Definition 2.6.6. Let $N \subset M$ be a pair of algebras over a field $\mathbb{K}$, and $E : M \to N$ a faithful conditional expectation. An E-extension of M is a pair $(L,f)$, where $L$ is an algebra containing $M$, $f \in L$, and

(i) $L$ is generated as an algebra by $M$ and $f$.

(ii) $f^2 = f$.

(iii) $fyf = E(y)f = fE(y)$ for all $y \in M$.

(iv) The morphism $\begin{cases} x \mapsto xf \text{ is injective.} \\ N \to L \end{cases}$

The model example of an E-extension is the fundamental construction $(\mathrm{End}_N^r(M),E)$, when $E$ is very faithful and $M$ is projective of finite type as a right $N$-module.

Lemma 2.6.7. *Let $(L,f)$ be an $E$-extension of $M$.*

(i) *Any element of $L$ has the form $y_0 + \sum_{j=1}^{n} y'_j f y''_j$, with $y_0, y'_j, y''_j \in M$. In particular $MfM$ is an ideal of $L$.*

(ii) *There is a unique conditional expectation $\overline{E} : L \to N$ extending $E$ and satisfying $\overline{E}(x)f = fxf$ for $x \in L$. Moreover $\overline{E}(x) = \overline{E}(xf) = \overline{E}(fx)$ for all $x \in L$.*

(iii) *For $x \in L$ there exist unique $b_1, b_2 \in M$ with $xf = b_1 f$ and $fx = fb_2$.*

Proof. (i) is immediate from the definition 2.6.6.

(ii) Let $\psi$ denote the isomorphism $x \mapsto xf$ from $N$ to $L$, whose range is exactly $fLf$. Then $\overline{E} : x \mapsto \psi^{-1}(fxf)$ has the desired properties.

(iii) If $x = y_0 + \sum_j y'_j f y''_j$, then $b_1 = y_0 + \sum_j y'_j E(y''_j)$ satisfies $xf = b_1 f$. If $b \in M$ and $bf = 0$, then for all $y \in M$, $0 = fybf = E(yb)f = \psi \circ E(yb)$. Since $E$ is faithful and $\psi$ injective, $b = 0$. This proves the existence and uniqueness of $b_1$. Proceed similarly for $b_2$. #

Remarks 2.6.8. (1) If $N \subsetneq M$, then $\overline{E}$ is never faithful since $f \neq \mathbf{1}$ and $\overline{E}((\mathbf{1}-f)x) = 0$ for all $x \in L$.

(2) Let $x \in L$. One has $\overline{E}(xy) = 0$ for all $y \in M$ if, and only if, $fx = 0$. Similarly $\overline{E}(yx) = 0$ for all $y \in M$ if, and only if, $xf = 0$.

Let us check the first assertion. Suppose $\overline{E}(xy) = 0$ for all $y \in M$. Then for all $y$,

$$0 = \overline{E}(xy) = \overline{E}(fxy) = \overline{E}(fb_2 y) = \overline{E}(b_2 y) = E(b_2 y).$$

Since $E$ is faithful, $b_2 = 0$ and $fx = fb_2 = 0$.

(3) If $N$, $M$ and $L$ are $*$-algebras, $E = E^*$, and $f = f^*$, then $\overline{E}$ is self adjoint, because $\psi$ is a $*$-morphism.

(4) If $N$ and $M$ are $C^*$-algebras, $L$ is a $*$-subalgebra of a $C^*$-algebra, $E = E^*$ and $f = f^*$, then $\overline{E}$ is positive. Indeed $x \mapsto fxf$ is positive and $\psi^{-1}$ is positive.

Proposition 2.6.9. *Assume that $M$ is projective of finite type as a right $N$-module and $E$ is very faithful. Let $(L,f)$ be an $E$-extension of $M$. Then*

$$\sigma \begin{cases} \mathrm{End}_N^r(M) \to L \\ \sum_j y'_j E y''_j \mapsto \sum y'_j f y''_j \end{cases}$$

*defines a (non-unital) isomorphism of $\mathrm{End}_N^r(M)$ onto the ideal $MfM$ of $L$. Moreover there is a morphism of algebras $\varphi : L \to \mathrm{End}_N^r(M)$ such that $L = MfM \oplus \ker \varphi$ (direct sum of algebras).*

Proof. Identify $M$ with its image in $\mathrm{End}_N^r(M)$. Since by 2.6.3, $\sum y'_j \otimes y''_j \mapsto \sum y'_j E y''_j$ is an isomorphism of $M \otimes_N M$ onto $\mathrm{End}_N^r(M)$, the map $\sigma$ is well-defined, and it is an algebra morphism with image $MfM$, by definition 2.6.6. We set

$$\varphi \begin{cases} L \to \mathrm{End}_N^r(M) \\ y_0 + \sum_j y'_j f y''_j \mapsto y_0 + \sum_j y'_j E y''_j . \end{cases}$$

We have to check that $\varphi$ is well-defined. Let $x = y_0 + \sum_j y'_j f y''_j$ and $a = y_0 + \sum_j y'_j E y''_j$ with $y_0, y'_j, y''_j \in M$. Then for all $y', y'' \in M$

$$fy'xy''f = \{E(y'y_0y'') + \sum_j E(y'y'_j)E(y''_j y'')\}f,$$

while

$$Ey'ay''E = \{E(y'y_0y'') + \sum_j E(y'y'_j)E(y''_j y'')\}E.$$

If $x = 0$, then $Ey'ay''E = 0$ for all $y', y'' \in M$, so $MEMaMEM = 0$; but $\mathrm{End}_N^r(M) = MEM$ by 2.6.3, and since this algebra has a unit, $a = 0$.

It is clear that $\varphi$ is a surjective algebra morphism (indeed $\varphi(MfM) = \mathrm{End}_N^r(M)$), and that $\varphi \circ \sigma$ is the identity. Hence $\sigma$ is injective and $L = MfM \oplus \ker \varphi$ as vector spaces. Since both $MfM$ and $\ker\varphi$ are ideals in $L$, this is actually a direct sum of algebras. #

## 2.7. Markov traces on pairs of multi-matrix algebras.

Let $N \subset M$ be a pair of multi-matrix algebras and let $\lambda : M \to L = \mathrm{End}^{r}_{N}(M)$ be the pair obtained by the fundamental construction. If $E : M \to N$ is a faithful conditional expectation, we know from Corollary 2.6.4 that $L$ is generated as a vector space by elements of the form $\lambda(x)E\lambda(y)$ with $x,y \in M$. Any trace $\mathrm{Tr} : L \to \mathbb{K}$ satisfies

$$\mathrm{Tr}(\lambda(x)E\lambda(y)) = \mathrm{Tr}(\lambda(yx)E) = \mathrm{Tr}(E\lambda(yx)E) = \mathrm{Tr}(\lambda(E(yx))E),$$

for all $x,y \in M$, and hence $\mathrm{Tr}$ is determined by its values on elements of the form $\lambda(x)E$ for $x \in N$.

Let $\mathrm{tr}$ be a faithful trace on $M$ with faithful restriction to $N$ and let $E : M \to N$ be the conditional expectation defined in Proposition 2.6.2. Let $\beta \in \mathbb{K}$. Define $\mathrm{tr}$ to be a Markov trace of modulus $\beta$ if there exists a trace $\mathrm{Tr} : L \to \mathbb{K}$ such that

$$\left.\begin{aligned} \mathrm{Tr}(\lambda(x)) &= \mathrm{tr}(x) \\ \beta\,\mathrm{Tr}(\lambda(x)E) &= \mathrm{tr}(x) \end{aligned}\right\} \text{ for all } x \in M.$$

Observe that this relation implies $\beta \neq 0$, because $\mathrm{tr}$ is faithful. If such a $\mathrm{Tr}$ exists, it is unique in the following strong sense.

Lemma 2.7.1. *Let $N \subset M$ be a pair of multi-matrix algebras and let $\beta \in \mathbb{K}^*$. Let $\mathrm{tr}$ and $E$ be as above. Then there exists at most one trace $\mathrm{Tr}$ on $L$ such that*

$$\beta\,\mathrm{Tr}(\lambda(y)E) = \mathrm{tr}(y) \quad \textit{for all } y \in N.$$

*If such a $\mathrm{Tr}$ exists, then it is faithful and satisfies*

$$\beta\,\mathrm{Tr}(\lambda(x)E) = \mathrm{tr}(x) \quad \textit{for all } x \in M.$$

*If $\overleftarrow{r}$ is the vector describing $\mathrm{Tr}$ and $\overleftarrow{t}$ the vector describing $\mathrm{tr}|_N$, then $\overleftarrow{r}\beta = \overleftarrow{t}$.*

Proof. We use the notation of Corollary 2.6.4. If such a trace $\mathrm{Tr}$ exists, then for each $j$,

$$\begin{aligned} \beta\, r_j &= \beta \mathrm{Tr}(\lambda(f_j)E) && \text{(by 2.6.4.c)} \\ &= \mathrm{tr}(f_j) = t_j, && j = 1,,\cdots,,n, \end{aligned}$$

so that $\overleftarrow{r}\beta = \overleftarrow{t}$. Uniqueness and faithfulness of $\mathrm{Tr}$ follow. Finally

$$\beta\,\mathrm{Tr}(\lambda(x)E) = \beta\,\mathrm{Tr}(E\lambda(x)E) = \beta\,\mathrm{Tr}(\lambda(E(x))E) = \mathrm{tr}(E(x)) = \mathrm{tr}(x)$$

for all $x \in M$. #

Proposition 2.7.2. *Let $\beta \in \mathbb{K}^*$, let $N \subset M$ be a multi-matrix algebra pair with inclusion matrix $\Lambda$ and let $\lambda : M \to L$ be the pair obtained by the fundamental construction. Let the decompositions into factors be*

$$N = \bigoplus_{j=1}^{n} q_j N \subset M = \bigoplus_{i=1}^{m} p_i M \xrightarrow{\lambda} L = \bigoplus_{j=1}^{n} \rho(q_j)L$$

$$q_j N \cong \mathrm{Mat}_{\nu_j}(\mathbb{K}) \qquad p_i M \cong \mathrm{Mat}_{\mu_i}(\mathbb{K}) \qquad \rho(q_j)L \cong \mathrm{Mat}_{\kappa_j}(\mathbb{K}),$$

$$\vec{\nu} = (\nu_1,\cdots,\nu_n) \qquad \vec{\mu} = (\mu_1,\cdots,\mu_m) \qquad \vec{\kappa} = (\kappa_1,\cdots,\kappa_n)$$

*so that in particular*

$$\Lambda\vec{\nu} = \vec{\mu} \qquad \Lambda^t\vec{\mu} = \vec{\kappa}$$

*Let $\mathrm{tr}$ be a faithful trace on $M$ with faithful restriction to $N$ and associated conditional expectation $E : M \to N$. Let $\overleftarrow{s} \in \mathbb{K}^m$ and $\overleftarrow{t} \in \mathbb{K}^n$ be the corresponding vectors, so that in particular $\overleftarrow{t} = \overleftarrow{s}\Lambda$. Finally, let $\beta \in \mathbb{K}^*$.*

*Then the following are equivalent.*

(i) *$\mathrm{tr}$ is a Markov trace of modulus $\beta$.*

(ii) *$\overleftarrow{s}(\Lambda\Lambda^t) = \beta\,\overleftarrow{s}$ and $\overleftarrow{t}(\Lambda^t\Lambda) = \beta\,\overleftarrow{t}$.*

*In particular, if $\mathrm{char}(\mathbb{K}) = 0$ and if $\beta$ is the modulus of some Markov trace on $M$, then $\beta$ is a totally positive algebraic number; that is, $\beta > 0$ for any imbedding of $\mathbb{Q}(\beta)$ in $\mathbb{C}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathrm{Tr}$ be as in the definition of a Markov trace, and let $\overleftarrow{r} \in \mathbb{K}^n$ be the corresponding vector. Then $\overleftarrow{t} = \overleftarrow{r}\Lambda^t\Lambda$ because $\mathrm{Tr}$ extends $\mathrm{tr}$, and $\overleftarrow{t} = \beta\overleftarrow{r}$ by the previous lemma, so that $\beta\,\overleftarrow{t} = \overleftarrow{t}\Lambda^t\Lambda$. One has also $\overleftarrow{s} = \overleftarrow{r}\Lambda^t$, so that

$$\overleftarrow{s}\Lambda\Lambda^t = \overleftarrow{r}\Lambda^t\Lambda\Lambda^t = \beta\,\overleftarrow{r}\Lambda^t = \beta\,\overleftarrow{s}.$$

(ii) $\Rightarrow$ (i) Set $\overleftarrow{r} = \beta^{-1}\overleftarrow{t}$ and let $\mathrm{Tr} : L \to \mathbb{K}$ be the corresponding trace. Then $\mathrm{Tr}$ extends $\mathrm{tr}$ because

$$\overleftarrow{r}\Lambda^t = \beta^{-1}\overleftarrow{t}\Lambda^t = \beta^{-1}\overleftarrow{s}\Lambda\Lambda^t = \overleftarrow{s}.$$

Consider the linear map $\tilde{\tau} : N \to \mathbb{K}$ defined by $\tilde{\tau}(y) = \beta\,\mathrm{Tr}(\lambda(y)E)$; it is a trace, because $E$ is N–linear and idempotent. If $f_j$ denotes some minimal idemotent in $q_jN$, one has

$$\tilde{\tau}(f_j) = \beta\mathrm{Tr}(\lambda(f_j)E) = \beta r_j = t_j,\quad j = 1,,\cdots,n$$

by Corollary 2.6.4.c and the definition of $\overleftarrow{r}$, so that $\tilde{\tau} = \mathrm{tr}|_N$. Thus Tr satisfies the Markov condition $\beta\mathrm{Tr}(\lambda(x)E) = \mathrm{tr}(x)$ for all $x \in M$ by the previous lemma.

Finally matrices of the form $\Lambda^t\Lambda$ have totally positive eigenvalues, when $\mathrm{char}(\mathbb{K}) = 0$. #

Remarks.

(1) Take $\Lambda = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\overleftarrow{s} = (3,1)$, so that $\overleftarrow{t} = (4,4)$. Then $\overleftarrow{t}\Lambda^t\Lambda = 4\overleftarrow{t}$, but $\overleftarrow{s}\Lambda\Lambda^t$ is not a scalar multiple of $\overleftarrow{s}$. This shows that one cannot delete the first equality in condition (ii). Although $\overleftarrow{t}\Lambda^t\Lambda = \overleftarrow{t}\beta$ follows from $\overleftarrow{s}\Lambda\Lambda^t = \overleftarrow{s}\beta$ (because $\overleftarrow{t} = \overleftarrow{s}\Lambda$), we prefer to state (ii) in a symmetric form.

(2) *We stress that* $\beta > 0$ *holds without any positivity assumption on* tr, *in case* $\mathrm{char}(\mathbb{K}) = 0$.

Theorem 2.7.3. *Let* $\mathbb{K}$ *be a field extension of* $\mathbb{R}$. *Let* $N \subset M$ *be a pair of multi–matrix algebras over* $\mathbb{K}$ *with inclusion matrix* $\Lambda$, *and with* $Z_M \cap Z_N = \mathbb{K}$. *Let* $\beta \in \mathbb{K}^*$.

*A necessary and sufficient condition for the existence of a positive Markov trace of modulus* $\beta$ *on* M *is that* $\beta = [M{:}N] = \|\Lambda\|^2$. *Any two positive Markov traces on* M *are proportional.*

Proof. Since $Z_M \cap Z_N = \mathbb{K}$, it follows that $\Lambda$ is indecomposable and $\Lambda\Lambda^t$ is irreducible (2.3.1f and 1.3.2b). Recall that $[M{:}N] = \|\Lambda\|^2$ by Theorem 2.1.1.

If tr is a positive Markov trace of modulus $\beta$ on M, then $\beta = \|\Lambda\Lambda^t\| = [M{:}N]$ by the previous proposition and Perron–Frobenius theory.

Conversely, set $\beta = [M{:}N]$. Let $\overleftarrow{s}$ be a Perron–Frobenius vector such that $\overleftarrow{s}\,\Lambda\Lambda^t = \beta\overleftarrow{s}$. Let $\overleftarrow{t} = \overleftarrow{s}\Lambda$; it follows as in remark (1) above that $\overleftarrow{t}\Lambda^t\Lambda = \beta\,\overleftarrow{t}$. Hence if tr is the (positive) trace corresponding to the vector $\overleftarrow{s}$, then tr is a Markov trace of modulus $\beta$ by 2.7.2.

The final statement follows from the uniqueness of the Perron–Frobenius eigenvector for $\Lambda\Lambda^t$. #

A crucial property of a Markov trace tr on a pair $N \subset M$ is that the trace Tr on $L = \mathrm{End}_N^r(M)$ entering the definition of the Markov property is again a Markov trace on $M \subset L$. More precisely:

Proposition 2.7.4. *Let* tr *be a Markov trace of modulus* $\beta$ *on a multi–matrix pair* $N \subset M$, *set* $L = \mathrm{End}_N^r(M)$ *as usual, let* $\mathrm{Tr} : L \to \mathbb{K}$ *be the extension of* tr *to a trace on* L *as in Lemma 2.7.1, and let* $D : L \to \lambda(M)$ *be the conditional expectation defined by* Tr *and* tr. *Then*

(a) Tr *is a Markov trace of modulus* $\beta$ *(with respect to* $\lambda : M \to L$*)*;
(b) $\beta\, D(E) = \mathbf{1}$;
(c) $\beta\, D\overline{\lambda}(E)D = D$, where $\overline{\lambda}(\cdot)$ means left multiplication on L;
(d) $\beta\overline{\lambda}(E)D\overline{\lambda}(E) = \overline{\lambda}(E)$.

Proof. (a) Let $\overleftarrow{s}$ and $\overleftarrow{t}$ be the vectors defining the trace tr on M and N respectively. As tr is a Markov trace of modulus $\beta$, one has

$$\overleftarrow{s}\Lambda\Lambda^t = \beta\,\overleftarrow{s},\quad \overleftarrow{t}\Lambda^t\Lambda = \beta\,\overleftarrow{t}$$

by Proposition 2.7.2. From the proof of 2.7.1, we know that Tr is described by $\overleftarrow{r} = \beta^{-1}\overleftarrow{t}$. Consequently

$$\overleftarrow{r}\Lambda^t\Lambda = \beta\,\overleftarrow{r},\quad \overleftarrow{s}\Lambda\Lambda^t = \beta\,\overleftarrow{s}$$

and (a) now follows from 2.7.2.

(b) The bilinear form $(u,v) \mapsto \mathrm{Tr}(uv)$ is nondegenerate on L and its restriction to $\lambda(M)$ is nondegenerate; thus $L = \lambda(M) \oplus \lambda(M)^\perp$, where orthogonality is meant with respect to this bilinear form. For all $x \in M$ one has

$$\mathrm{Tr}(\beta\, E\lambda(x) - \lambda(x)) = \beta\,\mathrm{Tr}(\lambda(x)E) - \mathrm{Tr}(\lambda(x)) = \mathrm{tr}(x) - \mathrm{tr}(x) = 0$$

so that $\beta\, E - \mathbf{1} \in \lambda(M)^\perp$. As D is the orthogonal projection of L onto $\lambda(M)$, this implies $D(\beta E) = \mathbf{1}$.

(c) By M–linearity of D one has $D\overline{\lambda}(E)D = \overline{\lambda}(D(E))D$, so (c) follows from (b).

(d) Choose $x,y \in M$ and set $u = \lambda(x)E\lambda(y) \in L$. The maps from M to M,

$$E\lambda(x)E\lambda(y) : z \mapsto E(xE(yz))$$
$$E\lambda(E(x))\lambda(y) : z \mapsto E(E(x)yz)$$
$$\lambda(E(x))E\lambda(y) : z \mapsto E(x)E(yz)$$

are equal by (N,N)–linearity of E. By (M,M)–linearity of D one has

$$\overline{\lambda}(E)D\overline{\lambda}(E)u = ED(E\lambda(x)E\lambda(y)) = ED(\lambda(E(x))E\lambda(y))$$
$$= E\lambda(E(x))D(E)\lambda(y).$$

Consequently, using (a),

$$\beta\bar{\lambda}(E)D\bar{\lambda}(E)u = E\lambda(E(x))\lambda(y) = Eu = \bar{\lambda}(E)u,$$

which proves (d). #

*This completes the proof of Theorems 2.1.3 and 2.1.4.*

We now analyze the role of Markov traces for towers. Changing our notation slightly, we consider a multi-matrix pair $M_0 \subset M_1$, the tower $(M_k)_{k\geq 0}$ it generates, and a trace $tr = tr_1$ on $M_1$, which is a Markov trace of modulus $\beta$ on the pair $M_0 \subset M_1$. We denote by $tr_2$ the extension of the trace to $M_2$ denoted previously by $Tr$, and by

$E_1 = E : M_1 \to M_0$, $E_1 \in M_2$

$E_2 = D : M_2 \to M_1$, $E_2 \in M_3$

the associated conditional expectations. According to Proposition 2.7.4, the process of extending a Markov trace on $M_k$ to $M_{k+1}$ iterates; namely, if

$E_k$: $M_k \to M_{k-1}$ is the conditional expectation associated to $tr_k$ and $tr_{k-1}$, and $tr_{k+1} : M_{k+1} \to \mathbb{K}$ is the unique extension of $tr_k$ satisfying $\beta\, tr_{k+1}(xE_k) = tr_k(x)$ for all $x \in M_k$ (see 2.7.1),

then $tr_{k+1}$ is also a Markov trace, and the process can continue. Note that $M_{k+1}$ is the algebra generated by $M_k$ and $E_k$, for short $M_{k+1} = \langle M_k, E_k\rangle$. Denote by $M_\infty$ the inductive limit (union) of the nested sequence

$$M_0 \subset M_1 \subset \cdots \subset M_k \subset M_{k+1} \subset \cdots.$$

This is a $\mathbb{K}$-algebra with unit which is the union of its finite dimensional semi-simple subalgebras, and which has a finite dimensional center isomorphic to $Z_M \cap Z_N$. The union of the $tr_k$'s constitutes a trace $tr : M_\infty \to \mathbb{K}$ which is nondegenerate (namely, $tr(xy) = 0$ for all $y \in M_\infty$ implies $x = 0$). If $\mathbb{K} \supset \mathbb{R}$ and $tr = tr_1$ is positive, then $tr$ is also positive in the sense that $tr(\epsilon) > 0$ for any non zero idempotent $\epsilon$ in $M_\infty$. If this holds, and if moreover $Z_M \cap Z_N \cong \mathbb{K}$, then $tr$ is the unique positive trace on $M_\infty$, up to normalization; see Remark (5) at the end of Section 2.5.

<u>Proposition 2.7.5</u>. *Let* $M_0 \subset M_1$ *be a pair of multi-matrix algebras and let* $tr : M_1 \to \mathbb{K}$ *be a Markov trace of modulus* $\beta$. *With the notation above one has*

(a) $\beta E_i E_j E_i = E_i$ *for* $i,j \geq 1$ *with* $|i-j| = 1$;

(b) $E_i E_j = E_j E_i$ *for* $i,j \geq 1$ *with* $|i-j| \geq 2$;

(c) $\beta\, tr(wE_k) = tr(w)$ *for all* $w \in M_k$. *In particular, if* $tr$ *is normalized by* $tr(\mathbf{1}) = 1$, *then* $tr(E_k) = \beta^{-1}$ *for all* $k \geq 1$.

<u>Proof</u>. Statements (a) and (c) follow from (a), (c) and (d) of Proposition 2.7.4. If $j \geq i+2$, then $E_i \in M_{j-1}$, and (b) follows because $E_j$ is $M_{j-1}$-linear. #

*Observe that this Proposition contains Theorem 2.1.6.*

<u>2.7.6. The path model for $M_\infty$ and the idempotents $E_i$</u>. Let $M_0 \subset M_1$ be a pair of multi-matrix algebras and let

$$M_0 \subset M_1 \subset \cdots \subset M_k \subset M_{k+1} \subset \cdots$$

be the tower generated by iterating the fundamental construction. Let $\tilde{B}$ be the augmented Bratteli diagram of the tower and

$$A_0 \subset A_1 \subset \cdots \subset A_k \subset A_{k+1} \subset \cdots$$

the chain of path algebras associated to $\tilde{B}$ as in 2.3.11. Having identified $M_0 \subset M_1$ with $A_0 \subset A_1$, we can obtain an explicit sequence of isomorphisms $\alpha_k : M_k \to A_k$ with $\alpha_{k+1}\big|_{M_k} = \alpha_k$ for all $k$, by iterating the procedure of 2.4.6.

If $tr$ is a Markov trace of modulus $\beta$ on $M_1$, then $tr$ extends uniquely to a trace on $M_\infty$ which is faithful on each $M_k$ and which has the Markov property: if $E_k : M_k \to M_{k-1}$ is the conditional expectation determined by the trace, then $\beta tr(E_k x) = tr(x)$ for all $x \in M_k$. If $t_j^{(k)}$ denotes the weights of the trace on the $k^{th}$ floor of B, then $t_j^{(k)} = \beta^{-1} t_j^{(k-2)}$ for all $k$ and $j$. We also write $tr$ for the corresponding trace on $A_\infty = \bigcup_k A_k$.

Assuming (just for the sake of having definite formulae) that $\mathbb{K}$ is quadratically closed, we can choose the isomorphisms $\{\alpha_k\}$ so that $e_k = \alpha_k(E_k) =$

$$\sum_{\substack{\xi,\eta\in\Omega_{[k-1,k]}\\ \xi_{[k-1]}=\eta_{[k-1]}}} \frac{\sqrt{t^{(k)}(\xi_{[k]})}\sqrt{t^{(k)}(\eta_{[k]})}}{t^{(k-1)}(\xi_{[k-1]})} T_{(\xi,S_k\xi),(\eta,S_k\eta)},$$

where $S_k$ denotes reflection of an edge through the $k^{th}$ floor of B. In fact we know that this choice determines $\{\alpha_k\}$ completely because of the decomposition 2.6.4.(a). Then $\{e_k\}$ is a sequence of idempotents (self-adjoint projections on $\ell^2(\Omega)$ in case $\mathbb{K}=\mathbb{C}$ and tr is positive) satisfying (a)-(c) of 2.7.5.

Iterating the decomposition (2.6.5.6), we can write any matrix unit $T_{\alpha,\beta}$ in $A_k$ as a monomial in the matrix units of $A_1$ and the idempotents $e_1,\cdots,e_{k-1}$. For example for $(\alpha,\beta)\in R_3$ $(T_{\alpha,\beta}\in A_3)$ one finds

$$T_{\alpha,\beta} = C(\alpha,\beta)T_{\alpha_1],(\gamma^1,\tilde{\alpha}_2)}e_1e_2\,T_{(\gamma^1,\tilde{\alpha}_3),(\gamma^2,\tilde{\beta}_3)}e_1\,T_{(\gamma^2,\tilde{\beta}_2),\beta_1]},$$

where $\tilde{\alpha}_i$ denotes the edge in $B(M_0\subset M_1)$ directly below the edge $\alpha_i$, and $\gamma^1,\gamma^2$ are arbitrary edges in $\Omega_{0]}$ with the appropriate endpoints. The constant $C(\alpha,\beta)$ can be evaluated by computing $tr(T_{\alpha,\beta}\cdot T_{\beta,\alpha})$, using the fact that $e_kxe_k = E_k(x)e_k = e_kE_k(x)$ $(x\in A_k)$ and the Markov property of tr.

Let $A_{tr,k}(M_0\subset M_1)$ be the subalgebra of $M_k$ generated by $\mathbb{1},E_1,\cdots,E_{k-1}$. Our next goal is to understand the structure of these algebras. We shall see that, when the modulus $\beta$ of the Markov trace tr lies in a certain generic set, these algebras depend only on $\beta$ and k, and not on any other data pertaining to the inclusion $M_0\subset M_1$ or the trace tr. For $\beta$ in this generic set, $A_{tr,k}(M_0\subset M_1)$ is isomorphic to an abstractly defined algebra $\mathcal{A}_{\beta,k}$, whose structure we describe in detail in the next section. For non-generic $\beta$, new phenomena can occur, and our knowledge is much less satisfactory in this case; see Section 2.9. The following two sections borrow heavily from [Jo 1].

## 2.8 – The algebras $\mathcal{A}_{\beta,k}$ for generic $\beta$

For any integer $k\geq 1$ and for any number $\beta\neq 0$ in the basic field $\mathbb{K}$, let $\mathcal{A}_{\beta,k}$ be the algebra abstractly defined (as an associative algebra over $\mathbb{K}$) by

the generators $\epsilon_1,\epsilon_2,\cdots,\epsilon_{k-1}$ and the unit $\mathbb{1}$

the relations $\epsilon_i^2=\epsilon_i$

$\beta\epsilon_i\epsilon_j\epsilon_i=\epsilon_i$ if $|i-j|=1$

$\epsilon_i\epsilon_j=\epsilon_j\epsilon_i$ if $|i-j|\geq 2$

(Observe k indexes the algebra generated by idempotents up to k-1; this agrees with the usual convention for Artin's braid groups, but is *not* as in [Jo1] or [Jo2].)

A monomial in $\mathcal{A}_{\beta,k}$ is a product $\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_q}$ where each $\epsilon_{i_j}$ is one of $\epsilon_1,\cdots,\epsilon_{k-1}$; the unit $\mathbb{1}$ of $\mathcal{A}_{\beta,k}$ is a monomial (the empty product).

Proposition 2.8.1. *Any monomial* $w\in\mathcal{A}_{\beta,k}$ *may be written in one reduced form*

$$\beta^{-r}(\epsilon_{i_1}\epsilon_{i_1-1}\cdots\epsilon_{j_1})(\epsilon_{i_2}\epsilon_{i_2-1}\cdots\epsilon_{j_2})\cdots(\epsilon_{i_p}\epsilon_{i_p-1}\cdots\epsilon_{j_p})$$

*where* $r\in\mathbb{N}$ *is an appropriate integer and where*

$$1\leq i_1<i_2<\cdots<i_p\leq k-1$$
$$1\leq j_1<j_2<\cdots<j_p\leq k-1$$
$$i_1\geq j_1, i_2\geq j_2,\cdots,i_p\geq j_p$$
$$0\leq p\leq k-1.$$

*Moreover* $\dim_{\mathbb{K}}\mathcal{A}_{\beta,k}\leq\frac{1}{k+1}\binom{2k}{k}$.

Proof. Consider an integer m with $0\leq m\leq k-1$; we prove the first part of the lemma by induction on m for a monomial w in $\{\epsilon_1,\cdots,\epsilon_m\}$. As this is obvious for monomials with $m\leq 1$, we may assume that $m\geq 2$ and that the claim holds for m-1.

Suppose w is a monomial in which $\epsilon_m$ appears at least twice. Then w has one of the forms

$$w = w_1\epsilon_m a\epsilon_m w_2$$

or

$$w = w_1\epsilon_m a\epsilon_{m-1}b\epsilon_m w_2,$$

where a,b are monomials in $\{\epsilon_1,\cdots,\epsilon_{m-2}\}$. As $\epsilon_m$ commutes with these, w equals either $\qquad w_1\epsilon_m a w_2$

or $\qquad w_1 a\beta^{-1}\epsilon_m b w_2,$

and the number of $\epsilon_m$'s has been reduced. Consequently we may assume that $w$ involves exactly one $\epsilon_m$.

Let $w = w_1\epsilon_m w_2$ with $w_1$ and $w_2$ monomials in $\{\epsilon_1,\cdots,\epsilon_{m-1}\}$. Using first the induction hypothesis on $w_2$ and then the commutation $\epsilon_m\epsilon_j = \epsilon_j\epsilon_m$ for $j \le m-2$, we can reduce to the case that $w = w_1\epsilon_m\epsilon_{m-1}\cdots\epsilon_n$, with $w_1$ a reduced monomial finishing, say, with $\epsilon_\ell$. If $\ell \ge n$ one has

$$\begin{aligned}\epsilon_\ell\epsilon_m\epsilon_{m-1}\cdots\epsilon_n &= \epsilon_m\cdots\epsilon_{\ell+2}(\epsilon_\ell\epsilon_{\ell+1}\epsilon_\ell)\epsilon_{\ell-1}\cdots\epsilon_n\\ &= \beta^{-1}\epsilon_\ell\epsilon_{\ell-1}\cdots\epsilon_n\epsilon_m\epsilon_{m-1}\cdots\epsilon_{\ell+2}.\end{aligned}$$

Consequently we may assume that $\ell < n$, so that $w$ is of the form

$$w = \beta\epsilon_{i_1}\cdots\epsilon_{j_1}\epsilon_{i_2}\cdots\epsilon_{j_2}\cdots\epsilon_{i_p}\cdots\epsilon_{j_p} \quad (i_p = m,\ j_p = n)$$

with all desired relations for the $i$'s and the $j$'s. This ends the induction argument.

We now count the number of reduced monomials, following Chapter III in [Fel]. By a path in the lattice $\mathbb{Z}^2$, we mean here an oriented connected polygonal line with vertices at integral points and with edges being either horizontal and directed to the right or vertical and directed upwards. A path starting at $(a,b)$ and ending at $(c,d)$ has $c-a+d-b$ unit edges, $c-a$ horizontal ones and $d-b$ vertical ones. The number of these paths is consequently the binomial coefficient

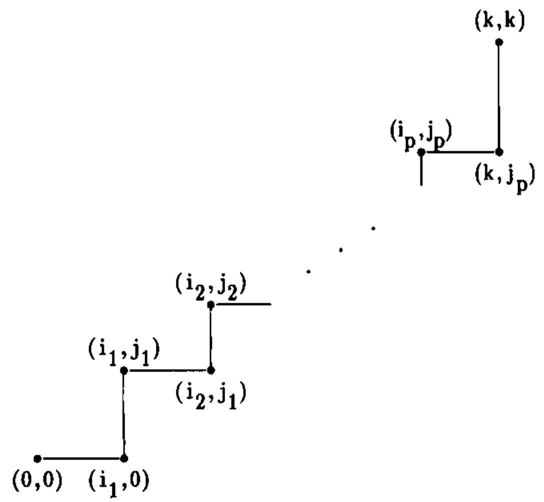
$$N\begin{Bmatrix}c,d\\a,b\end{Bmatrix} = \begin{bmatrix}c-a+d-b\\c-a\end{bmatrix}.$$

Assume first $a > b$ and $c > d$. To each of these paths touching the main diagonal, associate the following "reflected" path: if $(j,j)$ is the diagonal point on the path with smallest $j$, replace the subpath from $(a,b)$ to $(j,j)$ by the reflected path (with respect to the diagonal) from $(b,a)$ to $(j,j)$ and leave the subpath from $(j,j)$ to $(c,d)$ unchanged. This defines a bijection between the set of paths from $(a,b)$ to $(c,d)$ which touch the diagonal and the set of paths from $(b,a)$ to $(c,d)$. Thus the number of paths from $(a,b)$ to $(c,d)$ which do <u>not</u> touch the main diagonal is $N\begin{Bmatrix}c,d\\a,b\end{Bmatrix} - N\begin{Bmatrix}c,d\\b,a\end{Bmatrix}$.

Assume now $a = b$ and $c = d = a+n$ for some $n > 0$. Consider the paths from $(a,a)$ to $(a+n,a+n)$ whose vertices are on or below the main diagonal. These are in bijection with paths from $(a+1,a)$ to $(a+n+1,a+n)$ which do not touch the main diagonal, and their number is

$$\begin{bmatrix}2n\\n\end{bmatrix} - \begin{bmatrix}2n\\n+1\end{bmatrix} = \frac{1}{n+1}\begin{bmatrix}2n\\n\end{bmatrix}.$$

Consider finally a sequence $(i_1,j_1,\cdots,i_p,j_p)$ corresponding to a reduced monomial in $\Lambda_{\beta,k}$. We may associate to this sequence the following path from $(0,0)$ to $(k,k)$, and any path from $(0,0)$ to $(k,k)$ which remains on or below the diagonal can be obtained in this way.




it follows that the number of reduced monomials is $\frac{1}{k+1}\begin{bmatrix}2k\\k\end{bmatrix}$. #

<u>Remark</u>. The Catalan numbers may be defined by

$$\tfrac{1}{2}(1-\sqrt{1-4\lambda}) = \sum_{n\ge 1} C_n\lambda^n = \sum_{n\ge 1}\tfrac{1}{n}\begin{bmatrix}2n-2\\n-1\end{bmatrix}\lambda^n.$$

With this notation, $\dim \Lambda_{\beta,k} \le C_{k+1}$. See e.g. $n^0$ 2.7.3 (page 111) of [GJ].

We shall also need the following computation. We agree that a binomial coefficient $\begin{bmatrix}a\\b\end{bmatrix}$ is zero if the integers $a,b$ satisfy $b < 0$ or $b > a$.

Lemma 2.8.2. *Let* $k \geq 1$ *be an integer and set* $m = [k/2]$, *the greatest integer less than or equal to* $k/2$. *Then*

$$\sum_{j=0}^{m} \left[\binom{k}{j} - \binom{k}{j-1}\right]^2 = \frac{1}{k+1}\binom{2k}{k}.$$

Proof. By comparison of the coefficients of $t^c$ on both sides of $(1+t)^a(1+t)^b = (1+t)^{a+b}$, one has

$$\sum_{j=0}^{c} \binom{a}{j}\binom{b}{c-j} = \binom{a+b}{c} \qquad (*)$$

for any integers $a,b,c \geq 0$. (See for example Section II.12 in [Fel].)

Assume first that $k$ is even: $k = 2m$. Setting $a = b = c = k$ in (*), one obtains

$$\begin{aligned}\sum_{j=0}^{m} \binom{k}{j}^2 &= \frac{1}{2}\sum_{j=0}^{k} \binom{k}{j}^2 + \frac{1}{2}\binom{k}{m}^2 \\ &= \frac{1}{2}\binom{2k}{k} + \frac{1}{2}\binom{k}{m}^2, \text{ and} \\ \sum_{j=0}^{m} \binom{k}{j-1}^2 &= \frac{1}{2}\binom{2k}{k} - \frac{1}{2}\binom{k}{m}^2.\end{aligned}$$

Setting $a = b = k$ and $c = k + 1$ in (*), one obtains

$$2\sum_{j=0}^{m} \binom{k}{j}\binom{k}{j-1} = \sum_{j=0}^{k} \binom{k}{j}\binom{k}{j-1} = \binom{2k}{k+1}$$

Thus

$$\sum_{j=0}^{m} \left[\binom{k}{j} - \binom{k}{j-1}\right]^2 = \binom{2k}{k} - \binom{2k}{k+1} = \frac{1}{k+1}\binom{2k}{k}.$$

For $k$ odd $(k = 2m+1)$, one obtains similarly

$$\begin{aligned}\sum_{j=0}^{m} \binom{k}{j}^2 &= \frac{1}{2}\sum_{j=0}^{k} \binom{k}{j}^2 = \frac{1}{2}\binom{2k}{k}, \\ \sum_{j=0}^{m} \binom{k}{j-1}^2 &= \frac{1}{2}\binom{2k}{k} - \binom{k}{m}^2, \text{ and} \\ 2\sum_{j=0}^{m} \binom{k}{j}\binom{k}{j-1} &= \sum_{j=0}^{k} \binom{k}{j}\binom{k}{j-1} - \binom{k}{m+1}\binom{k}{m} \\ &= \binom{2k}{k+1} - \binom{k}{m}^2;\end{aligned}$$

so the conclusion follows for $k$ odd as well. #

Define now a sequence $(P_k)_{k\geq 0}$ of polynomials in $\mathbb{Z}[\lambda]$ by

$$P_0 = 1, \; P_1 = 1, \; P_{k+1}(\lambda) = P_k(\lambda) - \lambda P_{k-1}(\lambda) \;\; (k \geq 1)$$

so that in particular

$$\begin{aligned}&P_2(\lambda) = 1-\lambda \;\; P_4(\lambda) = 1-3\lambda + \lambda^2 \;\; P_6(\lambda) = 1-5\lambda + 6\lambda^2 - \lambda^3 \\ &P_3(\lambda) = 1-2\lambda \;\; P_5(\lambda) = 1-4\lambda + 3\lambda^2 \;\; P_7(\lambda) = 1-6\lambda + 10\lambda^2 - 4\lambda^3.\end{aligned}$$

(Observe $P_k$ here is as in [Wen1], but as $P_{k-1}$ in [Jo1].)

Proposition 2.8.3. *Consider an integer* $k \geq 0$ *and set* $m = \left[\frac{k}{2}\right]$. *Then*

(i) *The polynomial* $P_k$ *is of degree* $m$. *Its leading coefficient is* $(-1)^m$ *if* $k = 2m$ *is even and* $(-1)^m(m+1)$ *if* $k = 2m+1$ *is odd.*

(ii) $P_k$ *has* $m$ *distinct roots which are given by* $\dfrac{1}{4\cos^2\left[\frac{j\pi}{k+1}\right]}$ for $j = 1,2,\cdots,m$.

(iii) *Assume* $k \geq 1$. *Let* $\lambda$ *be a real number with*

$$\frac{1}{4\cos^2\left[\frac{\pi}{k+2}\right]} < \lambda < \frac{1}{4\cos^2\left[\frac{\pi}{k+1}\right]}.$$

*Then* $P_1(\lambda) > 0, P_2(\lambda) > 0, \cdots, P_k(\lambda) > 0, P_{k+1}(\lambda) < 0$.

(iv) *Set* $Q_k(\lambda) = P_k(\lambda(\lambda+1)^{-2})$. *Then*

$$Q_k(\lambda) = \frac{\lambda^{k+1}-1}{(\lambda-1)(\lambda+1)^k}.$$

Proof. Claims (i) and (iv) are easily checked by induction.

For (ii), we compute in the ring $\mathbb{Q}[\lambda,\sqrt{1-4\lambda}\,]$ and proceed as in the proof of I.2.2. The difference equation for the $P_k$'s has an indicial equation $\mu^2-\mu+\lambda=0$ with roots

$$\mu_1=\tfrac{1}{2}(1+\sqrt{1-4\lambda}) \quad \mu_2=\tfrac{1}{2}(1-\sqrt{1-4\lambda})$$

so that $P_k=C\mu_1^k+D\mu_2^k$. By adjustment of the constants $C,D$ to fit $P_0,P_1$ we find $P_k=(\mu_1-\mu_2)^{-1}(\mu_1^{k+1}-\mu_2^{k+1})$ for each $k\geq 0$. Consider now a real number $\theta$ with $0<\theta<\pi/2$ and set $\lambda=\frac{1}{4\cos^2\theta}$, so that $\mu_1=\frac{e^{i\theta}}{2\cos\theta}$ and $\mu_2=\frac{e^{-i\theta}}{2\cos\theta}$ Then

$$P_k(\lambda)=\frac{\sin((k+1)\theta)}{2^k\cos^k(\theta)\sin\theta},$$

which vanishes when $\theta=\frac{j\pi}{k+1}$ with $j=1,2,\cdots,m$.

Claim (iii) is obvious for $k=1$, and we may assume $k\geq 2$. For $\ell\in\{2,\cdots,k\}$, the smallest root of $P_\ell$ is $\frac{1}{4\cos^2\left[\frac{\pi}{\ell+1}\right]}$, and $P_\ell(\lambda)>0$ for $\frac{1}{4\cos^2\left[\frac{\pi}{\ell+1}\right]}>\lambda$. As $\frac{1}{4\cos^2\left[\frac{\pi}{\ell+1}\right]}>\frac{1}{4\cos^2\left[\frac{\pi}{k+1}\right]}>\lambda$, one has $P_\ell(\lambda)>0$. The two smallest roots of $P_{k+1}$ are

$$\lambda_1=\frac{1}{4\cos^2\left[\frac{\pi}{k+2}\right]} \quad \lambda_2=\frac{1}{4\cos^2\left[\frac{2\pi}{k+2}\right]},$$

and $P_{k+1}<0$ on $]\lambda_1,\lambda_2[$. As $\frac{1}{4\cos^2\left[\frac{\pi}{k+1}\right]}<\lambda_2$ one has in particular $P_{k+1}(\lambda)<0$. #

Since the polynomials $P_k$ have coefficients in $\mathbb{Z}$, it makes sense to evaluate them at any number in our reference field $\mathbb{K}$. Given an integer $k\geq 1$, we define $\beta\in\mathbb{K}^*$ to be k-generic if

$$P_1(\beta^{-1})\neq 0, P_2(\beta^{-1})\neq 0,\cdots,P_k(\beta^{-1})\neq 0.$$

Say that $\beta$ is generic if it is k-generic for all $k$.

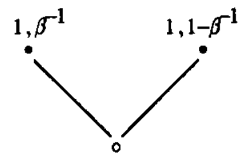
For example, any $\beta\in\mathbb{K}^*$ is 1-generic, and $\beta$ is 2-generic if and only if $\beta\neq 1$.

If $\mathbb{K}$ is not algebraic over its prime field, transcendental numbers are obviously generic. If $\mathbb{K}$ contains the reals, Proposition 2.8.3 (ii) shows also that any $\beta$ outside the interval $]0,4[$ is generic.

For $\beta\in\mathbb{K}^*$, let $q$ be a number distinct from 0 and $-1$, in $\mathbb{K}$ or possibly in some quadratic extension of $\mathbb{K}$, such that $\beta=q^{-1}(q+1)^2$. Claim (iv) of Proposition 2.8.3 shows that $\beta$ is not generic if and only if $\sum_{j=0}^{m}q^j=0$ for some integer $m\geq 2$. In particular, if $\mathbb{K}$ is a finite field, no $\beta$ is generic.

For generic $\beta\in\mathbb{K}^*$, we shall define inductively a nested sequence $(B_{\beta,k})_{k\geq 1}$ of associative $\mathbb{K}$-algebras with unit, and a normalized trace on each of these.

Set $B_{\beta,1}=\mathbb{K}$ and denote by $tr_1$ the tautological trace on $B_{\beta,1}$. Set $B_{\beta,2}=\mathbb{K}e_1\oplus\mathbb{K}(1-e_1)$ where $e_1$ is an idempotent, not zero. Define $tr_2:B_{\beta,2}\to\mathbb{K}$ by $tr_2(e_1)=\beta^{-1}$ and $tr_2(1-e_1)=1-\beta^{-1}$. Identify $B_{\beta,1}$ with the multiples of the identity in $B_{\beta,2}$. The Bratteli diagram of the pair $B_{\beta,1}\subset B_{\beta,2}$ is




(see the end of Section 2.5 for the notation).

In the next lemma, we set

$$\left\{{k \atop i}\right\}=\left[{k \atop i}\right]-\left[{k \atop i-1}\right]$$

Lemma 2.8.4. *Consider an integer* $n\geq 2$, *and assume* $\beta\in\mathbb{K}^*$ *is* n-*generic. Suppose there is given a nested sequence* $(B_{\beta,k})_{1\leq k\leq n}$ *of* $\mathbb{K}$-*algebras, together with traces* $tr_k:B_{\beta,k}\to\mathbb{K}$ *extending one another, such that the following hold for* $k\in\{2,\cdots,n\}$:

(i) $B_{\beta,k}$ *is generated by its unit, by elements* $e_1,\cdots,e_{k-2}$ *(all in* $B_{\beta,k-1}$*) and by* $e_{k-1}$. *Denote by* $B'_{\beta,k}$ *the two-sided ideal in* $B_{\beta,k}$ *generated by* $e_1,\cdots,e_{k-1}$.

(ii) *The generators satisfy the relations*

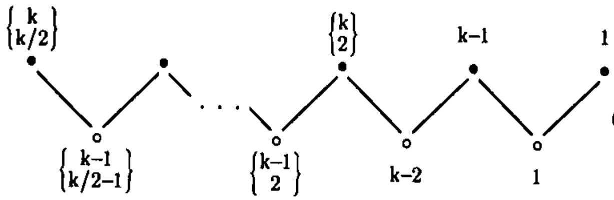
$$\begin{aligned}&e_i^2=e_i,\\&\beta e_ie_je_i=e_i &&\text{if } |i-j|=1,\\&e_ie_j=e_je_i &&\text{if } |i-j|\geq 2\end{aligned}$$

for all $i,j\in\{1,\cdots k-1\}$..

(iii) $B_{\beta,k}$ *is a direct sum of* $\left[\frac{k}{2}\right]+1$ *factors* $Q_j^k$, *with* $Q_j^k$ *isomorphic to the algebra of matrices of order* $\left\{{k \atop j}\right\}$, *for* $j=0,1,\cdots,\left[\frac{k}{2}\right]$. *One has* $B'_{\beta,k}=\bigoplus_{j>0}Q_j^k$. *Denote by* $d_k$

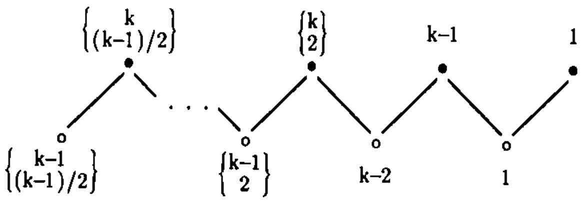
*the (unique) nonzero idempotent in* $Q_0^k$.

(iv) *The inclusion* $B_{\beta,k-1} \subset B_{\beta,k}$ *is described by the Bratteli diagram:*




*(k even),*

*or,*




*(k odd).*

*The index* $j$ *of the subfactors increases from right to left, so the white [resp. black] vertex on the extreme right represents* $Q_0^{k-1}$ *[resp.* $Q_0^k$*].*

(v) *The dimension of* $B_{\beta,k}$ *is* $\frac{1}{k+1}\left[{2k \atop k}\right]$.

(vi) *The trace satisfies* $tr_k(\mathbf{1}) = 1$ *and* $\beta tr_k(we_j) = tr_k(w)$ *for all* $w \in B_{\beta,j}$, $1 \le j \le k-1$.

(vii) *The value of* $tr_k$ *on a minimal idempotent of* $Q_j^k$ *is* $\beta^{-j}P_{k-2j}(\beta^{-1})$ *for* $j = 0,1,\cdots,\left[\frac{k}{2}\right]$, *so that in particular:*

(viii) $tr_k$ *is faithful.*

*Then one may define an algebra* $B_{\beta,n+1}$ *containing* $B_{\beta,n}$ *and a trace* $tr_{n+1}$ *on it, such that* (i) *to* (vii) *hold for* $k = n+1$. *Moreover, if* $\beta$ *is* $(n+1)$*–generic, then* (viii) *holds also for* $k = n+1$.

*Suppose in addition that* $\mathbb{K} = \mathbb{C}$, *that each* $B_{\beta,k}$ $(k \le n)$ *has a* $C^*$*–algebra structure making the idempotents* $e_i$ *self–adjoint projections, that* $\beta > 0$, *and that* $P_k(\beta^{-1}) > 0$ *for* $1 \le k \le n+1$. *Then* $B_{\beta,n+1}$ *also has a* $C^*$*–algebra structure making* $e_n$ *a self–adjoint projection, and the trace* $tr_{n+1}$ *is faithful and positive.*

<u>Proof</u> (see §5.1 in [Jo1]). During the proof, we write $B_k$ for $B_{\beta,k}$.

Both $tr_n$ and it restriction $tr_{n-1}$ to $B_{n-1}$ are nondegenerate by (vii), since $\beta$ is n–generic. Let $E_n : B_n \to B_{n-1}$ be the conditional expectation associated to $tr_n$ as in 2.6.2. Set $\tau = \beta^{-1}$.

Then $E_n e_{n-1} E_n = \tau E_n$. Indeed, for all $y \in B_n$ and all $z \in B_{n-1}$, one has, first by 2.6.2(i) and then by (vi)

$$tr_{n-1}(\{E_n(e_{n-1}E_n(y)) - \tau E_n(y)\}z) = tr_n(e_{n-1}E_n(yz)) - \tau tr_{n-1}(E_n(yz)) = 0.$$

Thus $E_n(e_{n-1}E_n(y)) = \tau E_n(y)$ because $tr_{n-1}$ is non–degenerate, and in particular $E_n(e_{n-1}) = \tau \mathbf{1}$..

Next we claim that

$$e_{n-1}E_n(e_{n-1}x) = \tau e_{n-1}x \quad (*)$$

for all $x \in B_n$.

Obviously $(*)$ holds for $x = \mathbf{1}$ because $E_n(e_{n-1}) = \tau \mathbf{1}$ by the previous claim.

Next we check that $(*)$ holds if $x = ye_{n-1}$ for some $y \in B_{n-1}$. First, if $y = y_1e_{n-2}y_2$ with $y_1, y_2 \in B_{n-2}$, then $e_{n-1}x = \tau y_1e_{n-1}y_2 = \tau e_{n-1}y_1y_2$, and

$$e_{n-1}E_n(e_{n-1}x) = \tau e_{n-1}E_n(e_{n-1})y_1y_2 = \tau^2 e_{n-1}y_1y_2 = \tau e_{n-1}x,$$

by $B_{n-2}$–linearity of $E_n$. If $y \in B_{n-2}$, then $e_{n-1}x = e_{n-1}y$, and again

$$e_{n-1}E_n(e_{n-1}x) = \tau e_{n-1}x.$$

Thus $(*)$ holds when $x = ye_{n-1}$, for any $y \in B_{n-2} + B_{n-2}e_{n-2}B_{n-2}$, namely for all $y \in B_{n-1}$.

Now using the $B_{n-1}$–linearity of $E_n$, we see finally that $(*)$ holds for all $x \in B_{n-1} + B_{n-1}e_{n-1}B_{n-1}$, namely for all $x \in B_n$.

Define $B'_{n+1}$ to be the algebra obtained from the pair $B_{n-1} \subset B_n$ by the fundamental construction, and set

$$B_{n+1} = B'_{n+1} \oplus \mathbb{K}d_{n+1}$$

where $d_{n+1}$ is a central idempotent. By Corollary 2.6.4, the two-sided ideal $B'_{n+1}$ is generated by $B_n$ and $E_n$. From now on, we write $e_n$ (an element in $B_{n+1}$) rather than $E_n$ (a mapping from $B_n$ onto $B_{n-1}$). Then $B_{n+1}$ is a multi-matrix algebra by 2.4.1 in which $e_1,\cdots,e_n$ generate $B'_{n+1}$, so that $\mathbf{1},e_1,\cdots,e_n$ generate $B_{n+1}$. We have checked (i) and (ii).

Define a map $J : B_n \to B'_{n+1} \oplus \mathbb{K}d_{n+1}$ by $J(x) = (x,0)$ if $x \in B'_n$ and $J(d_n) = (d_n,d_{n+1})$. (This is of course an abuse of notation: the first component of $J(d_n)$ is the element of $B'_{n+1} = \mathrm{End}_{B_{n-1}}(B_n)$ which is left (or right) multiplication by $d_n$!) Then $J$ is obviously an injective morphism, so that we may (and we shall) identify $B_n$ with a subalgebra of $B_{n+1}$. Now the shape of the diagram in (iv) follows from the induction hypothesis and Proposition 2.4.1b, and the dimensions from the relations

$$\left\{\begin{matrix} n+1 \\ (n+1)/2 \end{matrix}\right\} = \left\{\begin{matrix} n \\ (n+1)/2-1 \end{matrix}\right\} \qquad (n \text{ odd}),$$

and

$$\left\{\begin{matrix} n+1 \\ j \end{matrix}\right\} = \left\{\begin{matrix} n \\ j \end{matrix}\right\} + \left\{\begin{matrix} n \\ j-1 \end{matrix}\right\} \qquad (\text{all } n \text{ and } j).$$

This shows (iv), and consequently also (iii). Now (v) follows from Lemma 2.8.2.

Define the trace $tr_{n+1} : B_{n+1} \to \mathbb{K}$ by assigning the weight $\tau^j P_{n+1-2j}(\tau)$ to the factor $Q_j^{n+1}$, as desired for (vii). Let $f_j^k$ denote a minimal idempotent in $Q_j^k$. When $n$ is even and $j = n/2$ we have

$$tr_{n+1}(f^n_{n/2}) = tr_{n+1}(f^{n+1}_{n/2}) = \tau^{n/2}P_1(\tau) = \tau^{n/2},$$

while

$$tr_n(f^n_{n/2}) = \tau^{n/2}P_0(\tau) = \tau^{n/2}.$$

In all other cases we have

$$\begin{aligned} tr_{n+1}(f^n_j) &= tr_{n+1}(f^{n+1}_j) + tr_{n+1}(f^{n+1}_{j+1}) \\ &= \tau^j\{P_{n-2j+1}(\tau) + \tau P_{n-2j-1}(\tau)\} \\ &= \tau^j P_{n-2j}(\tau), \end{aligned}$$

by the three term recursion for the $P$'s. Consequently $tr_{n+1}$ extends $tr_n$, and in particular $tr_{n+1}(\mathbf{1}) = 1$. (This point shows precisely why the factor $Q_0^{n+1} = \mathbb{K}d_{n+1}$ had to be introduced in $B_{n+1}$!) Incidentally, this gives the relation

$$\sum_{j=0}^{\left[\frac{k}{2}\right]} \left\{\begin{matrix} k \\ j \end{matrix}\right\} \tau^j P_{k-2j}(\tau) = 1,$$

which could also be checked directly.

We next verify the relation

$$\beta tr_{n+1}(we_n) = tr_{n+1}(w) \quad (**)$$

for all $w \in B_n$.

We check this first for $w \in B_{n-1}$. We may then as well assume that $w$ is some minimal idempotent $f_j^{n-1}$ of $Q_j^{n-1}$, where $j$ is an integer with $0 \le j \le \left[\frac{n-1}{2}\right]$. But then we know from Corollary 2.6.4c that $f_j^{n-1}e_n$ is a minimal idempotent in $Q^{n+1}_{j+1}$. Consequently

$$\begin{aligned} tr_{n+1}(f_j^{n-1}e_n) &= \tau^{j+1}P_{n+1-2(j+1)}(\tau) \\ &= \tau\{\tau^j P_{n-1-2j}(\tau)\} = \tau tr_{n-1}(f_j^{n-1}) \end{aligned}$$

and (**) follows because $tr_{n+1}$ extends $tr_{n-1}$.

We now set $w = xe_{n-1}y$ for some $x,y \in B_{n-1}$. Then $e_nwe_n = xe_ne_{n-1}e_ny = \tau xe_ny$ by (ii) and, using the case of (**) already checked

$$tr_{n+1}(we_n) = \tau tr_{n+1}(yxe_n) = \tau^2 tr_{n+1}(yx).$$

On the other hand, by the induction hypothesis

$$tr_{n+1}(w) = tr_n(yxe_{n-1}) = \tau tr_n(yx).$$

Thus (**) holds for $w = xe_{n-1}y$.

Consequently (**) holds for all $w$ in $B_{n-1} + B_{n-1}e_{n-1}B_{n-1}$, namely for all $w \in B_n$. This proves (vi) and (vii). If $\beta$ is $(n+1)$-generic, then (viii) follows from (vii).

Finally, if $\mathbb{K} = \mathbb{C}$, and the $B_k$ are $C^*$-algebras for $k \le n$, then $B'_{n+1}$ also may be given a $C^*$-structure making the idempotent $e_n$ self-adjoint; see the discussion in Appendix IIa, or the remark under 2.6.5. Clearly $B_{n+1}$ also has a $C^*$-structure. Moreover the weights of the trace on $B_{n+1}$ are strictly positive by (vii). #

Theorem 2.8.5. *Consider an integer* $k \geq 1$ *and a number* $\beta \in \mathbb{K}^*$ *such that* $P_j(\beta^{-1}) \neq 0$ *for* $j \leq k-1$, *where* $(P_j)_{j\geq 1}$ *are the polynomials of Proposition 2.8.3.*

(a) $A_{\beta,k}$ *is a multi-matrix algebra of dimension* $\frac{1}{k+1}\begin{bmatrix}2k\\k\end{bmatrix}$, *isomorphic to* $\bigoplus_{j=0}^{m} M_{\left\{k \atop j\right\}}(\mathbb{K})$, *where* $m = \left[\frac{k}{2}\right]$ *and* $\left\{k \atop j\right\} = \begin{bmatrix}k\\j\end{bmatrix} - \begin{bmatrix}k\\j-1\end{bmatrix}$.

(b) *There exists a unique normalized trace* $tr_k : A_{\beta,k} \to \mathbb{K}$ *such that*

$$\beta tr_k(w\epsilon_j) = tr_k(w)$$

*whenever* $1 \leq j \leq k-1$ *and* $w$ *is in the subalgebra generated by* $\mathbf{1}, \epsilon_1, \cdots, \epsilon_{j-1}$. *Moreover* $tr_k$ *is faithful if* $P_k(\beta^{-1}) \neq 0$.

(c) *The natural map* $A_{\beta,k-1} \to A_{\beta,k}$ *is injective and* $tr_k$ *extends* $tr_{k-1}$.

(d) *If* $B_{\beta,k}$ *is as in Lemma 2.8.4, the assignment* $\epsilon_j \mapsto e_j$ $(1 \leq j \leq k-1)$ *extends to an isomorphism from* $A_{\beta,k}$ *onto* $B_{\beta,k}$.

(e) *The trace* $tr_k$ *on* $A_{\beta,k}$ *also satisfies*

$$\beta\, tr_k(\epsilon_j w) = tr_k(w)$$

*whenever* $1 \leq j \leq k-2$ *and* $w$ *is a word in* $\{\epsilon_{j+1}, \cdots, \epsilon_{k-1}\}$. *More generally we have*

$$tr_k(uv) = tr_k(u)tr_k(v)$$

*whenever* $u$ *is a word in* $\{\epsilon_1, \cdots, \epsilon_j\}$ *and* $w$ *is a word in* $\{\epsilon_{j+1}, \cdots, \epsilon_{k-1}\}$.

(f) *The map* $\epsilon_j \mapsto \epsilon_{k-j}$ *extends to a trace preserving automorphism* $\sigma_k$ *of* $A_{\beta,k}$. *Furthermore* $\sigma_k$ *is inner in case* $\mathbb{K}$ *contains a solution* $q$ *of the equation* $q^{-1}(q+1)^2 = \beta$.

Proof. Claims (i) and (ii) of the previous lemma show that the map of (d) is a morphism onto. Claim (v) of the lemma and Proposition 2.8.1 show that this morphism is injective. Consequently, assertions (a) and (c) and the existence of $tr_k$ in (b) follow from the lemma. But the relation in (b) together with the normalization $tr_k(\mathbf{1}) = 1$ and the trace property $tr_k(xy) = tr_k(yx)$ suffice to compute the trace on any word in the generators $\{\epsilon_i\}$ of $A_{\beta,k}$, so the trace is unique.

We prove by induction on $m$ $(j+1 \leq m \leq k-1)$ that the formula of (e) holds for $u \in \text{alg}\{\mathbf{1}, \epsilon_1, \cdots, \epsilon_j\}$ and $w \in \text{alg}\{\mathbf{1}, \epsilon_{j+1}, \cdots, \epsilon_m\}$. The case $m = j+1$ is clear from (b). Suppose that $m > j+1$ and that the result is verified for elements of $\text{alg}\{\mathbf{1}, \epsilon_{j+1}, \cdots, \epsilon_{m-1}\}$. It suffices then to deal with a reduced word $w = x\epsilon_m y$ where $x$ and $y$ are words in $\{\epsilon_{j+1}, \cdots, \epsilon_{m-1}\}$. Then $tr_k(w) = \beta^{-1}tr_k(xy)$, and $tr_k(uw) = tr_k(yux\epsilon_m) = \beta^{-1}tr_k(uxy) = tr_k(u)\,\beta^{-1}tr_k(xy)$, where the last step follows from the induction hypothesis.

Let $q$ be an element of $\mathbb{K}$, or of a quadratic extension of $\mathbb{K}$, satisfying $q^{-1}(1+q)^2 = \beta$. Define elements

$$\gamma_i = (q+1)\epsilon_i - \mathbf{1} \quad \text{and}$$
$$c_i = (\gamma_1\gamma_2\cdots\gamma_{i-1})\cdots(\gamma_1\gamma_2)\gamma_1$$

in $A_{\beta,k} \otimes_{\mathbb{K}} \mathbb{K}(q)$ for $1 \leq i \leq k-1$. These are invertible, with $\gamma_i^{-1} = (q^{-1}+1)\epsilon_i - \mathbf{1}$, and one verifies by induction that $c_j\gamma_i c_j^{-1} = \gamma_{j-1}$ and $c_j\epsilon_i c_j^{-1} = \epsilon_{j-1}$ for $i \leq j-1$. In particular, $\sigma_k : x \mapsto c_k x c_k^{-1}$ is the automorphism of part (f). This automorphism is trace preserving, because the trace $tr_k$ extends uniquely to $A_{\beta,k} \otimes_{\mathbb{K}} \mathbb{K}(q)$. #

Corollary 2.8.6. *Consider an integer* $k \geq 1$ *and an* arbitrary *number* $\beta \in \mathbb{K}^*$. *Let* $\varphi$ *be the homomorphism* $A_{\beta,k} \to A_{\beta,k+1}$ *which, for* $j \leq k-1$, *maps* $\epsilon_j$ *viewed as a generator of* $A_{\beta,k}$ *to* $\epsilon_j$ *(sic) viewed as a generator of* $A_{\beta,k+1}$.

(a) $A_{\beta,k}$ *is of dimension* $\frac{1}{k+1}\begin{bmatrix}2k\\k\end{bmatrix}$.

(b) $\varphi$ *is an injection and any element* $x \in A_{\beta,k+1}$ *can be written as* $x = \varphi(u) + \sum \varphi(u_i)\epsilon_k\varphi(w_i)$, where $u$, $v_i$, and $w_i$ are elements of $A_{\beta,k}$.

(c) *There is a sequence of traces* $tr_\ell : A_{\beta,\ell} \to \mathbb{K}$ $(1 \leq \ell \leq k)$ *such that* $tr_\ell(\mathbf{1}) = 1$, and $tr_{\ell+1}(\varphi(u) + \sum \varphi(u_i)\epsilon_\ell\varphi(w_i)) = tr_\ell(u) + \beta^{-1}\sum tr_\ell(v_i w_i)$ for all $u$, $u_i$, $w_i \in A_{\beta,\ell}$.

Proof. It is enough to prove the corollary for any extension of the field $\mathbb{K}$, so that we may assume $\mathbb{K}$ to contain infinitely many generic numbers.

Assume first that $\beta$ is generic. Then $A_{\beta,k}$ has a basis over $\mathbb{K}$ made of the $\frac{1}{k+1}\begin{bmatrix}2k\\k\end{bmatrix}$ reduced monomials (see 2.8.1 and 2.8.5a), say $(\epsilon_\sigma)_{\sigma\in S}$. The structure constants are defined by

$$\epsilon_\sigma\epsilon_\tau = \sum_{\mu\in S} c^\mu_{\sigma,\tau}\epsilon_\mu. \quad (*)$$

Proposition 2.8.1 shows that, for any given pair $(\sigma,\tau)$, all but one of the $c^\mu_{\sigma,\tau}$ vanish and the one non-zero $c^{\mu}_{\sigma,\tau}$ is a power of $\beta^{-1}$ depending on $\sigma$ and $\tau$. In particular there are monomials $c^{\mu}_{\sigma,\tau}(t) \in \mathbb{K}[t]$ such that $c^{\mu}_{\sigma,\tau}$ as above is just $c^{\mu}_{\sigma,\tau}(\beta^{-1})$ for any $\sigma,\tau,\mu \in S$.

Define now the "generic" algebra $\mathcal{A}_{gen,k}$ over the polynomial ring $\mathbb{K}[t]$ as the free $\mathbb{K}[t]$-module over $S$, with canonical basis denoted again by $(\epsilon_{\sigma})_{\sigma\in S}$, and with multiplication defined by

$$\epsilon_{\sigma}\epsilon_{\tau} = \sum_{\mu\in S} c^{\mu}_{\sigma,\tau}(t)\epsilon_{\mu} \qquad \sigma,\tau \in S.$$

The relations which express that this multiplication is associative are polynomial, and they hold when $t$ is specialized at $\beta^{-1}$ for any generic $\beta \in \mathbb{K}^*$, by Theorem 2.8.5. Hence they hold identically, and $\mathcal{A}_{gen,k}$ is a well-defined associative algebra. Indeed, it is the algebra with unit over $\mathbb{K}[t]$ abstractly defined by generators $\epsilon_1,\cdots,\epsilon_{k-1}$ and relations

$$\begin{aligned} &\epsilon_i^2 = \epsilon_i && \\ &\epsilon_i\epsilon_j\epsilon_i = t\epsilon_i && \text{if } |i-j| = 1 \\ &\epsilon_i\epsilon_j = \epsilon_j\epsilon_i && \text{if } |i-j| \geq 2 \end{aligned}$$

Consider finally an arbitrary $\beta \in \mathbb{K}^*$. Then $\mathcal{A}_{\beta,k}$ is isomorphic to $\mathcal{A}_{gen,k} \otimes_{\mathbb{K}[t]} \mathbb{K}$, where $\mathbb{K}$ is made a $\mathbb{K}[t]$-module by $c(t)\lambda = c(\beta^{-1})\lambda$ for $c(t) \in \mathbb{K}[t]$ and $\lambda \in \mathbb{K}$. This shows claim (a). That $\varphi$ is an injection follows similarly. As observed in the proof of (a), there exist bases of $\mathcal{A}_{\beta,k}$ and $\mathcal{A}_{\beta,k+1}$ consisting of the reduced monomials of 2.8.1, and claim (b) follows from this. We leave the details of part (c) to the reader; compare, however, 2.9.6. #

<u>Remark:</u> In general the traces $tr_{\ell}$ of claim (c) are not faithful; see Theorem 2.9.6.d.

Consider now the situation at the end of Section 2.7: One has a multi-matrix pair $M_0 \subset M_1$ and a Markov trace $tr : M_1 \to \mathbb{K}$ of modulus $\beta$; these generate a tower, and the conditional expectations $E_j : M_j \to M_{j-1}$ for $j = 1,\cdots,k-1$ generate (together with $\mathbb{1}$) a subalgebra $A_{tr,k}(M_0 \subset M_1)$ of $M_k$.

<u>Proposition 2.8.7.</u> *Suppose that* $\beta \in \mathbb{K}^*$ *satisfies* $P_j(\beta^{-1}) \neq 0$ *for* $i \leq j \leq k$.

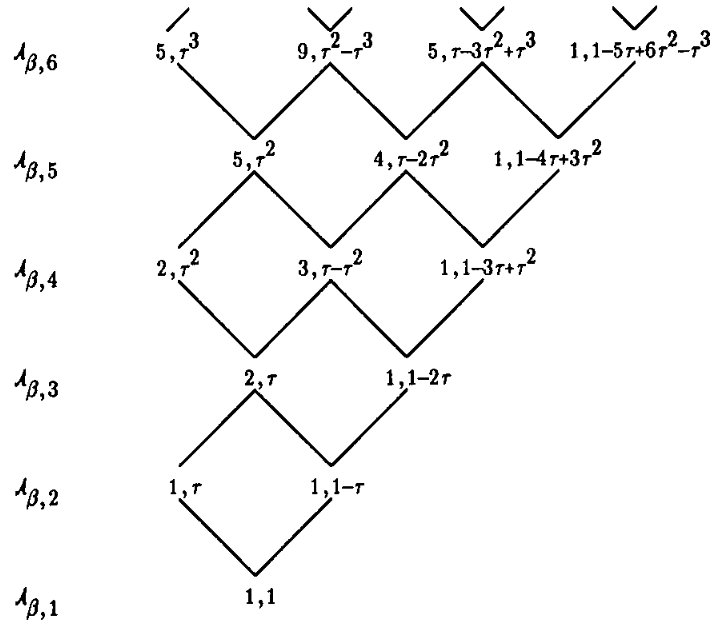
(a) *Suppose that* $\chi : \mathcal{A}_{\beta,k} \to C$ *is a surjective homomorphism of* $\mathbb{K}$*-algebras and that* $C$ *has a trace* $tr$ *satisfying* $\beta\, tr(w\chi(\epsilon_j)) = tr(w)$ *for* $i \leq j \leq k-1$ *and* $w \in \chi(\mathcal{A}_{\beta,j})$. *Then* $\chi$ *is an isomorphism and* $tr$ *is non-degenerate.*

(b) *In particular, with the notation above, the map* $\chi : \epsilon_j \mapsto E_j$ *extends to an isomorphism of* $\mathcal{A}_{\beta,k}$ *onto* $A_{tr,k}(M_0 \subset M_1)$, *and the restriction to* $A_{tr,k}(M_0 \subset M_1)$ *of the Markov trace* $tr : M_k \to \mathbb{K}$ *is non-degenerate.*

<u>Proof.</u> (a) It follows from 2.8.5(b) that $tr \circ \chi = tr_k$. Hence if $x \in \ker(\chi)$, then for all $y \in \mathcal{A}_{\beta,k}$ one has $tr_k(xy) = tr(\chi(x)\chi(y)) = 0$, so that $x = 0$, by the non-degeneracy of $tr_k$. Thus $\chi$ is an isomorphism and $tr$ is non-degenerate.

(b) By 2.7.5, the map $\chi$ extends to an homomorphism of $\mathcal{A}_{\beta,k}$ onto $A_{tr,k}(M_0 \subset M_1)$, and 2.7.5 together with 2.8.5(b) imply that $tr \circ \chi = tr_k$. Thus (b) follows from (a). #

Suppose $\beta \in \mathbb{K}^*$ is generic. The following picture sums up the structure of the traced algebras introduced in this section (with $\tau = \beta^{-1}$).



## 2.9. An approach to the non-generic case.

If $\beta \in \mathbb{K}^*$ is non-generic, then

(1) The algebra $\mathcal{A}_{\beta,k}$ defined by generators and relations as in Section 2.8 need not be semi-simple.

(2) Given a multi-matrix pair $M_0 \subset M_1$ and a Markov trace tr of modulus $\beta$ on $M_1$, the restriction of tr to $A_{tr,k}(M_0 \subset M_1)$, the algebra generated by $\{\mathbf{1}, E_1, \cdots, E_{k-1}\}$ in $M_k$, need not be faithful.

(3) Given a second such pair $\tilde{M}_0 \subset \tilde{M}_1$ and a Markov trace $\tilde{tr}$ of modulus $\beta$ on $\tilde{M}_1$, the algebras $A_{tr,k}(M_0 \subset M_1)$ and $A_{\tilde{tr},k}(\tilde{M}_0 \subset \tilde{M}_1)$ need not be isomorphic.

All this contrasts with the generic case described in 2.8.5 and 2.8.6. The modulus $\beta = 1$ illustrates these phenomena.

Example 2.9.1. The algebra $\mathcal{A}_{1,3}$ is not semi-simple. (This is a particular case of Theorem II.10 in Appendix II.c.)

Proof. Let $T = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$ be the algebra of 2-by-2 upper triangular matrices over $\mathbb{K}$. As T is not semi-simple, it suffices to show that T is a quotient of $\mathcal{A}_{1,3}$. But the assignment

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \epsilon_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \epsilon_2 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

extends to a homomorphism from $\mathcal{A}_{1,3}$ onto T. #

Example 2.9.2. Consider the pair $M_0 = \mathbb{C} \oplus \mathbb{C}$ imbedded in $M_1 = \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ with inclusion matrix $\Lambda = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, together with the trace tr on $M_1$ with weight vector (1,–1). Then tr is a Markov trace of modulus 1 on $M_1$. Consider also a pair $\tilde{M}_0 = \tilde{M}_1$ with any faithful trace $\tilde{tr}$ on $\tilde{M}_1$; then $\tilde{tr}$ is evidently a Markov trace of modulus 1 on $\tilde{M}_1$. We have

(2) The restriction of tr to $A_{tr,1}(M_0 \subset M_1)$ is not faithful.

(3) $A_{tr,k}(M_0 \subset M_1)$ and $A_{\tilde{tr},k}(\tilde{M}_0 \subset \tilde{M}_1)$ are non-isomorphic for all $k \geq 2$.

Proof. The matrix $\Lambda\Lambda^t = \Lambda^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ has eigenvectors (1,–1) and (1,1) with eigenvalues 1 and 9 respectively. The Perron–Frobenius eigenvalue 9 is also the index $[M_1 : M_0]$. But the other eigenvector (1,–1) also defines a Markov trace tr on $M_1$ with modulus $\beta = 1$.

Let $M_0 \subset M_1 \subset M_2 \cdots$ be the tower generated by $M_0 \subset M_1$. Since $M_k$ is generated as an algebra by $M_1$ and $A_{tr,k}(M_0 \subset M_1)$, if for some $n_0$ the algebras $A_{tr,n_0}$ and $A_{tr,n_0+1}$ were equal, then $M_{n_0} = M_{n_0+1}$ as well, and therefore $M_k = M_{n_0}$ for all $k \geq n_0$. But $\dim_{\mathbb{C}} M_k$ increases as $[M_1 : M_0]^k = 9^k$, by Proposition 2.4.2. Hence

$$A_{tr,k}(M_0 \subset M_1) \subsetneq A_{tr,k+1}(M_0 \subset M_1)$$

for all k. On the other hand $\tilde{M}_1 = \tilde{M}_n$ for all k and $A_{\tilde{tr},k}(\tilde{M}_0 \subset \tilde{M}_1) \cong \mathbb{C}$ for all k. This proves (3).

The algebra $A_{tr,2}(M_0 \subset M_1)$ is spanned by $\mathbf{1}$ and $E_1$, and is of dimension 2, since $A_{tr,2} \supsetneq A_{tr,1} \cong \mathbb{C}$. The trace tr on $M_2$ restricted to $A_{tr,2}$ is given by

$$\mathrm{tr}(a + bE_1) = a + b \qquad (a,b \in \mathbb{C}).$$

It is not faithful because

$$\mathrm{tr}((\mathbf{1}-E_1)(a+bE_1)) = a\,\mathrm{tr}(\mathbf{1}-E_1) = 0$$

for all $a,b \in \mathbb{C}$. #

We do not intend to make a detailed study of the algebras $A_{tr,k}(M_0 \subset M_1)$ when $\beta$ is not generic. But we want to describe the structure of the unique quotient of $\mathcal{A}_{\beta,k}$ on which the usual rules $\mathrm{tr}(\mathbf{1}) = 1$ and $\beta\,\mathrm{tr}(we_i) = \mathrm{tr}(w)$ for $w \in \mathrm{alg}\{\mathbf{1}, e_1, \cdots, e_{i-1}\}$ defines a faithful normalized trace. (Here $e_i$ denotes the image of $\epsilon_i$ in the quotient.)

The algebras $B_{\beta,k}$. For the rest of this section we fix a $\beta \in \mathbb{K}^*$ which is n-generic but not (n+1)-generic for some $n \geq 1$. That is $P_k(\beta^{-1}) \neq 0$ for $k \leq n$, but $P_{n+1}(\beta^{-1}) = 0$. We again define a nested sequence $(B_{\beta,k})_{k \geq 1}$ of multi-matrix algebras over $\mathbb{K}$, and a consistent family of normalized faithful traces $\mathrm{tr}_k$ on these algebras. For $k \leq n$, define $B_{\beta,k}$ and $\mathrm{tr}_k$ exactly as in Lemma 2.8.4; since $\beta$ is n-generic there is no problem in doing so. For $k \geq n$ define $B_{\beta,k+1}$ to be the algebra obtained by applying the fundamental construction to the pair $B_{\beta,k-1} \subset B_{\beta,k}$. Observe that $B_{\beta,n+1}$ is the same as $B'_{\beta,n+1}$ in 2.8.4. For $k \leq n+1$, define $\mathrm{tr}_k$ as in Lemma 2.8.4; then $\mathrm{tr}_{n+1}$ is also faithful because $P_{n+1}$ does not appear in the computation of the weights of the trace on $B_{\beta,n+1} = \bigoplus_{j>0} Q_j^{n+1}$. Also since $P_{n+1}(\tau) = 0$, the trace on $B_{\beta,n+1} = B'_{\beta,n+1}$ extends that on $B_{\beta,n}$; it thus follows from 2.8.4(vi) (with $k = n+1$) that $tr_n$ is a Markov trace of modulus $\beta$ on $B_{\beta,n-1} \subset B_{\beta,n}$. For $k \geq n+1$, we define $tr_k$ as in Proposition 2.7.4. Thus $tr_k$ is a Markov trace on $B_{\beta,k-1} \subset B_{\beta,k}$ for $k \geq n$, but *not* for $k < n$. Note that $B_{\beta,k}$ is a multi-matrix algebra generated by the identity and idempotents $\{e_1,\cdots,e_{k-1}\}$ satisfying the relations 2.8.4(ii); in fact these relations hold for $\{e_1,\cdots,e_n\}$ by 2.8.4 and for $\{e_n,e_{n+1},\cdots\}$ by 2.7.5. For $k \geq n+1$ the identity is contained in the algebra generated by $\{e_1,\cdots,e_{k-1}\}$, in contrast to the case of generic $\beta$; this follows from 2.6.4.

Note that if $\mathbb{K} = \mathbb{C}$ and $\beta = 4\cos^2(\pi/(n+2))$, then the algebras $B_{\beta,k}$ can be given a $C^*$-structure such that the generators $\{e_i\}$ are self-adjoint projections, and the trace is faithful and positive. This is shown in 2.8.4 for $k \leq n$. The assertion for $k \geq n+1$ follows, because the tower construction for a pair of finite dimensional $C^*$-algebras with a positive Markov trace produces a chain of $C^*$-algebras with a positive trace, and self-adjoint projections $e_i$; see the discussion in Appendix IIa.

Example 2.9.3. Let $\beta = 1$, so that $n = 1$. The definitions above (*cum grano salis*) give $B_{\beta,k} = B_{\beta,1} = \mathbb{K}$ for all $k \geq 1$.

Example 2.9.4. Assume that the characteristic of $\mathbb{K}$ is not 2 and let $\beta = 2$, so that $n = 2$. The structure of the algebras $B_{\beta,k}$ and of the traces $tr_k$ is shown in figure 2.9.4 below.

Example 2.9.5. Assume that $\mathbb{K}$ contains $\mathbb{Q}(\sqrt{5})$ and choose $\beta \in \{4\cos^2(\pi/5), 4\cos^2(2\pi/5)\}$, so that $n = 3$. The picture (with $\tau = \beta^{-1}$ satisfying $P_4(\tau) = \tau^2 - 3\tau + 1 = 0$) is given below in figure 2.9.5.

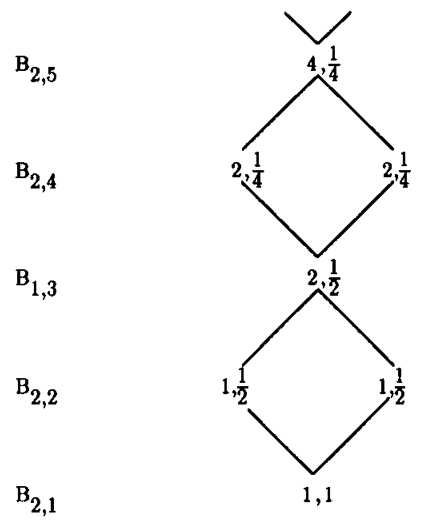



Figure 2.9.4.

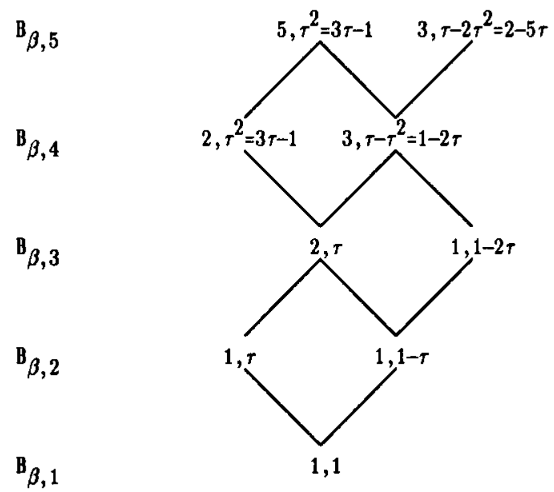



Figure 2.9.5.

In general, the picture for the $B_{\beta,k}$'s is obtained from that of the $\mathcal{A}_{\beta,k}$'s at the end of Section 2.8 by deleting the factor $Q_0^{n+1}$ (represented as the extreme right point in the $(n+1)$st row) as well as all factors above and to the right.

Theorem 2.8.5 gives a complete description of $\mathcal{A}_{\beta,k}$ when $\beta$ is (k–1)–generic. The following theorem indicates how part of the picture changes when $\beta$ is not generic. Recall that we may (and do) always identify $\mathcal{A}_{\beta,k}$ with a subalgebra of $\mathcal{A}_{\beta,k+1}$ (see Corollary 2.8.6) and that $B_{\beta,k}$ is also a subalgebra of $B_{\beta,k+1}$.

Theorem 2.9.6. *Consider an integer $n \geq 2$. Let $\beta \in \mathbf{K}^*$ be such that $P_j(\beta^{-1}) \neq 0$ for $j \leq n$ and $P_{n+1}(\beta^{-1}) = 0$, where $(P_j)_{j\geq 1}$ are the polynomials of Proposition 2.8.3. Then one has for all $k \geq 1$,*

(a) *$B_{\beta,k}$ is a multi–matrix algebra, and there exists a homomorphism $\pi_k$ of $\mathcal{A}_{\beta,k}$ onto $B_{\beta,k}$ mapping each generator $\epsilon_j$ onto $e_j$ $(1 \leq j \leq k-1)$.*

(b) *There exists a normalized trace $tr_k : B_{\beta,k} \to \mathbf{K}$ such that, for any $j \in \{1,\cdots,k-1\}$*

$$\beta tr_k(we_j) = tr_k(w)$$

*whenever $w$ is in the subalgebra $B_{\beta,j}$ of $B_{\beta,k}$. Moreover $tr_k$ is faithful and the restriction of $tr_k$ to $B_{\beta,j}$ is $tr_j$ for $j \leq k$.*

(c) *For $k \geq 2$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{A}_{\beta,k-1} & \longrightarrow & \mathcal{A}_{\beta,k} \\ \downarrow{\scriptstyle \pi_{k-1}} & & \downarrow{\scriptstyle \pi_k} \\ B_{\beta,k-1} & \longrightarrow & B_{\beta,k} \end{array}$$

(d) *There is a unique family of normalized traces $tr_k : \mathcal{A}_{\beta,k} \to \mathbf{K}$ such that*

$$(*) \qquad \left.\begin{array}{r} tr_k(x) = tr_{k-1}(x) \\ \beta\, tr_k(x\epsilon_{k-1}) = tr_{k-1}(x) \end{array}\right\} \; (x \in \mathcal{A}_{\beta,k-1}).$$

*If $I_{\beta,k}$ denotes the two sided ideal in $\mathcal{A}_{\beta,k}$ consisting of those $x$ such that $tr_k(xy) = 0$ for all $y \in \mathcal{A}_{\beta,k}$,* then *$I_{\beta,k} = \ker(\pi_k)$, so that $B_{\beta,k} \cong \mathcal{A}_{\beta,k}/I_{\beta,k}$.*

(e) *Suppose $(C_k)_{k\geq 1}$ is an increasing sequence of $\mathbf{K}$–algebras and $\psi_k : \mathcal{A}_{\beta,k} \to C_k$ are surjective homomorphisms such that $\psi_k\big|_{\mathcal{A}_{\beta,k-1}} = \psi_{k-1}$ for all k. Suppose further that each $C_k$ has a faithful normalized trace $tr_k : C_k \to \mathbf{K}$ satisfying*

$$tr_k\big|_{C_{k-1}} = tr_{k-1}, \text{ and}$$

$$\beta tr_k(w\psi_k(\epsilon_{k-1})) = tr_{k-1}(w)$$

*for $w \in C_{k-1}$. Then $C_k \cong B_{\beta,k} \cong \mathcal{A}_{\beta,k}/I_{\beta,k}$.*

(f) *The trace $tr_k$ on $\mathcal{A}_{\beta,k}$ also satisfies*

$$\beta\, tr_k(\epsilon_j w) = tr(w)$$

*whenever $1 \leq j \leq k-2$ and $w$ is an element of $\mathrm{alg}\,\{\mathbf{1},e_{j+1},\cdots,e_{k-1}\}$. More generally, we have*

$$tr_k(uw) = tr_k(u)\, tr_k(w)$$

*whenever $u \in \mathrm{alg}\,\{\mathbf{1},e_1,\cdots,e_j\}$ and $w \in \mathrm{alg}\,\{\mathbf{1},e_{j+1},\cdots,e_{k-1}\}$.*

(g) *The map $\epsilon_j \mapsto \epsilon_{k-j}$ extends to a trace preserving automorphism $\sigma_k$ of $\mathcal{A}_{\beta,k}$, and $e_j \mapsto e_{k-j}$ extends to a trace preserving automorphism $\overline{\sigma}_k$ of $B_{\beta,k}$. These automorphisms are inner in case $\mathbf{K}$ contains an element $q$ satisfying $q^{-1}(q+1)^2 = \beta$.*

Proof. Claims (a) to (c) follow from the construction of the $B_{\beta,k}$ above. The traces $tr_k|B_{\beta,k} \circ \pi_k$ on $\mathcal{A}_{\beta,k}$ satisfy $(*)$. The uniqueness statements in (b) and (d) are proved as in 2.8.5(b). We have $tr_k(xy) = tr_k(\pi_k(x)\pi_k(y))$, so that if $x \in \ker(\pi_k)$, then $x \in I_{\beta,k}$. Conversely if $x \in I_{\beta,k}$, then $\pi_k(x) = 0$ by faithfulness of $tr_k$ on $B_{\beta,k}$. This proves (d), and (e) follows similarly. Statement (f) is proved as 2.8.5(e), and statement (g) as 2.8.5(f). #

Corollary 2.9.7. *Suppose that $\mathbf{K} \supset \mathbf{R}$, that $M_0 \subset M_1$ is a pair of multi–matrix algebras over $\mathbf{K}$, and that tr is a positive Markov trace on $M_1$ of modulus $\beta = [M_1:M_0]$. Then $\Lambda_{tr,k}(M_0 \subset M_1)$ is isomorphic to $B_{\beta,k}$ for all $k \geq 1$.*

Proof. This follows from 2.8.5 and 2.8.7 when $\beta$ is generic, so we suppose that $\beta$ is non–generic. Let $(M_k)_{k\geq 1}$ be the tower of algebras generated by $M_0 \subset M_1$, and tr the extension of the trace to $\bigcup_k M_k$, as described in Section 2.7. Both $\beta = \|\Lambda_{M_0}^{M_1}\|^2$ and the weights of the trace are real and positive; see 2.7.3. Using the path model (2.4.6 and 2.6.5), we see that it is possible to choose a system of matrix units $T_{\xi,\eta}$ for the algebra $M_k$ so that the idempotents $E_i$ $(1 \le i \le k-1)$ are positive linear combinations of certain minimal idempotents $T_{\xi,\xi}$; see especially 2.6.5.2 and 2.6.5.4. Let $M_k^{\mathbb{R}}$ be the $\mathbb{R}$-linear span of the matrix units generating $M_k$. Thus $M_k^{\mathbb{R}}$ is a multi-matrix algebra over $\mathbb{R}$, and $M_k = M_k^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{K}$. Let $A_k^{\mathbb{R}}$ be the $\mathbb{R}$-subalgebra of $M_k^{\mathbb{R}}$ generated by $\{\mathbf{1}, E_1, \cdots, E_{k-1}\}$. The trace tr restricts to a positive $\mathbb{R}$-valued trace on $M_k^{\mathbb{R}}$. Note that $A_k^{\mathbb{R}}$ is closed under the $\mathbb{R}$-linear involution $*$ of $M_k^{\mathbb{R}}$ defined by $T_{\xi,\eta}^* = T_{\eta,\xi}$. Positivity of the trace implies that $\mathrm{tr}(x^*x) > 0$ for all non-zero $x \in M_k$, and as this holds in particular for $x \in A_k^{\mathbb{R}}$, we conclude that $\mathrm{tr}|_{A_k^{\mathbb{R}}}$ is faithful. It follows by linear algebra that tr is also faithful on $A_{\mathrm{tr},k}(M_0 \subset M_1) = A_k^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{K}$, and therefore 2.9.6(e) implies the conclusion. #

*The proof of Theorem 2.1.8 is now complete.*

<u>Theorem 2.9.8.</u> ([Jo2]). *Let* $n \ge 2$ *be an integer and suppose that* $\beta \in \mathbb{K}^*$ *is* n*-generic but not* $(n+1)$*-generic. Then the generating function* $f_n(x)$ *for* $(\dim_{\mathbb{K}}(B_{\beta,k+1}))_{k\ge 0}$ *is*

$$f_n(z) = \sum_{k=0}^{\infty} \dim_{\mathbb{K}}(B_{\beta,k+1})z^k = \frac{P_{n-1}(z)}{P_{n+1}(z)},$$

*where the* $P_j$ *are the polynomials of Proposition 2.8.3.*

<u>Proof.</u> Set $\Lambda_n = \Lambda_{B_{\beta,n-1}}^{B_{\beta,n}}$ and $b_k^n = \dim_{\mathbb{K}}(B_{\beta,k})$. Also let $\xi^{(n,k)}$ be the vector of dimensions of the multi-matrix algebra $B_{\beta,k}$. Note that the Bratteli diagram for $B_{\beta,n-1} \subset B_{\beta,n}$ is the Coxeter graph $A_{n+1}$, with a particular bicoloration and labelling of the vertices. (See 2.8.4(iv) for the picture, substituting n for k.) Thus for n odd $\Lambda_n$ is the $\frac{n+1}{2}$-by-$\frac{n+1}{2}$ Jordan block

$$\begin{bmatrix} 1 & 0 & & & \\ 1 & 1 & \ddots & & \\ 0 & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & 0 & 1 & 1 \end{bmatrix},$$

while for n even $\Lambda_n$ is the $(\frac{n}{2}+1)$-by-$\frac{n}{2}$ matrix

$$\begin{bmatrix} 1 & 0 & & \\ 1 & 1 & \ddots & \\ 0 & 1 & \ddots & \\ & \ddots & \ddots & 1 & 0 \\ & & \ddots & 1 & 1 \\ & & & 0 & 1 \end{bmatrix}.$$

In order to accomodate vectors and matrices of different sizes, we adopt the convention that $\mathbb{R}^d$ imbeds in $\mathbb{R}^{d+1}$ via

$$(\xi_1, \cdots, \xi_d)^t \mapsto (0, \xi_1, \cdots, \xi_d)^t.$$

With this convention we have <u>for n odd</u>

$$\xi^{(n,k)} = \begin{cases} (\Lambda_n\Lambda_n^t)^{(k-1)/2}\xi & \text{for } k \text{ odd} \\ \Lambda_n^t(\Lambda_n\Lambda_n^t)^{k/2-1}\xi & \text{for } k \text{ even,} \end{cases}$$

where $\xi = (0,0,\cdots 0,1)^t$. Hence

$$b_k^n = \|\xi^{(n,k)}\|^2 = \langle(\Lambda_n\Lambda_n^t)^{k-1}\xi|\xi\rangle \quad (n \text{ odd}). \tag{2.9.8.1}$$

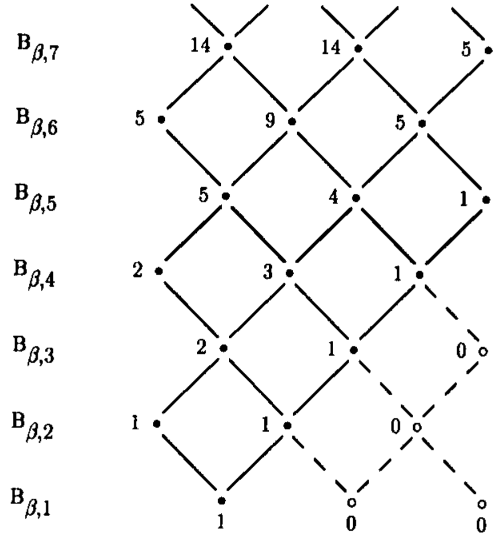
The corresponding formulae *for* n *even* are

$$\xi^{(n,k)} = \begin{cases} (\Lambda_n^t\Lambda_n)^{(k-1)/2}\xi & \text{for } k \text{ odd} \\ \Lambda_n(\Lambda_n^t\Lambda_n)^{k/2-1}\xi & \text{for } k \text{ even.} \end{cases}$$

Hence

$$b_k^n = \langle(\Lambda_n^t\Lambda_n)^{k-1}\xi|\xi\rangle \quad (n \text{ even}). \tag{2.9.8.2}$$

One can visualize these results quite easily by adding to the Bratteli diagram of the chain $(B_{\beta,k})_{k\ge 1}$ some "phantom" vertices with zero dimension. The picture for $n = 5$, for example, is




Recall also that our labelling of the vertices on each floor increases from right to left. Since $\Lambda_n\xi = \xi$ (n odd) and $\Lambda_n^t\xi = \xi$ (n even), (2.9.8.1) and (2.9.8.2) give

(2.9.8.3) $$b_k^n = \langle(\Lambda_n^t\Lambda_n)^k\xi|\xi\rangle \quad (n \text{ odd}), \text{ and}$$

(2.9.8.4) $$b_k^n = \langle(\Lambda_n\Lambda_n^t)^k\xi|\xi\rangle \quad (n \text{ even}).$$

Finally one verifies that

(2.9.8.5) $$\Lambda_{n+1}^t\Lambda_{n+1} - \Lambda_n^t\Lambda_n = E \quad (n \text{ odd}), \text{ and}$$

(2.9.8.6) $$\Lambda_{n+1}\Lambda_{n+1}^t - \Lambda_n\Lambda_n^t = E \quad (n \text{ even}),$$

where $E$ is the orthogonal projection onto $\mathbb{R}\xi$, in the Euclidean space of the appropriate dimension.

We claim that the functions $(f_n(z))_{n\geq 2}$ satisfy the first order difference equation

$$f_{n+1}(z) - [zf_n(z)+1] = zf_{n+1}(z)[zf_n(z)+1].$$

First consider the case that $n$ is odd. Then $zf_n(z) + 1 =$

$$= \sum_{k=0}^{\infty} b_{k+1}^n z^{k+1} + 1 = \sum_{k=1}^{\infty} b_k^n z^k + 1$$
$$= \sum_{k=1}^{\infty} \langle(\Lambda_n^t\Lambda_n)^k\xi|\xi\rangle z^k + \langle\xi|\xi\rangle$$
$$= \sum_{k=0}^{\infty} \langle(\Lambda_n^t\Lambda_n)^k z^k\xi|\xi\rangle,$$

using 2.9.8.3. Setting $B = \Lambda_n^t\Lambda_n$ we have

$$zf_n(z) + 1 = \langle(\mathbf{1}-Bz)^{-1}\xi|\xi\rangle.$$

Similarly using 2.9.8.2 and setting $A = \Lambda_{n+1}^t\Lambda_{n+1}$, we have

$$f_{n+1}(z) = \sum_{k=0}^{\infty} b_{k+1}^{n+1} z^k = \sum_{k=0}^{\infty} \langle(\Lambda_{n+1}^t\Lambda_{n+1})^k z^k\xi|\xi\rangle$$
$$= \langle(\mathbf{1}-Az)^{-1}\xi|\xi\rangle.$$

The difference $f_{n+1}(z) - [zf_n(z)+1]$ is computed using 2.9.8.5, and the resolvent identity:

$$\begin{aligned} f_{n+1}(z) - [zf_n(z)+1] &= \langle[(\mathbf{1}-Az)^{-1}-(\mathbf{1}-Bz)^{-1}]\xi|\xi\rangle \\ &= \langle(\mathbf{1}-Az)^{-1}z(\Lambda_{n+1}^t\Lambda_{n+1}-\Lambda_n^t\Lambda_n)(\mathbf{1}-Bz)^{-1}\xi|\xi\rangle \\ &= z\langle(\mathbf{1}-Az)^{-1}E(\mathbf{1}-Bz)^{-1}\xi|\xi\rangle \\ &= z\langle(\mathbf{1}-Az)^{-1}\xi|\xi\rangle\langle(\mathbf{1}-Bz)^{-1}\xi|\xi\rangle \\ &= zf_{n+1}(z)[zf_n(z)+1]. \end{aligned}$$

The case $n$ even is entirely similar.

Next we observe that the functions $s_n(z) = \dfrac{P_{n-1}(z)}{P_{n+1}(z)}$ satisfy the same difference equation. First note that

$$zs_n + 1 = \frac{zP_{n-1}+P_{n+1}}{P_{n+1}} = \frac{P_n}{P_{n+1}},$$

by the second-order difference equation for the $P_j$. Hence

$$s_{n+1} - [zs_n+1] = \frac{P_n}{P_{n+2}} - \frac{P_n}{P_{n+1}}$$
$$= \frac{P_n}{P_{n+2}P_{n+1}}(P_{n+1}-P_{n+2}) = \frac{zP_n^2}{P_{n+2}P_{n+1}},$$

using the defining relation for the $P_j$ again. But this last expression is $zs_{n+1}[zs_n+1]$.

Since $(f_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ satisfy the same first order difference equation, it suffices now to check that $f_2 = s_2$. But $b_{k+1}^2 = 2^k$ for all $k$, so $f_2(z) = \sum_{k=0}^{\infty} 2^k z^k = \frac{1}{1-2z}$, while $s_2(z) = \frac{P_1(z)}{P_3(z)} = \frac{1}{1-2z}$. #

## 2.10. A digression on Hecke algebras.

As a general reference for this section, we use [BLie], especially exercises $\geq 22$ in §IV.2. See also [CR], §11D. For the origin of the term "Hecke algebra", see p. xi in [Lus].

2.10.a - The complex Hecke algebra defined by $GL_n(q)$ and its Borel subgroup.

If $G$ is a finite group and $G_0$ is a subgroup, the complex Hecke algebra $H(G,G_0)$ of the pair $G_0 \subset G$ is the commutant of the natural representation of $G$ on the complex vector space $\mathbb{C}[G/G_0]$ of functions from $G/G_0$ to $\mathbb{C}$.

We denote by $\mathbb{C}[G]$ the algebra of complex functions on $G$, with the convolution product. We identify $\mathbb{C}[G/G_0]$ with the subspace of this algebra consisting of functions $\varphi$ with $\varphi(gh) = \varphi(g)$ for $g \in G$ and $h \in G_0$, and we denote by $\mathbb{C}[G_0\backslash G/G_0]$ the subalgebra of $\mathbb{C}[G]$ of $G_0$-bi-invariant functions.

Proposition 2.10.1. *The algebras* $H(G,G_0)$ *and* $\mathbb{C}[G_0\backslash G/G_0]$ *are isomorphic.*

Proof. More generally, consider first an associative algebra $A$ with unit, an idempotent $e \in A$, and the left $A$-module $Ae$. It is easy to check that the map $x \mapsto \rho(x)$ = right multiplication by $x$ is an anti-isomorphism from $eAe$ to $\mathrm{End}_A(Ae)$.

Now let $A = \mathbb{C}[G]$; for each $g \in G$, denote by $\delta_g$ the characteristic function of $\{g\}$. Set $e = \frac{1}{|G_0|}\sum_{h\in G_0} \delta_h$. Then $Ae = \mathbb{C}[G/G_0]$ and $eAe = \mathbb{C}[G_0\backslash G/G_0]$, so that $H(G,G_0)$ and $\mathbb{C}[G_0\backslash G/G_0]$ are anti-isomorphic. But $\mathbb{C}[G]$ has a canonical anti-isomorphism $\varphi \mapsto \check{\varphi}$, defined by $\check{\varphi}(g) = \varphi(g^{-1})$, which restricts to $\mathbb{C}[G_0\backslash G/G_0]$, so the proposition follows. #

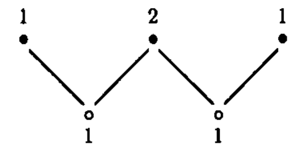
Corollary 2.10.2. *Let* $e$ *be the central idempotent in* $\mathbb{C}[G_0]$ *corresponding to the trivial representation* $G_0 \to GL_1(\mathbb{C})$, *and denote by* $p_1,\cdots,p_m$ *the minimal central idempotents in* $\mathbb{C}[G]$. *Then*

$$H(G,G_0) \approx \oplus\, ep_i\mathbb{C}[G]p_ie$$

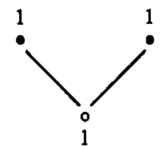
*where the direct sum is over the* $i$*'s with* $ep_i \neq 0$. *The Bratteli diagram for the pair* $\mathbb{C} \subset H(G,G_0)$ *is that part of the Bratteli diagram for the pair* $\mathbb{C}[G_0] \subset \mathbb{C}[G]$ *which lies above the vertex corresponding to* $e$.

Proof. This follows from Section 2.3. (See Corollary 11.26 of [CR] for a generalization.) #

As a first example, consider the permutation groups $\mathfrak{S}_2 \subset \mathfrak{S}_3$; the diagram for $\mathbb{C}[\mathfrak{S}_2] \subset \mathbb{C}[\mathfrak{S}_3]$ is




Then $\mathbb{C} \subset H(\mathfrak{S}_3,\mathfrak{S}_2)$ is described by




In particular $H(\mathfrak{S}_3,\mathfrak{S}_2) \approx \mathbb{C} \oplus \mathbb{C}$. It is easy to check that there are two double cosets in $\mathfrak{S}_2\backslash\mathfrak{S}_3/\mathfrak{S}_2$. One shows similarly that $H(\mathfrak{S}_{k+1},\mathfrak{S}_k) \approx \mathbb{C} \oplus \mathbb{C}$ for any integer $k \geq 1$.

But the case of main interest here is when $q$ is a prime power, $G = GL_n(q)$ for some $n \geq 2$, and $G_0$ is the (Borel) subgroup $B$ of upper triangular matrices. (The letter $q$ will no longer denote an idempotent below.) Identifying the double cosets is a special case of the "Bruhat decomposition"

$$GL_n(q) = \coprod_{w \in W} BwB$$

where $W$ is the "Weyl group", namely here the symmetric group $\mathfrak{S}_n$ embedded in $GL_n(q)$ as permutation matrices (see §IV.2 in [BLie]). Thus to each permutation $w \in \mathfrak{S}_n$ there is associated an element $a_w$ of the Hecke algebra $H(G,G_0)$, which is the characteristic function of $BwB$ divided by the order of $B$. For $i = 1,2,\cdots,n-1$, let $s_i$ be the element of $W$ given by the matrix

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}$$

where the first diagonal 0 is the (i,i)th entry, and set $g_i = a_{s_i}$.

Proposition 2.10.3. *With the notation above, one has*

(a) $g_i^2 = (q-1)g_i + q$ $\qquad i = 1,\cdots,n-1$

(b) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ $\qquad i = 1,\cdots,n-2$

(c) $g_i g_j = g_j g_i$ *if* $|i-j| \geq 2$ $\qquad i,j = 1,\cdots,n-1$

*Furthermore the elements* $g_i$ $(1 \leq i \leq n-1)$ *generate the Hecke algebra* $H(GL_n(q),B)$.

Proof. (see [BLie] as well as Propositions 11.30 and 11.34 in [CR]). For each permutation $w \in \mathfrak{S}_n$, set $C(w) = BwB$. Let $a_w \in \mathbb{C}[B\backslash G/B]$ be the quotient by $|B|$ of the characteristic function of $C(w)$; then $(a_w)_{w \in \mathfrak{S}_n}$ is a $\mathbb{C}$-basis of the Hecke algebra. For $w,w',w'' \in \mathfrak{S}_n$ and for $g \in C(w'')$, one has

$$\begin{aligned}(a_w * a_{w'})(g) &= \sum_{h \in G} a_w(h) a_{w'}(h^{-1}g) \\ &= \frac{1}{|B|^2} |C(w) \cap gC(w')^{-1}| \qquad (*)\end{aligned}$$

If $C(w) \cap gC(w')^{-1}$ is not empty, there exist $b_1,\cdots,b_4 \in B$ with $b_1 w b_2 = g b_3 w'^{-1} b_4$, so that $g \in C(w)C(w')$.

For $s$ in the set $S = \{s_1,\cdots,s_{n-1}\}$ of generators of $\mathfrak{S}_n$, we need to compute $|C(s)|$. Observe more generally that, for any $h \in G$, the map

$$\begin{cases} B/(B \cap hBh^{-1}) \to (BhB)/B \\ \text{class of } b \mapsto \text{class of } bh \end{cases}$$

is well defined (if $b,b'$ are in the same class modulo $B \cap hBh^{-1}$, there exists $b'' \in B$ with $b' = bhb''h^{-1}$, and $b'hB = bhB$) and bijective. Then the number of left classes modulo $B$ in $BhB$ is the index $[B : B \cap hBh^{-1}]$. It follows that

$$|C(s)| = |B|\,[B : B \cap sBs] = |B|q.$$

Let us compute $(a_w * a_{w'})(g)$ when $w = w' = s$. As $C(s)C(s) = B \cup C(s)$ this is zero unless $g \in B \cup C(s)$. For $g \in B$ one has by $(*)$

$$(a_s * a_s)(g) = \frac{1}{|B|^2}|C(s)| = q a_1(g).$$

As $a_1$ is a convolution unit in $\mathbb{C}[B\backslash G/B]$, this implies

$$a_s * a_s = \lambda a_s + q$$

for some $\lambda \in \mathbb{C}$. Introduce the restriction $\mu$ to $\mathbb{C}[B\backslash G/B]$ of the augmentation homomorphism $\mathbb{C}[G] \to \mathbb{C}$, mapping $\varphi$ to $\sum_{g \in G} \varphi(g)$. Then

$$\mu(a_s * a_s) = \frac{1}{|B|^2}|C(s)|^2 = q^2$$

$$\mu(\lambda a_s + q) = \lambda \frac{1}{|B|}|C(s)| + q = (\lambda+1)q$$

and consequently $\lambda = q - 1$. This shows (a).

Introduce the length function $\ell : \mathfrak{S}_n \to \{0,1,2,\cdots\}$ with respect to the generators $S$. Then

$$a_s * a_w = a_{sw} \quad \text{if } \ell(sw) > \ell(w) \qquad (**)$$

Indeed, if $\ell(sw) > \ell(w)$, then $C(s)C(w) = C(sw)$ by n° IV.2.4 in [BLie], so that $a_s * a_w$ is a scalar multiple of $a_{sw}$ by $(*)$. Let $g,h \in C(s)$ and $u,v \in C(w)$ with $gu = hv$; then $vu^{-1} = h^{-1}g \in C(s)C(s) = B \cup C(s)$; but $vu^{-1} \in C(s)$ would imply $v \in C(s)C(w) =$

$C(sw)$, which is incompatible with $v \in C(w)$; hence $g \in hB$, and thus any element in $C(sw)$ can be written in exactly $|B|$ ways as a product of one element in $C(s)$ by one in $C(w)$. This shows that $a_s * a_w = a_{sw}$. It follows in particular that $\{a_s\}$ generates $H(GL_n(q),B)$.

Consider finally $s,t \in S$ with $(st)^3 = 1$. Then $\ell(s) = 1$, $\ell(st) = 2$, $\ell(sts) = 3$ and thus $a_s * a_t * a_s = a_{sts}$ by (**). Similarly $a_t * a_s * a_t = a_{tst}$, and (b) holds. Claim (c) follows in the same way. #

Now remember that the symmetric group in $n$ letters has a presentation with generators the transpositions $s_i = (i,i+1)$ for $1 \le i \le n-1$ and relations

$$s_i^2 = 1 \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad s_i s_j = s_j s_i \text{ if } |i-j| \ge 2.$$

There is an easy proof of this which shows at the same time that the abstract algebra generated by $n-1$ generators subjected to the relations of 2.10.3 is of dimension at most $n!$. (See the beginning of §4 in [HKW].) For $q$ a prime power, it follows then that the relations of 2.10.3 give a presentation of the Hecke algebra $H(GL_n(q),B)$. But we shall see that it is important to consider a more general family of algebras, defined for all $q \neq 0$.

## 2.10.b – The Hecke algebras $H_{q,n}$.

Let $\mathbb{K}$ again be an arbitrary field. Consider an integer $n \ge 1$ and a parameter $q \in \mathbb{K}$. We define $H_{q,n}$ to be the associative $\mathbb{K}$-algebra with unit presented by

generators: $g_1, g_2, \cdots, g_{n-1}$

relations: as in 2.10.3.

Proposition 2.10.4. *One has* $\dim_{\mathbb{K}} H_{q,n} = n!$ *for all* $q \in \mathbb{K}$ *and for all* $n \ge 1$.

Proof. We take for granted the presentation of $\mathfrak{S}_n$ in terms of the transpositions $\{s_i\}$. Each of the $n!$ elements $\pi$ of $\mathfrak{S}_n$ can be written uniquely as a reduced word $w$ in the $\{s_i\}$ with

(i) minimum length among all words representing $\pi$,

(ii) the largest $s_i$ in $w$ appearing only once, and moved as far to the right as possible, and

(iii) all subwords of $w$ reduced according to criteria (i) and (ii).

The corresponding $n!$ words in the generators $\{g_i\}$ of $H_{q,n}$ span $H_{q,n}$ linearly, because the Hecke algebra relations 2.10.3(a)–(c) can be used

(i) to reduce the length of a word in the $\{g_i\}$ (i.e., to write it as a linear combination of shorter words), and

(ii) to reduce the number of occurences of the largest $g_i$ in a word, and to move it to the right,

whenever the corresponding operation can be performed on the corresponding word in the $\{s_i\}$. It follows that $\dim_{\mathbb{K}} H_{q,n}$ is at most $n!$. On the other hand, we will exhibit below a sufficient family of inequivalent irreducible representation of $H_{q,n}$ to obtain the other inequality. See [HKW,§4] for a more explicit proof. #

For convenience we take $\mathbb{K} = \mathbb{C}$ in the following discussion. For $q$ a prime power, $H_{q,n}$ is the same as $H(GL_n(q),B)$ in 2.10.a, and is in particular semi-simple. But we have no reason *a priori* to believe that there is any relationship between these algebras for different values of $q$. Also, the decomposition of any $H_{q,n}$ as a direct sum of matrix algebras is not obvious, each summand corresponding to some irreducible representation of $GL_n(q)$.

Observe however that, if we put $q = 1$, we recognize $H_{1,n}$ as the algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group, so $H_{1,n}$ is semi-simple. A necessary and sufficient condition for semi-simplicity of $H_{q,n}$ is the non-degeneracy of the Killing trace $x \longmapsto tr(\lambda(x))$, where $tr$ denotes the trace on $\mathrm{End}_{\mathbb{C}}(H_{q,n})$. (For a finite dimensional $\mathbb{C}$-algebra $A$, the radical $\mathrm{rad}(A)$ coincides with $A^\perp := \{x \in A: \ tr(\lambda(xy)) = 0 \text{ for all } y \in A\}$. In fact, both $\mathrm{rad}(A)$ and $A^\perp$ are ideals which contain every nil ideal, and to show equality one shows that each is a nil ideal.) From the proof of Proposition 2.10.4 one obtains a basis $\{g_\sigma : \sigma \in \mathfrak{S}_n\}$ of $H_{q,n}$ and polynomial structure constants $p^\mu_{\sigma,\tau}(q)$ such that $g_\sigma g_\tau = \sum_\mu p^\mu_{\sigma,\tau}(q)\, g_\mu$. It follows that degeneracy is determined by a polynomial equation in $q$, so for all but a finite set of $q \in \mathbb{C}$ ($n$ fixed), $H_{q,n}$ is semi-simple of dimension $n!$. Also $H_{q,n-1}$ embeds in $H_{q,n}$ via the obvious identification of the generators $g_i$ for $1 \le i \le n-2$.

We now argue intuitively, though extremely plausibly. For the values of $q$ for which $H_{q,n-1}$ and $H_{q,n}$ are semi- simple, the inclusion $H_{q,n-1} \subset H_{q,n}$ is completely described by a vector of integers (for the dimensions of the factors in $H_{q,n}$) and an integer valued matrix (the inclusion matrix). As these should vary continuously with $q$, they should be independent of $q$ for these values. In particular they can be determined by examining the case $q = 1$. But then they are determined entirely by the dimensions of the different representations of $\mathfrak{S}_{n-1}$ and $\mathfrak{S}_n$ and the restriction rule from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$. For this reason we shall now describe this structure. In 2.10d we will identify a certain singular set