent in the factor,

as well as with the dimensions. Thus, in the situation of Proposition 2.5.1, and for $\Lambda = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$, the diagram is



(Note that on any Bratteli diagram it is actually superfluous to record the dimensions except on the first floor. Similarly on a finite Bratteli diagram it is superfluous to record the weights of a trace except on the top floor, but on an infinite Bratteli diagram it is not (in general) superfluous to record the traces, since the traces on the higher floors are not determined by those on the floors below.)

(5) Suppose $\mathbb{K} \supset \mathbb{R}$, and $N \subset M$ is a pair of multi–matrix algebras over $\mathbb{K}$ with $Z_N \cap Z_M = \mathbb{K}$ and with inclusion matrix Λ. Let $(M_k)_{k\geq 0}$ be the tower obtained by iterating the fundamental construction. Then it follows from Perron–Frobenius theory that there is a unique positive normalized $(tr(\mathbf{1}) = 1)$ trace on $M_\infty = \bigcup_k M_k$.

In fact, let $\vec{t}^{(0)}$ be the Perron–Frobenius eigenvector for $\Lambda^t\Lambda$, normalized by $\sum t_j^{(0)}\nu_j = 1$. Define $\vec{t}^{(2k)} = \|\Lambda\|^{-2k}\,\vec{t}^{(0)}$ and $\vec{t}^{(2k-1)} = \vec{t}^{(2k)}\Lambda^t$ $(k \geq 1)$. Then $\{\vec{t}^{(k)}\}_{k\geq 1}$ defines a consistent family of traces on the algebras M_k, since $\vec{t}^{(k+1)}\Lambda_{M^k}^{M^{k+1}} = \vec{t}^{(k)}$ for all k.

An argument similar to one given in the proof of 2.4.2. shows that $\bigcap_{r\geq 0} (\Lambda^t\Lambda)^r(\mathbb{R}_+^n)$ consists of Perron–Frobenius eigenvectors for $\Lambda^t\Lambda$. Suppose tr is any positive normalized trace on M_∞ and $\vec{s}^{(k)}$ is the vector determining tr on M_k. Then for all k and r,

$$\vec{s}^{(2k+2r)}(\Lambda^t\Lambda)^r = \vec{s}^{(2k)},$$

whence $\vec{s}^{(2k)}$ is a Perron–Frobenius eigenvector for $\Lambda^t\Lambda$. Since $\vec{s}^{(2k)}(\Lambda^t\Lambda)^k = \vec{s}^{(0)}$, we have $\vec{s}^{(2k)} = \|\Lambda\|^{-2k}\,\vec{s}^{(0)}$.

2.6. Conditional expectations.

We are primarily interested in the following situation:

(1) $N \subset M$ is a pair of multi–matrix algebras.

(2) M has a faithful trace with faithful restriction to N.

(3) $E : M \to N$ is the orthogonal projection of M onto N with respect to the inner product determined by the trace.

However, to clarify somewhat the roles played by semi- simplicity, the pair of faithful traces, and the conditional expectation E, we begin in a more general setting.

A <u>conditional expectation</u> from a $\mathbb{K}$-algebra M onto a subalgebra N is an (N,N)-linear map whose restriction to N is the identity. Recall that such a map E is faithful if for each non-zero $x \in M$ there is a $y \in M$ such that $E(xy) \neq 0$. For example, if M is a factor, $M = \mathrm{Mat}_\mu(\mathbb{K})$, where $\mathbb{K}$ has charactersitic 0 or μ is relatively prime to $\mathrm{char}(\mathbb{K})$, then the trace on M normalized by $\mathrm{trace}(\mathbf{1}) = 1$ is a faithful conditional expectation of M onto $\mathbb{K}$.

Consider $\mathrm{Hom}_N^r(M,N)$, the set of right N-linear maps from M to N, with its left N-module structure defined by $(x\varphi)(y) = x\varphi(y)$ $(x \in N,\ y \in M,\ \varphi \in \mathrm{Hom}_N^r(M,N))$. We associate to a conditional expectation $E : M \to N$ the left N-linear map $E^b : M \to \mathrm{Hom}_N^r(M,N)$ defined by $E^b(x)(y) = E(xy)$ for $x,y \in M$. Then E is faithful if and only if E^b is injective. We say that E is <u>very faithful</u> if E^b is an isomorphism.

<u>Lemma 2.6.1</u>. *Let* $N \subset M$ *be a pair of finite dimensional* $\mathbb{K}$*–algebras. Suppose* N *has a faithful* $\mathbb{K}$*–linear functional. Then any faithful expectation* E *from* M *to* N *is very faithful.*

<u>Proof</u>. Choose a faithful functional $\tau : N \longrightarrow \mathbb{K}$ and set $\sigma = \tau \circ E$. If $x \in M$ is such that $\sigma(xx') = 0$ for all $x' \in M$, then $\sigma(xyz) = \tau(E(xy)z) = 0$ for all $y \in M$ and for all $z \in N$, so that $E(xy) = 0$ by faithfulness of τ and $x = 0$ by that of E. Thus σ is faithful. It follows that any $\mathbb{K}$-linear map $M \longrightarrow \mathbb{K}$ is of the form $x \mapsto \sigma(ax)$ for some $a \in M$, since M is finite dimensional.

Consider a right N-linear map $\varphi : M \to N$. There exists $a \in M$ with $\tau\varphi(x) = \sigma(ax)$ for all $x \in M$. Define $\psi : M \to N$ by $\psi = E^b(a)$; i.e., $\psi(x) = E(ax)$. We claim that $\psi = \varphi$. It is enough to check that $\lambda\psi = \lambda\varphi$ for any $\mathbb{K}$-linear $\lambda : N \to \mathbb{K}$. But as τ is faithful, such a λ is given by $y \mapsto \tau(yb)$ for some $b \in N$. Now one has for all $x \in M$

$$\lambda\psi(x) = \tau(E(ax)b) = \tau E(axb), \text{ and}$$

$$\lambda\varphi(x) = \tau(\varphi(x)b) = \tau(\varphi(xb)) = \sigma(axb) = \tau E(axb). \quad \#$$

<u>Remarks</u>. (1) If N is a multi-matrix algebra, then N has a faithful $\mathbb{K}$-linear functional.

(2) Let V be a $\mathbb{K}$-vector space and define a multiplication on $A = \mathbb{K} \oplus V$ by $(\lambda,v)(\lambda',v') = (\lambda\lambda', \lambda v' + \lambda' v)$. The result is a $\mathbb{K}$-algebra for which any subspace of $0 \oplus V$ is an ideal. Suppose $\dim V \geq 2$. If $\varphi : A \to \mathbb{K}$ is any $\mathbb{K}$-linear functional, then $\ker(\varphi) \cap V$ is a non-zero ideal in $\ker(\varphi)$. So A has no faithful linear functional.

But if V is one–dimensional, spanned by v, then the functional $(a,bv) \mapsto a + b$ is faithful on A.

The next proposition concerns the existence of faithful conditional expectations.

<u>Proposition 2.6.2</u>. *Let* $N \subset M$ *be a pair of* $\mathbb{K}$*–algebras with* N *finite dimensional, and let* $\mathrm{tr} : M \to \mathbb{K}$ *be a faithful trace with faithful restriction to* N. *Then there exists a unique* $\mathbb{K}$*–linear map* $E : M \to N$ *such that*

(i) $\mathrm{tr}(E(x)) = \mathrm{tr}(x)$ $\quad x \in M$

(ii) $E(y) = y$ $\quad y \in N$

(iii) $E(xy) = E(x)y$ $\quad x \in M,\ y \in N$.

Moreover E *is a faithful conditional expectation from* M *to* N, *namely*

(iv) $E(yx) = yE(x)$ $\quad x \in M,\ y \in N$

(v) $E(xy) = 0$ *for all* y *implies* $x = 0$.

If M *is finite dimensional, then* E *is very faithful; that is*

(vi) $E^b : M \to \mathrm{Hom}_N^r(M,N)$ *defined by* $a \mapsto (x \mapsto (E(ax))$ *is an isomorphism.*

<u>Proof</u>. We consider M together with the nondegenerate symmetric $\mathbb{K}$-bilinear form $(x,z) \mapsto \mathrm{tr}(xz)$ and with the associated orthogonality relation. As tr and $\mathrm{tr}|_N$ are faithful one has $M = N \oplus N^\perp$.

We begin by checking uniqueness. Let $E : M \to N$ satisfy (i) to (iii). As E is defined on N by (ii), it is enough to check that $E = 0$ on $N^\perp$. Let $t \in N^\perp$. For any $y \in N$ one has by (iii) and (i)

$$\mathrm{tr}(E(t)y) = \mathrm{tr}(E(ty)) = \mathrm{tr}(ty) = 0$$

so that $E(t) \perp N$. But $E(t)$ is also in N, so that $E(t) = 0$.

To prove existence, define E to be the projection of M onto N along $N^\perp$. It is obvious that (ii) holds. For $x \in M$, one has $E(x)-x$ orthogonal to N and hence to $\mathbf{1}$, so (i) holds.

Note that $N^\perp$ is a right N-module because of the trace property of tr. Namely if $y,y' \in N$ and $z \in N^\perp$. Then

$$\mathrm{tr}(y'(zy)) = \mathrm{tr}((yy')z) = 0,$$

so $zy \in N^\perp$. Now $xy - E(xy)$ and $x - E(x)$ are in $N^\perp$, and hence also $xy - E(x)y \in N^\perp$. The difference

$$(xy-E(xy)) - (xy-E(x)y) = E(x)y-E(xy)$$

is in $N^\perp \cap N = (0)$, which proves (iii). One obtains (iv) similarly.

Since $\mathrm{tr} = \mathrm{tr} \circ E$, the faithfulness of E follows from that of tr. Finally, if M is finite dimensional, then E is very faithful by Lemma 2.6.1. #

Remark. Conditions (i)–(iii) are equivalent to the single condition

$$\mathrm{tr}(E(x)y) = \mathrm{tr}(xy) \quad \text{for } x \in M \text{ and } y \in N,$$

as the reader may verify.

The relevance of conditional expectations for the *fundamental construction* comes from the following fact.

Proposition 2.6.3. *Let* M, N *be* $\mathbb{K}$*-algebras with* $\mathbf{1} \in N \subset M$; *set* $L = \mathrm{End}_N^r(M)$ *and let* $\lambda : M \to L$ *denote the inclusion. Assume moreover that*

(i) *the right* N*-module* M *is projective of finite type, and*

(ii) *there exists a very faithful conditional expectation* E *from* M *to* N.

Then L *is generated by* M *and* E *(viewed as a map from* M *to* M*). More precisely,* L *is generated as a* $\mathbb{K}$*–vector space by elements of the form* $\lambda(x)E\lambda(y)$ *with* $x,y \in M$. *Furthermore, the map* $x \otimes y \mapsto \lambda(x)E\lambda(y)$ *from* $M \otimes_N M$ *to* $\mathrm{End}_N^r(M)$ *is an isomorphism.*

Proof. Hypothesis (ii) says that $E^b : M \to M^* = \mathrm{Hom}_N^r(M,N)$ is an isomorphism. As projective modules of finite type are flat (see [BAC 1], page 28), the $\mathbb{K}$-linear map

$$\mathrm{id}_M \otimes E^b : M \otimes_N M \longrightarrow M \otimes_N M^*$$

is an isomorphism. Let

$$\theta \begin{cases} M \otimes_N M^* \to L \\ x \otimes x^* \mapsto (z \mapsto x x^*(z)) \end{cases}$$

be the canonical homomorphism. By (i), it is an isomorphism (see, e.g., [BA 2], page 111). Consequently, the composition

$$\Phi = \theta(\mathrm{id}_M \otimes E^b) : M \otimes_N M \to L$$

is an isomorphism. Routine computations show that

$$\Phi(x \otimes y) = \lambda(x)E\lambda(y) \qquad x,y \in M$$

$$\Phi(x \otimes y)\Phi(z \otimes t) = \Phi(xE(yz) \otimes t) \quad x,y,z,t \in M.$$

The proposition follows from the first of these. #

Remarks.

(1) It could be that M is projective of finite type as a right N–module but not as a left N–module, as observed in [BA 8], page 53.

(2) In the situation of the previous proposition can we conclude that L is projective of finite type over M (as a *right* $\lambda(M)$–module)?

For pairs of multi–matrix algebras, the situation regarding pairs of faithful traces and conditional expectations is the following:

(1) If $\mathrm{char}\,\mathbb{K} = 0$, then for any pair of multi–matrix algebras $N \subset M$ over $\mathbb{K}$, there exist faithful traces on M with faithful restriction to N (2.5.1), hence faithful conditional expectations $E : M \to N$ (2.6.2).

(2) Whenever $E : M \to N$ is a faithful conditional expectation, it is very faithful, since N always has a faithful functional (2.6.1).

(3) If $\mathrm{char}\,\mathbb{K} > 0$, M need not have a faithful trace with faithful restriction to N. For example there is no pair of faithful traces for $\mathbb{F}_2 \subset \mathrm{Mat}_2(\mathbb{F}_2)$. Note that nevertheless $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + b + c$ defines a faithful conditional expectation $\mathrm{Mat}_2(\mathbb{F}_2) \to \mathbb{F}_2$.

Corollary 2.6.4. *Consider a pair of multi–matrix algebras*

$$\mathbf{1} \in N = \bigoplus_{j=1}^{n} q_j N \subset M = \bigoplus_{i=1}^{m} p_i M$$

as well as

$$L = \bigoplus_{j=1}^{n} \rho(q_j)L = \mathrm{End}_N^r(M).$$

Suppose there is a faithful conditional expectation $E : M \to N$. *Then*

(a) L *is generated as a* $\mathbb{K}$*-vector space by elements* $\lambda(x)E\lambda(y)$ *for* $x,y \in M$;

(b) *The* $\mathbb{K}$*–linear map* $\varphi : N \to ELE$ *defined by* $\varphi(x) = \lambda(x)E$ *is an isomorphism of algebras.*

(c) *If* f_j *is a minimal idempotent in the factor* q_jN, *then* $\lambda(f_j)E$ *is a minimal idempotent in the factor* $\rho(q_j)L$.

Proof. (a) Condition (i) of Proposition 2.6.3 is fulfilled because any module over a semi–simple algebra is projective, and condition (ii) is fulfilled by Lemma 2.6.1.

To prove (b), first note that φ is a morphism because E is an idempotent which commutes with $\lambda(x)$ for all $x \in N$. If $x \in N$ and $\varphi(x) = 0$, then also $x = \varphi(x)(\mathbf{1}) = 0$, so φ is injective. Finally φ is surjective by part (a).

For $j \in \{1,\cdots,n\}$, the idempotent $\rho(q_j)E = \lambda(q_j)E$ is not zero and lies in the factor $\rho(q_j)L$. The resulting reduced factor is $\rho(q_j)ELE = \rho(q_j)\lambda(N)E$. As φ is an isomorphism, its restriction φ_j to q_jN is also an isomorphism onto $\rho(q_j)ELE$. It follows that the idempotent $\varphi_j(f_j) = \lambda(f_j)E$ is minimal in the factor $\rho(q_j)ELE$. But if e $\in$ L is an non-zero idempotent in L dominated by $\lambda(f_j)E$, and thus also by $\lambda(q_j)E = \rho(q_j)E$, then $e = \rho(q_j)E\, e\, \rho(q_j)E \in \rho(q_j)ELE$, and therefore $e = \lambda(f_j)E$. In other words, $\lambda(f_j)E$ is also minimal in L. #

<u>Remark:</u> The following instructive proof of 2.6.4.a was given by Wenzl [Wen3]. First note that the map φ of 2.6.4.b is an injective homomorphism. Now consider the subalgebra A of L generated by $\lambda(M)$ and E, and note that

$$A = \{\ \lambda(y_0) + \sum_i \lambda(x_i)E\lambda(y_i) : x_i, y_i \in M\},\ \text{and}$$
$$EAE = \varphi(N) \cong N$$

If ψ is a non-zero element of rad(A), then there exist $x, y \in M$ such that $E(y\psi(x)) \neq 0$ (using the faithfulness of E). But then $E\lambda(y)\psi\lambda(x)E = \lambda(E(y\psi(x))E = \varphi(E(y\psi(x)))$ is a non-zero element of $\mathrm{rad}(A) \cap EAE = \mathrm{rad}(EAE)$, a contradiction since EAE is isomorphic to the semi-simple algebra N. Thus A is semi-simple. Note that $A' = \lambda(M)' \cap \{E\}' = \rho(N)$, so $A'' = \rho(N)' = L$, where primes denote centralizers in $\mathrm{End}_K(M)$. Since A is semi-simple, $A = A'' = L$. Finally observe that $\lambda(M)E\lambda(M) = \{\sum_i \lambda(x_i)E\lambda(y_i) : x_i, y_i \in M\}$ is an ideal in L, and if ψ is a central projection in L orthogonal to this ideal, then for all $x, y \in M$,

$$0 = (E\lambda(y)\psi)(x) = E(y\psi(x)).$$

Hence $\psi = 0$ by faithfulness of E, so $L = \lambda(M)E\lambda(M)$. #

<u>2.6.5 Reprise of 2.6.4 using the path model</u>. Let N, M, and L be as in 2.6.4. Suppose tr is a faithful trace on M with faithful restriction to N, and let $E : M \to N$ be the conditonal expectation determined by tr, as in 2.6.2. Let B be the Bratteli diagram for $N \subset M \subset L$, and let $\tilde{B}$ be the augmented diagram, as in 2.3.11 and 2.4.6. We identify $N \subset M$ with the pair of path algebras $A_0 \subset A_1$, but we distinguish between $L = \mathrm{End}_N^r(M)$ and the isomorphic path algebra A_2. Let $\overleftarrow{s}$ and $\overleftarrow{t}$ be the vectors determing the trace tr on M and N. We also regard $\overleftarrow{s}$ and $\overleftarrow{t}$ as functions of vertices on the 0^{th} and 1^{st} floors respectively: $t(v_j^0) = t_j$ and $s(v_i^1) = s_i$. Recall the *-operation

which reflects edges through the first floor. We define the reflection on vertices as well:

$$(v_i^k)^* = v_i^{2-k} \quad (0 \le k \le 2 \quad 1 \le i \le m(k))$$

We first give a formula for $E \in \mathrm{End}_N^r(M)$. Recall that E is determined by the requirement $\mathrm{tr}(E(z)x) = \mathrm{tr}(zx)$, for $z \in M$ and $x \in N$. If $(\xi,\eta) \in R_1$ and $(\alpha,\beta) \in R_0$, so that $T_{\xi,\eta} \in M$ and $T_{\alpha,\beta} \in N$, then one verifies that

$$\mathrm{tr}(T_{\xi,\eta}T_{\alpha,\beta}) = \delta(\eta_0,\alpha_0)\delta(\xi_0,\beta_0)\delta(\xi_1,\eta_1)s(\xi_{[1]}),$$

while

$$\mathrm{tr}(\delta(\xi_1,\eta_1)T_{\xi_0,\eta_0}T_{\alpha,\beta}) = \delta(\eta_0,\alpha_0)\delta(\xi_0,\beta_0)\delta(\xi_1,\eta_1)t(\xi_{[0]}).$$

Hence

$$\text{(2.6.5.1)} \qquad E(T_{\xi,\eta}) = \frac{s(\xi_{[1]})}{t(\xi_{[0]})}\delta(\xi_1,\eta_1)T_{\xi_0,\eta_0} \quad ((\xi,\eta) \in R_1).$$

(Remark that $\xi_{[1]} = \eta_{[1]}$ and $\xi_{[0]} = \eta_{[0]}$ if $E(T_{\xi,\eta}) \neq 0$, so the expression is not so asymmetric as it may first appear.)

Let

$$U(T_{\xi,\eta}) = c(\eta_1)(\xi_0,\xi_1,\eta_1^*) \otimes \eta_0 \quad ((\xi,\eta) \in R_1),$$

and $F = U^{-1}$, as in 2.4.6. Next we compute $e = U \circ E \circ F$, the image of E in A_2. For $\xi \otimes \eta_0$, an elementary tensor in V_j for some j,

$$\begin{aligned} e\xi \otimes \eta_0 = e(\xi \otimes \eta_0) &= \frac{1}{c(\xi_2^*)} U(E(T_{(\xi_0,\xi_1),(\eta_0,\xi_2^*)})) \\ &= \frac{1}{c(\xi_2^*)} \frac{s(\xi_{[1]})}{t(\xi_{[0]})} \delta(\xi_1,\xi_2^*)\, U(T_{\xi_0,\eta_0}) \\ &= \frac{1}{c(\xi_1)} \frac{s(\xi_{[1]})}{t(\xi_{[0]})} \delta(\xi_1,\xi_2^*)\, U\Big[\sum_{\substack{\lambda \\ \lambda_{[0]}=\xi_{[0]}}} T_{(\xi_0,\lambda),(\eta_0 \circ \lambda)} \Big] \end{aligned}$$

$$= \Big[\delta(\xi_1,\xi_2^*)\frac{s(\xi_{[1]})}{t(\xi_{[0]})}\sum_{\substack{\lambda \\ \lambda_{[0]}=\xi_{[0]}}}\frac{c(\lambda)}{c(\xi_1)}(\xi_0,\lambda,\lambda^*)\Big]\otimes\eta_0.$$

It follows that

$$e = \sum_{\substack{\xi,\lambda\in\Omega_{[0,1]} \\ \xi_{[0]}=\lambda_{[0]}}}\frac{c(\lambda)}{c(\xi)}\frac{s(\xi_{[1]})}{t(\xi_{[0]})}T_{(\lambda,\lambda^*),(\xi,\xi^*)}. \tag{2.6.5.2}$$

Remark. If $\mathbb{K}=\mathbb{C}$, and the trace tr is positive, we prefer to use the inner product $\langle x,y\rangle = tr(xy^*)$ on M, where $*$ is the natural $*$ operation on the path algebra M, rather than the bilinear form $(x,y)\mapsto tr(xy)$. (The orthogonal projection $E: M\to N$ is unaffected by the change.) We give V the inner product for which $\biguplus_j \Omega^j_{2]}\otimes\Omega^j_{0]}$ is an orthonormal basis. Then the choice

$$U(T_{\xi,\eta}) = \sqrt{s(\xi_{[1]})}\,(\xi_0,\xi_1,\eta_1^*)\otimes\eta_0 \tag{2.6.5.3}$$

makes U into a unitary operator from M onto V. In this case e is given by

$$e = \sum_{\substack{\xi,\lambda\in\Omega_{[0,1]} \\ \xi_{[0]}=\lambda_{[0]}}}\frac{\sqrt{s(\xi_{[1]})}\,\sqrt{s(\lambda_{[1]})}}{t(\xi_{[0]})}T_{(\lambda,\lambda^*),(\xi,\xi^*)}. \tag{2.6.5.4}$$

Then e is a self–adjoint projection in the C^*–algebra A_2. This formula for e is due to Sunder [Sun] and Ocneanu [Ocn]. The formulae (2.6.5.3) and (2.6.5.4) are also sensible if $\mathbb{K}$ is any quadratically closed field.

We know from 2.6.4 that any $\varphi\in \mathrm{End}^r_N(M)$ has a decomposition $\varphi=\sum_i \lambda(x_i)\,E\,\lambda(y_i)$ where $x_i,y_i\in M$, but so far we have not considered how to compute such a decomposition. Since the isomorphism $\alpha:\varphi\mapsto U\circ\varphi\circ F$ of $\mathrm{End}^r_N(M)$ onto A_2 takes $\lambda(x)$ to x $(x\in M)$, it suffices to decompose $z\in A_2$ into a sum $z=\sum_i x_i e y_i$ with $x_i,y_i\in M$. For (α,γ) and $(\delta,\beta)\in R_1$ (so $T_{\alpha,\gamma}$ and $T_{\delta,\beta}\in M$) one computes from (2.6.5.2) that

$$T_{\alpha,\gamma}eT_{\delta,\beta} = \delta(\gamma_0,\delta_0)\frac{c(\gamma_1)}{c(\delta_1)}\frac{s(\delta_{[1]})}{t(\delta_{[0]})}T_{(\alpha_0,\alpha_1,\gamma_1^*),(\beta_0,\beta_1,\delta_1^*)}.$$

Hence for $(\alpha,\beta)\in R_2$,

$$T_{\alpha,\beta} = \frac{c(\beta_2^*)t(\beta_{[2]}{}^*)}{c(\alpha_2^*)s(\beta_{[1]})}T_{(\alpha_0,\alpha_1),(\gamma_0,\alpha_2^*)}\,e\,T_{(\gamma_0,\beta_2^*),(\beta_0,\beta_1)}, \tag{2.6.5.5}$$

where γ_0 is an arbitrary edge in $\Omega_{0]}$ with $\gamma_{[0]}=\mathrm{end}(\beta)^*=\mathrm{start}(\beta_2^*)$. In particular if we use the convention (2.6.5.3), and formula (2.6.5.4) we get

$$T_{\alpha,\beta} = \frac{t(\beta_{[2]}{}^*)}{\sqrt{s(\alpha_{[1]})}\sqrt{s(\beta_{[1]})}}T_{(\alpha_0,\alpha_1),(\gamma_0,\alpha_2^*)}\,e\,T_{(\gamma_0,\beta_2^*),(\beta_0,\beta_1)}. \tag{2.6.5.6}$$

Another way to write this is

$$T_{\alpha,\beta} = t(\mathrm{end}(\beta)^*)F(\alpha\otimes\gamma_0)\,e\,F(\beta\otimes\gamma_0)^*.$$

As an exercise in using (2.6.5.6) we compute a decomposition for the minimal central idempotent $\rho(q_j)$ in $\mathrm{End}^r_N(M)$. We have

$$\begin{aligned}\alpha(\rho(q_j)) = \tilde{q}_j &= \sum_{\xi\in\Omega^j_{2]}}T_{\xi,\xi} \\ &= \sum_{\xi\in\Omega^j_{2]}}\frac{t_j}{s(\xi_{[1]})}T_{(\xi_0,\xi_1),(\gamma_0,\xi_2^*)}\,e\,T_{(\gamma_0,\xi_2^*),(\xi_0,\xi_1)},\end{aligned}$$

for any $\gamma_0\in\Omega^j_{0]}$. Taking the average over the ν_j elements of $\Omega^j_{0]}$, we arrive at

$$\rho(q_j) = \frac{t_j}{\nu_j}\sum_{\substack{(\xi,\eta)\in R_1 \\ \eta_{[0]}=v^0_j}}\frac{1}{s(\xi_{[1]})}\lambda(T_{\xi,\eta})\,E\lambda(T_{\eta,\xi}).$$

In the remainder of this section we discuss, following [Wen3] and [BW], the notion of an extension of an algebra with respect to a conditional expectation. This type of structure appears frequently in Chapter 4.