PREFACE

THIS monograph is the belated fulfilment of an undertaking made some years ago to publish a self-contained account of Hall polynomials and related topics.

These polynomials were defined by Philip Hall in the 1950s, originally as follows. If M is a finite abelian p-group, it is a direct sum of cyclic subgroups, of orders $p^{\lambda_1}, p^{\lambda_2},\dots,p^{\lambda_r}$ say, where we may suppose that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r$. The sequence of exponents $\lambda = (\lambda_1,\dots,\lambda_r)$ is a partition, called the *type* of M, which describes M up to isomorphism. If now μ and ν are partitions, let $g^{\lambda}_{\mu\nu}(p)$ denote the number of subgroups N of M such that N has type μ and M/N has type ν. Hall showed that $g^{\lambda}_{\mu\nu}(p)$ is a polynomial function of p, with integer coefficients, and was able to determine its degree and leading coefficient. These polynomials are the Hall polynomials.

More generally, in place of finite abelian p-groups we may consider modules of finite length over a discrete valuation ring $\mathfrak{o}$ with finite residue field: in place of $g^{\lambda}_{\mu\nu}(p)$ we have $g^{\lambda}_{\mu\nu}(q)$, where q is the number of elements in the residue field.

Next, Hall used these polynomials to construct an algebra which reflects the lattice structure of the finite $\mathfrak{o}$-modules. Let $H(q)$ be a free $\mathbf{Z}$-module with basis (u_λ) indexed by the set of all partitions λ, and define a multiplication in $H(q)$ by using the $g^{\lambda}_{\mu\nu}(q)$ as structure constants, i.e.

$$u_\mu u_\nu = \sum_\lambda g^{\lambda}_{\mu\nu}(q) u_\lambda.$$

It is not difficult to show (see Chapter II for the details) that $H(q)$ is a commutative, associative ring with identity, which is freely generated (as $\mathbf{Z}$-algebra) by the generators $u_{(1^r)}$ corresponding to the elementary $\mathfrak{o}$-modules.

Symmetric functions now come into the picture in the following way. The ring of symmetric polynomials in n independent variables is a polynomial ring $\mathbf{Z}[e_1,\dots,e_n]$ generated by the elementary symmetric functions $e_1,\dots,e_n$. By passing to the limit with respect to n, we obtain a ring $\Lambda = \mathbf{Z}[e_1, e_2,\dots]$ of symmetric functions in infinitely many variables. We might therefore map $H(q)$ isomorphically onto Λ by sending each generator $u_{(1^r)}$ to the elementary symmetric function e_r. However, it turns out that a better choice is to define a homomorphism $\psi : H(q) \to \Lambda \otimes \mathbf{Q}$ by $\psi(u_{(1^r)}) = q^{-r(r-1)/2} e_r$ for each $r \geqslant 1$. In this way we obtain a family of symmetric functions $\psi(u_\lambda)$, indexed by partitions. These symmetric functions are essentially the Hall–Littlewood functions, which are the subject of Chapter III. Thus the combinatorial lattice properties of finite $\mathfrak{o}$-modules are reflected in the multiplication of Hall–Littlewood functions.

The formalism of symmetric functions therefore underlies Hall's theory, and Chapter I is an account of this formalism—the various types of symmetric functions, especially the Schur functions (S-functions), and the relations between them. The character theory of the symmetric groups, as originally developed by Frobenius, enters naturally in this context. In an appendix we show how the S-functions arise 'in nature' as the traces of polynomial functors on the category of finite-dimensional vector spaces over a field of characteristic 0.

In the past few years, the combinatorial substructure, based on the 'jeu de taquin', which underlies the formalism of S-functions and in particular the Littlewood–Richardson rule (Chapter I, §9), has become much better understood. I have not included an account of this, partly from a desire to keep the size of this monograph within reasonable bounds, but also because Schützenberger, the main architect of this theory, has recently published a complete exposition [46].

The properties of the Hall polynomials and the Hall algebra are developed in Chapter II, and of the Hall–Littlewood symmetric functions in Chapter III. These are symmetric functions involving a parameter t, which reduce to S-functions when $t=0$ and to monomial symmetric functions when $t=1$. Many of their properties generalize known properties of S-functions.

Finally, Chapters IV and V apply the formalism developed in the previous chapters. Chapter IV is an account of J. A. Green's work [14] on the characters of the general linear groups over a finite field, and we have sought to bring out, as in the case of the character theory of the symmetric groups, the role played by symmetric functions. Chapter V is also about general linear groups, but this time over a non-archimedean local field rather than a finite field, and instead of computing characters we compute spherical functions. In both these contexts Hall's theory plays a decisive part.

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