Quantum groups:
A survey of definitions, motivations, and results

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Introduction

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Introduction

(1.1) Goal of this survey

The theory of quantum groups began its development in about 1982-1985. It is now 10 years since the 1986 ICM address of V.G. Drinfel'd ignited a wild frenzy of research activity in this area and things related to it. During this time quantum groups have become a "household" term in Lie theory in much the same way that Kac-Moody Lie algebras did in the 1970's. Given that quantum groups are now a part of every day Lie theory it seems desirable that there are treatments of the subject which are accessible to graduate students.

It has been my goal to produce a survey which is accessible to graduate students, and which contains the necessary background and the main results in the theory. I have chosen to make this a compendium of motivation, definitions and results. A secondary goal has been to write this in a relatively small space (long works are usually too daunting) and with this in mind I have chosen not to include any proofs. In many cases, providing a full proof would require introducing and developing some fairly sophisticated tools.

My main focus in these notes is to give a description of what the Drinfel'd-Jimbo quantum groups are, how one arrives at them and why they are natural. In the last chapter I shall explain how the Drinfel'd-Jimbo quantum groups are applied to get link invariants such as the Jones polynomial.

(1.2) References for quantum groups

Drinfel'd's paper in the proceedings of the ICM 1986 is a dense summary of many of the amazing results that he had obtained. This paper still remains a basic reference.

[Dr] V.G. Drinfeld, *Quantum Groups*, in Proceedings of the International Congress of Mathematicians, A.M. Gleason ed., pp. 798-820, American Mathematical Society, Providence 1987.

Between 1987 and 1995 literally thousands of papers on quantum groups have been published. The book by V. Chari and A. Pressley which appeared in 1994 has 70 pages of references in minuscule type! Instead of wading through this mass of literature I have decided to only refer you to the books on quantum groups which have begun to appear recently, as follows:

[CP] V. Chari and A. Pressley, "A Guide to Quantum Groups", Cambridge University Press, Cambridge, 1994.

[Ja] J. Jantzen, "Lectures on Quantum Groups", Graduate Studies in Mathematics Vol. **6**, American Mathematical Society, 1995.

[Jo] A. Joseph, "Quantum groups and their Primitive Ideals", Ergebnisse der Mathematik und ihrer Grenzgebiete; **3** Folge, Bd. **29**, Springer-Verlag, New York-Berlin, 1995.

[Ka] C. Kassel, “Quantum groups”, Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995.

[Lu] G. Lusztig, “Introduction to Quantum Groups”, Progress in Mathematics **110**, Birkhauser, Boston, 1993.

[Ma] S. Majid, “Foundations of quantum group theory”, Cambridge University Press, 1995.

[SS] S. Shnider and S. Sternberg, “Quantum groups: From Coalgebras to Drinfel'd Algebras”, Graduate Texts in Mathematical Physics Vol. **2**, International Press, Cambridge, MA 1995.

I recommend [CP] for obtaining a basic understanding of what quantum groups are, where they came from, what the main results are, and what was known as of about the end of 1993. It contains only easy proofs and sketches of more involved proofs, very often referring the reader to the original papers for the full details of proofs. This book, however, is very useful for understanding what is going on. The recent book [Ja] is written specifically for graduate students. It has an excellent choice of topics, thorough descriptions of the motivations at each stage and detailed proofs. The book [SS] treats the deformation theory aspect of quantum groups in detail and the book [Lu] is the only one that covers the connection between the quantum group and perverse sheaves.

(1.3) Some missing topics and where to find them

There are many beautiful things in the theory of quantum groups that we won't even have time to mention. A *few* of these are:

(a) Canonical and crystal bases and the Littelmann path model for representations, see [Jo] Chapt. 5-6 and [Ja] Chapt. 9-11.

(b) Yangians, see [CP] Chapt. 12.

(c) Quasi-Hopf algebras and twisting, see [CP] Chapt. 16 and [SS] Chapt. 8.

(d) The Knizhnik-Zamalodchikov equation and hypergeometric functions, see [CP] Chapt. 16, [Ka] Chapt. 19 and [SS] Chapt. 12.

(e) Lie bialgebras, Poisson Lie groups, and symplectic leaves, see [CP] Chapt. 1.

(f) Representations at roots of unity and the connection to representations of algebraic groups over a finite field, see [CP] Chapt. 11 and [AJS].

(g) The connection between representations of quantum groups at roots of unity and representations of affine Lie algebras at negative level, see [CP] Chapt. 11 and Chapt. 16 and [KL].

(1.4) Further references for the background topics

Chapters I-IV consist of background material needed for the material on quantum groups. These chapters are:

I. Hopf algebras and braided tensor categories

II. Lie algebras and enveloping algebras

III. Deformations of Hopf algebras

IV. Perverse Sheaves

The following book contains a very nice up-to-date account of the theory of Hopf algebras, and it also includes some useful things on quantum groups.

[Mo] S. Montgomery, "Hopf Algebras and their Actions on Rings", Regional Conference Series in Mathematics **82**, American Mathematical Society, 1992.

The book by Chari and Pressley [CP] contains a nice introduction to monoidal categories and braided monoidal categories.

The following little book is a beautiful summary of the main results in semisimple Lie theory.

[Se] J.-P. Serre, "Complex Semisimple Lie algebras", Springer-Verlag, New York, 1987.

Comprehensive accounts of the theory of Lie algebras and enveloping algebras can be found in Bourbaki and in the book by Dixmier.

[Bou] N. Bourbaki, "Groupes et Algèbres de Lie, Chapitres I-VIII", Masson, Paris, 1972.

[Dix] J. Dixmier, "Enveloping algebras", Amer. Math. Soc. (1994); originally published in French by Gauthier-Villars, Paris 1974 and in English by North Holland, Amsterdam 1977.

The following are standard (and very useful) texts in Lie theory.

[Hu] J. Humphreys, "Introduction to Lie algebras and representation theory", Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin, (3rd printing) 1980.

[K] V. Kac, "Infinite dimensional Lie algebras", Birkhauser, Boston, 1983.

The most comprehensive reference for modern deformation theory, especially in regard to deformations of Hopf algebras, is the book by Shnider and Sternberg [SS] listed above. The book [CP] also contains a very informative chapter on deformation theory.

Unfortunately, to my knowledge, there is no good introductory text on the theory of perverse sheaves. The classical reference is the following monograph.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982), Soc. Math. France.

On the other hand, much of the background material to perverse sheaves, such as homological algebra and sheaf theory is classical and appears in many books. The first few chapters of the following book contain an introduction to these topics.

[KS] M. Kashiwara and P. Schapira, "Sheaves on Manifolds", Grundlehren der mathematischen Wissenschaften **292**, Springer-Verlag, New York-Berlin, 1980.

(1.5) On reading these notes

I advise the reader to begin immediately with Chapter V and find out what a quantum group is. One can always peek back at the earlier chapters and find out the definitions later. This makes it more fun and provides good motivation for learning the earlier background material. It also avoids getting bogged down before one even gets to the quantum group.

In a number of places I have chosen to make these notes "nonlinear". There have been some occasions when I have decided to repeat some definition or some statement. Also in a few places, I have used some terms and notations that have not been defined yet, with an appropriate reference to the place later in the text where the definitions and notations can be found. I have done this with the intention of making each section a somewhat complete set of ideas without disrupting any particular section with a myriad of lengthy definitions. Even though we may wish it so, ideas in mathematics are not really linear and this has been reflected in these notes. *The reader should feel free to skip around in the notes whenever the inclination arises.*

I have included a complete table of contents in the hope that it will be helpful to the reader as a tool for finding definitions and for organizing and motivating the structures. For the same reason I have given every small section a title. This way the reader can follow the process of the development, as well as the details. Think of the table of contents as a flow chart for the mathematics.

(1.6) Disclaimer

Even though the theory of quantum groups is less than 15 years old I shall not undertake the complicated task of giving appropriate references and credits concerning the sources of the theorems and their first proofs. I refer the reader to the above books on quantum groups for this information.

Let me stress that none of the theorems stated in this manuscript are due to me with two possible exceptions. Chapt. I Proposition (5.5) and Chapt. VII Theorem (5.2) are more general than I know of in the existing literature. Chapt. I Proposition (5.5) is well known in the context of the quantum group and I am only pointing out here that the well known proof, see [Ta] Prop. 2.2.1, works for any quantum double. Chapt. VII Theorem (5.2) is a nontrivial, but very natural, extension of well known results which appear, for example, in [Ja] Chapt. 8. The crucial part of the proof is similar to the proof of [Ja] Lemma 8.3.

I have tried to indicate, at the beginning of each chapter, where one can find proofs of the theorems stated in that chapter. In many instances I have had to make minor changes in notations and statements in order to be consistent with the definitions that I have given. Especially since I have not included proofs the reader should be watchful and open to the possibility that there may be some minor errors.

(1.7) Acknowledgements

First and foremost I thank Hans Wenzl for introducing me to the world of quantum groups and encouraging me to pursue research in related topics. He taught me the basics of quantum group theory and notes from a course he gave at University of California, San Diego have been tremendously useful over the years. I thank all of the audience members in my course in quantum groups at University of Wisconsin during Spring of 1994 for their interest, their suggestions and for coming so early in the morning.

I thank Gus Lehrer for inviting me to Australia, for making my year there a wonderful one and for suggesting my name for various invitations via which these notes have come into being. I thank Chuck Miller and John Cossey for the invitation to give a series of lectures on quantum groups to graduate students at the Workshop on Algebra, Geometry and Topology at Australian National University in Canberra during January 1996. These notes are an expanded version of the notes I distributed there. I thank Michael Murray and Alan Carey for inviting me to speak at the Australian Lie Groups Conference 1996 and for inviting me to contribute to these proceedings. Finally, I thank Dave Benson for some very helpful proofreading.

I. Hopf algebras and quasitriangular Hopf algebras

Let k be a field. Unless otherwise specified all maps between vector spaces over k are assumed to be k-linear and, if V is a vector space over k, then $\mathrm{id}_V: V \to V$ denotes the identity map from V to V.

The proofs of most of the statements in this chapter can be found in [Mo]. The proof that the antipode is an antihomomorphism (2.1) is given in [Sw] 4.0.1. The statement of Theorem (5.3), giving the construction of the quantum double, is given explicitly in [D1] §13, and the proof can be found in [Ma] p. 287-289. A statement similar to Proposition (5.5) is in [Ta] Prop. 2.2.1 and the proof is similar to the proof given there.

1. SRMCwMFFs

(1.1) Definition of an algebra

An *algebra over* k is a vector space A over k with a *multiplication*

$$\begin{array}{rccc} m: & A\otimes A & \longrightarrow & A \\ & a\otimes b & \longmapsto & a\cdot b = ab \end{array}$$

and an *identity element* $1_A \in A$ such that

(a) m is *associative*, i.e. $(ab)c = a(bc)$, for all $a, b, c \in A$, and

(b) $1_A \cdot a = a \cdot 1_A = a$, for all $a \in A$.

Equivalently, an *algebra over* k is a vector space A over k with a *multiplication* $m: A \otimes A \to A$ and a *unit* $\iota: k \to A$ such that

(a) m is *associative*, i.e. $m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)$, and

(b) (*unit condition*) $m \circ (\iota \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes \iota) = \mathrm{id}_A$.

The relationship between the identity $1_A \in A$ and the unit $\iota: k \to A$ is $\iota(1) = 1_A$. If we are being precise we should denote an algebra over k by a triple (A, m, ι) or $(A, m, 1_A)$ but we shall usually be lazy and simply write A.

(1.2) Definition of a module

Let A be an algebra over k. An *A-module* is a vector space M over k with an A-action

$$\begin{array}{ccc} A\otimes M & \longrightarrow & M \\ a\otimes m & \longmapsto & a\cdot m = am \end{array}$$

such that

(a) $(ab)m = a(bm)$, for all $a, b \in A$ and $m \in M$, and

(b) $1_A m = m$, for all $m \in M$.

Let M and N be A-modules. An *A-module morphism from M to N* is a map $\varphi: M \to N$ such that

$$\varphi(am) = a\varphi(m), \quad \text{for all } a \in A \text{ and } m \in M.$$

The set of A-module morphisms from M to N is denoted $\mathrm{Hom}_A(M, N)$. An A-module is *finite dimensional* if it is finite dimensional as a vector space over k.

(1.3) Motivation for SRMCwMFFs

Our interest will be in special algebras for which the category of finite dimensional A-modules has a lot of nice structure. We want to be able to take the tensor product of two A-modules and get a new A-module, we want to be able to take the dual of an A-module and get a new A-module and we want to have a 1-dimensional "trivial" A-module.

(1.4) Definition of SRMCwMFFs

Let A be an algebra over k. The category of finite dimensional A-modules is a *strict rigid monoidal category such that the forgetful functor is monoidal* (a SRMCwMFF for short) if

(a) For every pair M, N of finite dimensional A-modules there is a given A-module structure on $M \otimes N$,

(b) For every finite dimensional A-module M there is a given A-module structure on $M^* = \mathrm{Hom}_k(M, k)$,

(c) There is a distinguished one-dimensional A-module $\mathbf{1}$ with a distinguished basis element $1 \in \mathbf{1}$,

and the following conditions are satisfied:

(1) For all finite dimensional A-modules M, N, and P,

$$(M \otimes N) \otimes P = M \otimes (N \otimes P)$$

as A-modules*.

(2) The maps

$$\begin{array}{ccc} \mathbf{1} \otimes M & \xrightarrow{\sim} & M \\ 1 \otimes m & \longmapsto & m \end{array} \quad \text{and} \quad \begin{array}{ccc} M \otimes \mathbf{1} & \xrightarrow{\sim} & M \\ m \otimes 1 & \longmapsto & m \end{array}$$

are A-module isomorphisms.

(3) For each finite dimensional A-module M, the maps

$$\begin{array}{ccc} M^* \otimes M & \xrightarrow{\sim} & \mathbf{1} \\ \varphi \otimes m & \longmapsto & \varphi(m) \cdot 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{1} & \xrightarrow{\sim} & M \otimes M^* \\ 1 & \longmapsto & \sum_i m_i \otimes \varphi_i \end{array}$$

are A-module morphisms.

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* Strictly speaking we can only identify $(M \otimes N) \otimes P$ and $M \otimes (N \otimes P)$ up to coherent natural isomorphisms. If we are being precise this is crucial, but conceptually these two spaces are "equal".

In condition (3) the set $\{m_i\}$ is a basis of M and the set $\{\varphi_i\}$ is the dual basis in M^*, i.e. $\varphi_i \in M^*$ is such that $\varphi_i(m_j) = \delta_{ij}$ for all i, j.

The distinguished one-dimensional A-module $\mathbf{1}$ is called the *trivial A* module.

2. Hopf algebras

(2.1) Definition of Hopf algebras

A *Hopf algebra* is a vector space A over k with

a multiplication,	$m: A \otimes A \longrightarrow A$,
a comultiplication,	$\Delta: A \longrightarrow A \otimes A$,
a unit,	$\iota: k \longrightarrow A$,
a counit,	$\epsilon: A \longrightarrow k$, and
an antipode,	$S: A \longrightarrow A$,

such that

(1) m is *associative*,

$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A),$$

(2) Δ is *coassociative*,

$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta,$$

(3) (unit condition),

$$m \circ (\mathrm{id}_A \otimes \iota) = m \circ (\iota \otimes \mathrm{id}_A) = \mathrm{id}_A,$$

(4) (counit condition),

$$(\mathrm{id}_A \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A,$$

(5) Δ is an algebra homomorphism,

$$\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta),$$

(6) ϵ is an algebra homomorphism,

$$\epsilon \circ m = \epsilon \otimes \epsilon,$$

(7) (antipode condition),

$$m \circ (\mathrm{id}_A \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}_A) \circ \Delta = \iota \circ \epsilon.$$

In condition (5) the algebra structure on $A \otimes A$ is given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd, \quad \text{for all } a, b, c, d \in A,$$

and the map τ is given by

$$\begin{array}{cccc} \tau\colon & A \otimes A & \longrightarrow & A \otimes A \\ & a \otimes b & \longmapsto & b \otimes a. \end{array}$$

In condition (6) we have identified the vector space $k \otimes k$ with k. One can show that the antipode $S\colon A \to A$ is always an anti-homomorphism,

$$S(ab) = S(b)S(a), \quad \text{for all } a, b \in A.$$

(2.2) Sweedler notation for the comultiplication

Let A be a Hopf algebra over k. If $a \in A$ we write

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$$

to express $\Delta(a)$ as an element of $A \otimes A$. This unusual notation is called Sweedler notation and is a standard notation for working with Hopf algebras. Don't let it bother you, we are simply trying to write $\Delta(a)$ so that it looks like an element of $A \otimes A$, without having to go through the rigmarole of actually choosing a basis in A.

(2.3) Hopf algebras give us SRMCwMFFs!

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a Hopf algebra over k.

(a) If M_1 and M_2 are A-modules define an A-module structure on $M_1 \otimes M_2$ by

$$a(m_1 \otimes m_2) = \Delta(a)(m_1 \otimes m_2) = \sum_a a_{(1)}m_1 \otimes a_{(2)}m_2\ ,$$

for each $a \in A$, $m_1 \in M_1$, and $m_2 \in M_2$.

(b) Define $\mathbf{1}$ to be the vector space $\mathbf{1} = k \cdot 1$ and define an action of A on $\mathbf{1}$ by

$$a \cdot 1 \;=\; \epsilon(a) \cdot 1, \quad \text{for each } a \in A.$$

(c) If M is a finite dimensional A-module define an A-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(a\varphi)(m) = \varphi(S(a)m), \quad \text{for each } a \in A,\ \varphi \in M^*, \text{ and } m \in M.$$

The point is that if A is a Hopf algebra then, with the definitions in (a)-(c) above, the category of finite dimensional A-modules is very nice; it is a strict rigid monoidal category such that the forgetful functor is monoidal.

(2.4) Group algebras are Hopf algebras

Let G be a group. The *group algebra of G over k* is the vector space kG of finite k-linear combinations of elements of G,

$$kG = \left\{ \sum_g c_g g \mid c_g \in k \text{ and all but a finite number of } c_g = 0 \right\} ,$$

with multiplication given by the k-linear extension of the multiplication in G. A *G-module* is a kG-module.

(a) If M_1 and M_2 are G-modules define a G-module structure on $M_1 \otimes M_2$ by

$$g(m_1 \otimes m_2) = gm_1 \otimes gm_2, \quad \text{for all } g \in G,\ m_1 \in M_1, \text{ and } m_2 \in M_2.$$

(b) The *trivial* G-module is the 1-dimensional vector space $\mathbf{1}$ with G-action given by

$$g \cdot v = v, \quad \text{for all } g \in G,\ v \in \mathbf{1}.$$

(c) If M is a finite dimensional G-module define a G-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(g\varphi)(m) = \varphi(g^{-1}m) , \quad \text{for all } g \in G,\ m \in M, \text{ and } \varphi \in M^*.$$

With these definitions the category of finite dimensional G-modules is a strict monoidal category such that the forgetful functor is monoidal.

The group algebra kG is a Hopf algebra if we define

(a) a comultiplication, $\Delta: kG \to kG \otimes kG$, by

$$\Delta(g) = g \otimes g, \quad \text{for all } g \in G,$$

(b) a counit, $\epsilon: kG \to k$, by

$$\epsilon(g) = 1, \quad \text{for all } g \in G,$$

(c) and an antipode, $S: kG \to kG$, by

$$S(g) = g^{-1}, \quad \text{for all } g \in G.$$

(2.5) Enveloping algebras of Lie algebras are Hopf algebras

Let $\mathfrak{g}$ be a Lie algebra over k and let $\mathfrak{U}\mathfrak{g}$ be its enveloping algebra. (See II (1.1) and II (4.2) for definitions of Lie algebras and enveloping algebras.)

(a) If M_1 and M_2 are $\mathfrak{g}$-modules we define a $\mathfrak{g}$-module structure on $M_1 \otimes M_2$ by

$$x(m_1 \otimes m_2) = xm_1 \otimes m_2 + m_1 \otimes xm_2, \quad \text{for all } x \in \mathfrak{g},\ m_1 \in M_1, \text{ and } m_2 \in M_2.$$

(b) The *trivial* $\mathfrak{g}$-module is the 1-dimensional vector space $\mathbf{1}$ with $\mathfrak{g}$-action given by

$$xv = 0\ , \quad \text{for all } x \in \mathfrak{g},\ v \in \mathbf{1}.$$

(c) If M is a finite dimensional $\mathfrak{g}$-module we define a $\mathfrak{g}$-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(x\varphi)(m) = \varphi(-xm), \quad \text{for all } x \in \mathfrak{g},\ \varphi \in M^*, \text{ and } m \in M.$$

With these definitions the category of finite dimensional $\mathfrak{g}$-modules is a strict rigid monoidal category such that the forgetful functor is monoidal.

The enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ is a Hopf algebra if we define

(a) a comultiplication, $\Delta\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$, by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \text{for all } x \in \mathfrak{g},$$

(b) a counit, $\epsilon\colon \mathfrak{U}\mathfrak{g} \to k$, by

$$\epsilon(x) = 0, \quad \text{for all } x \in \mathfrak{g},$$

(c) and an antipode, $S\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g}$, by

$$S(x) = -x, \quad \text{for all } x \in \mathfrak{g}.$$

(2.6) Definition of the adjoint action of a Hopf algebra on itself

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a Hopf algebra. The vector space A is an A-module where the action of A on A is given by

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ a \otimes b & \longmapsto & \sum_a a_{(1)} b S(a_{(2)}) \end{array}, \quad \text{where} \quad \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}.$$

The linear transformation of A determined by the action of an element $a \in A$ is denoted ad_a. Thus,

$$\mathrm{ad}_a(b) = \sum_a a_{(1)} b S(a_{(2)}), \quad \text{for all } b \in A.$$

(2.7) Motivation for the definition of the adjoint action

Let M be an A-module and let $\rho: A \to \mathrm{End}(M)$ be the corresponding representation of A, i.e. the map

$$\begin{array}{rccc} \rho: & A & \longrightarrow & \mathrm{End}(M) \\ & a & \longmapsto & \rho(a) \end{array}$$

where $\rho(a)$ is the linear transformation of M determined by the action of a. Note that $\mathrm{End}(M) \cong M \otimes M^*$ as a vector space. On the other hand $M \otimes M^*$ is an A-module. If we view A as an A-module under the adjoint action then the composite map

$$\rho: A \to \mathrm{End}(M) \cong M \otimes M^*$$

is a homomorphism of A-modules.

(2.8) Definition of an ad-invariant bilinear form on a Hopf algebra

Let A be a Hopf algebra with antipode S and let M be an A-module. A bilinear form

$$\begin{array}{rccc} (,): & M \otimes M & \to & k \\ & m \otimes n & \mapsto & (m,n) \end{array} \quad \text{is } \textit{invariant} \text{ if} \quad (am_1, m_2) = (m_1, S(a)m_2),$$

for all $a \in A$, $m_1, m_2 \in M$. This is equivalent to the condition that the map $(,)$ is a homomorphism of A-modules when we identify k with the trivial A-module $\mathbf{1}$..

A bilinear form

$$(,): A \otimes A \to k \quad \text{is } \textit{ad-invariant} \text{ if} \quad (\mathrm{ad}_a(b_1), b_2) = (b_1, \mathrm{ad}_{S(a)}(b_2)),$$

for all $a, b_1, b_2 \in A$. In other words, the bilinear form is invariant if we view A as an A-module via the adjoint action.

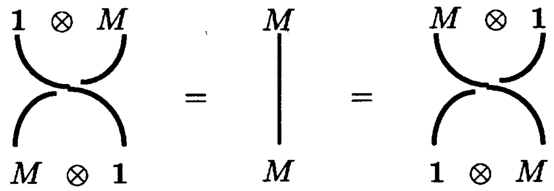
3. Braided SRMCwMFFs

(3.1) Motivation for braided SRMCwMFFs

Our interest here will be in even more special algebras for which the category of finite dimensional A-modules is "braided". Specifically, we want the two tensor product modules $M \otimes N$ and $N \otimes M$ to be isomorphic.

(3.2) Definition of braided SRMCwMFFs

Let A be an algebra over k. The category of finite dimensional A-modules is a *braided strict rigid monoidal category such that the forgetful functor is monoidal* (a braided SRMCwMFF for short) if it is a strict rigid monoidal category such that the forgetful functor is monoidal and

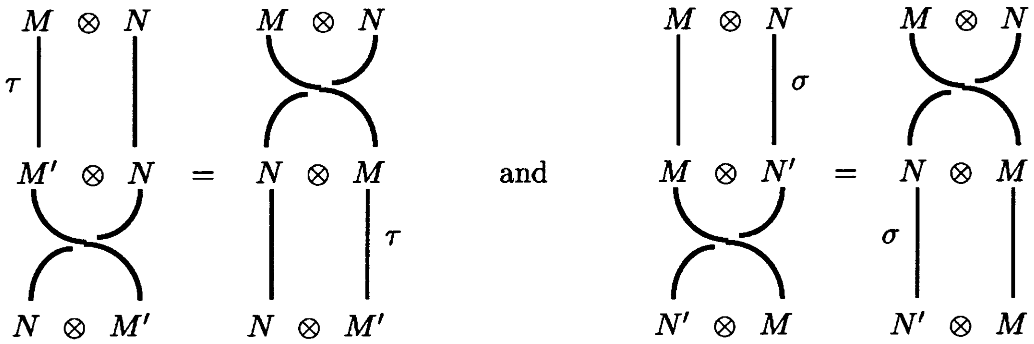


(3.4) What "natural isomorphism" means

Let M, M', N, N' be A-modules and let $\tau: M \to M'$ and $\sigma: N \to N'$ be A-module isomorphisms. Then the naturality condition on the isomorphisms $\check{R}_{M,N}$ means that the following diagrams commute.

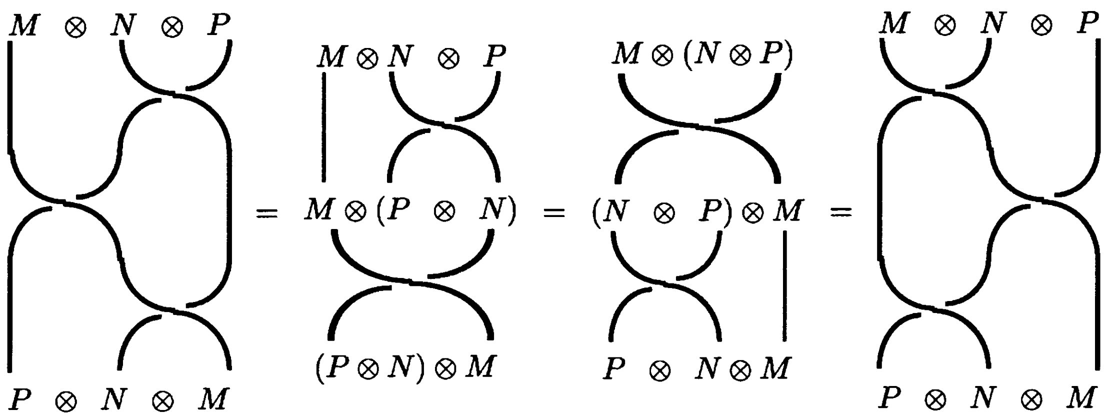
$$\begin{array}{ccc} M\otimes N & \xrightarrow{\tau\otimes \mathrm{id}_N} & M'\otimes N \\ {\scriptstyle \check{R}_{M,N}}\big\downarrow & & \big\downarrow{\scriptstyle \check{R}_{M',N}} \\ N\otimes M & \xrightarrow{\mathrm{id}_N\otimes\tau} & N\otimes M' \end{array} \qquad \begin{array}{ccc} M\otimes N & \xrightarrow{\mathrm{id}_M\otimes \sigma} & M\otimes N' \\ {\scriptstyle \check{R}_{M,N}}\big\downarrow & & \big\downarrow{\scriptstyle \check{R}_{M,N'}} \\ N\otimes M & \xrightarrow{\sigma\otimes\mathrm{id}_M} & N'\otimes M \end{array}$$

Pictorially we have



(3.5) The braid relation

The relations in (3.3) imply the following relation which is usually called the *braid relation.*



where the middle equality is a consequence of the naturality property and the fact that the map $\check{R}_{N,P}$ is an isomorphism.

4. Quasitriangular Hopf algebras

(4.1) Motivation for quasitriangular Hopf algebras

In addition to the definition of a braided SRMCwMFF the following observations help to motivate the definition of a quasitriangular Hopf algebra.

Let $(A, m, \Delta, \epsilon, \iota, S)$ be a Hopf algebra and let τ be the k-linear map

$$\begin{array}{rccc} \tau\colon & A \otimes A & \longrightarrow & A \otimes A \\ & a \otimes b & \longmapsto & b \otimes a. \end{array}$$

Let $\Delta^{\mathrm{op}} = \tau \circ \Delta$ so that, if $a \in A$ and

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then} \quad \Delta^{\mathrm{op}}(a) = \sum_a a_{(2)} \otimes a_{(1)}.$$

Then $(A, m, \Delta^{\mathrm{op}}, \iota, \epsilon, S^{-1})$ is a Hopf algebra.

The map $\tau : A \otimes A \to A \otimes A$ is an algebra automorphism of $A \otimes A$ (the algebra structure on $A \otimes A$ is as given in (2.1)) and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \mathrm{id}_A}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ A & \xrightarrow{\Delta^{\mathrm{op}}} & A \otimes A \end{array}$$

Sometimes we are lucky and we can replace τ by an *inner* automorphism.

(4.2) Definition of quasitriangular Hopf algebras

A *quasitriangular Hopf algebra* is a pair $(A, \mathcal{R})$ where A is a Hopf algebra and $\mathcal{R}$ is an invertible element of $A \otimes A$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \text{for all } a \in A, \text{ and}$$

$$(\Delta \otimes \mathrm{id}_A)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (\mathrm{id}_A \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where, if $\mathcal{R} = \sum a_i \otimes b_i$, then

$$\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1, \quad \mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \quad \text{and} \quad \mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i.$$

(4.3) Quasitriangular Hopf algebras give braided SRMCwMFFs

Let $(A, \mathcal{R})$ be a quasitriangular Hopf algebra. For each pair of finite dimensional A-modules M, N define

$$\begin{array}{rccc} \check{R}_{M,N}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum b_i n \otimes a_i m, \end{array}$$

where $\mathcal{R} = \sum a_i \otimes b_i \in A \otimes A$. Then the category of finite dimensional A-modules is a braided strict rigid monoidal category such that the forgetful functor is monoidal.

5. The quantum double

(5.1) Motivation for the quantum double

In general it can be very difficult to find quasitriangular Hopf algebras, especially ones where the element $\mathcal{R}$ is different from $1 \otimes 1$. The construction in (5.3) says that, given a Hopf algebra A, we can sort of paste it and its dual A^* together to get a quasitriangular Hopf algebra $D(A)$ and that the $\mathcal{R}$ for this new quasitriangular Hopf algebra is both a natural one and is nontrivial.

(5.2) Construction of the Hopf algebra $A^{*\mathrm{coop}}$

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a finite dimensional Hopf algebra over k. Let $A^* = \mathrm{Hom}_k(A, k)$ be the dual of A. There is a natural bilinear pairing $\langle , \rangle\colon A^* \otimes A \longrightarrow k$ between A and A^* given by

$$\langle \alpha, a \rangle = \alpha(a), \quad \text{for all } \alpha \in A^* \text{ and } a \in A.$$

Extend this notation so that if $\alpha_1, \alpha_2 \in A^*$ and $a_1, a_2 \in A$ then

$$\langle \alpha_1 \otimes \alpha_2, a_1 \otimes a_2 \rangle = \langle \alpha_1, a_1 \rangle \langle \alpha_2, a_2 \rangle.$$

We make A^* into a Hopf algebra, which is denoted $A^{*\mathrm{coop}}$, by defining a multiplication and a comultiplication Δ on A^* via the equations

$$\langle \alpha_1 \alpha_2, a \rangle = \langle \alpha_1 \otimes \alpha_2, \Delta(a) \rangle \quad \text{and} \quad \langle \Delta^{\mathrm{op}}(\alpha), a_1 \otimes a_2 \rangle = \langle \alpha, a_1 a_2 \rangle,$$

for all $\alpha, \alpha_1, \alpha_2 \in A^*$ and $a, a_1, a_2 \in A$. The definition of Δ^{op} is in (4.1).

(a) The identity in $A^{*\mathrm{coop}}$ is the counit $\epsilon\colon A \to k$ of A.

(b) The counit of $A^{*\mathrm{coop}}$ is the map

$$\begin{array}{rccc} \epsilon\colon & A^* & \to & k \\ & \alpha & \mapsto & \alpha(1), \end{array}$$

where 1 is the identity in A.

(c) The antipode of $A^{*\mathrm{coop}}$ is given by the identity $\langle S(\alpha), a \rangle = \langle \alpha, S^{-1}(a) \rangle$, for all $\alpha \in A^*$ and all $a \in A$.

(5.3) Construction of the quantum double

We want to paste the algebras A and $A^{*\mathrm{coop}}$ together in order to make a quasitriangular Hopf algebra $D(A)$. There are three main steps.

(1) We paste A and $A^{*\mathrm{coop}}$ together by letting

$$D(A) = A \otimes A^{*\mathrm{coop}}.$$

Write elements of $D(A)$ as $a\alpha$ instead of as $a \otimes \alpha$.

(2) We want the multiplication in $D(A)$ to reflect the multiplication in A and the multiplication in $A^{*\mathrm{coop}}$. Similarly for the comultiplication.

(3) We want the $\mathcal{R}$-matrix to be

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is the dual basis in A^*.

The condition in (2) determines the comultiplication in $D(A)$,

$$\Delta(\alpha a) = \Delta(\alpha)\Delta(a) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)},$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}$. The condition in (2) doesn't quite determine the multiplication in $D(A)$. We need to be able to expand products like $(a_1\alpha_1)(a_2\alpha_2)$. If we knew

$$\alpha_1 a_2 = \sum_j b_j\beta_j, \quad \text{for some elements } \beta_j \in A^{*\mathrm{coop}} \text{ and } b_j \in A,$$

then we would have

$$(a_1\alpha_1)(a_2\alpha_2) = \sum_j (a_1 b_j)(\beta_j\alpha_2)$$

which is a well defined element of $D(A)$. Miraculously, the condition in (3) and the equation

$$\mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a), \quad \text{for all } a \in A,$$

force that if $\alpha \in A^{*\mathrm{coop}}$ and $a \in A$ then, in $D(A)$,

$$\alpha a = \sum_{\alpha,a} \langle \alpha_{(1)}, S^{-1}(a_{(1)})\rangle \langle \alpha_{(3)}, a_{(3)}\rangle a_{(2)}\alpha_{(2)}, \quad \text{and}$$

$$a\alpha = \sum_{\alpha,a} \langle \alpha_{(1)}, a_{(1)}\rangle \langle \alpha_{(3)}, S^{-1}(a_{(3)})\rangle \alpha_{(2)}a_{(2)},$$

where, if Δ is the comultiplication in $D(A)$,

$$(\Delta\otimes\mathrm{id})\circ\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}\otimes a_{(3)},\quad \text{and}\quad (\Delta\otimes\mathrm{id})\circ\Delta(\alpha)=\sum_\alpha \alpha_{(1)}\otimes\alpha_{(2)}\otimes\alpha_{(3)}.$$

These relations completely determine the multiplication in $D(A)$. This construction is summarized in the following theorem.

Theorem. *Let A be a finite dimensional Hopf algebra over k and let $A^{*\mathrm{coop}}$ be the Hopf algebra $A^*=\mathrm{Hom}_k(A,k)$ except with opposite comultiplication. Then there exists a unique quasitriangular Hopf algebra $(D(A),\mathcal{R})$ given by*

(1) The k-linear map

$$\begin{array}{ccc} A\otimes A^{*\mathrm{coop}} & \longrightarrow & D(A) \\ a\otimes\alpha & \longmapsto & a\alpha \end{array}$$

is bijective.

(2) $D(A)$ contains A and $A^{*\mathrm{coop}}$ as Hopf subalgebras.

(3) The element $\mathcal{R}\in D(A)\otimes D(A)$ is given by

$$\mathcal{R}=\sum_i b_i\otimes b^i\,,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is dual basis in $A^{*\mathrm{coop}}$.

In condition (2), A is identified with the image of $A\otimes 1$ under the map in (1) and $A^{*\mathrm{coop}}$ is identified with the image of $1\otimes A^{*\mathrm{coop}}$ under the map in (1).

(5.4) If A is an infinite dimensional Hopf algebra

It is sometimes possible to do an analogous construction when A is infinite dimensional if one is careful about what the dual of A is and how to express the (now infinite) sum $\mathcal{R}=\sum_i b_i\otimes b^i$. To get an idea of how this is done see VII (7.1) and [Lu] Chapt. 4.

(5.5) An ad-invariant pairing on the quantum double

Proposition. *Let $(A,m,\Delta,\iota,\epsilon,S)$ be a Hopf algebra. The bilinear form on the quantum double $D(A)$ of A which is defined by*

$$\langle a\alpha,b\beta\rangle=\langle\beta,S(a)\rangle\langle\alpha,S^{-1}(b)\rangle,\quad \textit{for all } a,b\in A \textit{ and all } \alpha,\beta\in A^{*\mathrm{coop}},$$

satisfies

$$\langle ad_u(x),y\rangle=\langle x,\mathrm{ad}_{S(u)}(y)\rangle,\qquad \textit{for all } u,x,y\in D(A).$$

The proposition says that the bilinear form is ad-invariant, as defined in (2.8). This bilinear form is *not* necessarily symmetric,

$$\langle y,x\rangle=\langle x,S^2(y)\rangle,\quad \text{for all } x,y\in D(A).$$

An ideal $\mathfrak{a}$ of $\mathfrak{g}$ is *solvable* if there exists a positive integer n such that $D^n\mathfrak{a} = 0$. The *radical* of $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$. A finite dimensional Lie algebra is *semisimple* if its radical is 0.

(1.4) Definition of simple modules for a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over a field k. A $\mathfrak{g}$*-module* is a vector space V over k with a $\mathfrak{g}$-action

$$\begin{array}{ccc} \mathfrak{g}\otimes V & \longrightarrow & V \\ x\otimes v & \longmapsto & x\cdot v = xv \end{array}$$

such that

$$[x,y]\cdot v = x(yv) - y(xv), \quad \text{for all } x,y\in\mathfrak{g}, \text{ and } v\in V.$$

A *representation* of $\mathfrak{g}$ on a vector space V is a map

$$\begin{array}{rccc} \rho: & \mathfrak{g} & \longrightarrow & \mathrm{End}(V) \\ & x & \longmapsto & \rho(x) \end{array} \quad \text{such that} \quad \rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x),$$

for all $x,y\in\mathfrak{g}$. Every $\mathfrak{g}$-module V determines a representation of $\mathfrak{g}$ on V (and vice versa) by the formula

$$\rho(x)v = xv, \quad \text{for all } x\in\mathfrak{g}, \text{ and } v\in V.$$

A *submodule* of a $\mathfrak{g}$-module V is subspace $W\subseteq V$ such that $xw\in W$ for all $x\in\mathfrak{g}$ and $w\in W$. A *simple* or *irreducible* $\mathfrak{g}$*-module* is a $\mathfrak{g}$-module V such that the only submodules of V are 0 and V. A $\mathfrak{g}$-module V is *completely decomposable* if V is a direct sum of simple submodules.

(1.5) Definition of the adjoint representation of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k. The vector space $\mathfrak{g}$ is a $\mathfrak{g}$-module where the action of $\mathfrak{g}$ on $\mathfrak{g}$ is given by

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ x\otimes y & \longmapsto & [x,y]. \end{array}$$

The linear transformation of $\mathfrak{g}$ determined by the action of an element $x\in\mathfrak{g}$ is denoted ad_x. Thus,

$$\mathrm{ad}_x(y) = [x,y], \quad \text{for all } y\in\mathfrak{g}.$$

The representation

$$\begin{array}{rccc} \mathrm{ad}: & \mathfrak{g} & \longrightarrow & \mathrm{End}(\mathfrak{g}) \\ & x & \longmapsto & \mathrm{ad}_x \end{array}$$

is the *adjoint representation* of $\mathfrak{g}$.

(1.6) Definition of the Killing form

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k. The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $\langle,\rangle : \mathfrak{g} \times \mathfrak{g} \to k$ given by

$$\langle x, y\rangle = Tr(\mathrm{ad}_x \mathrm{ad}_y), \quad \text{for all } x, y \in \mathfrak{g}.$$

The Killing form $\langle,\rangle$ is *invariant*, i.e.

$$\langle [x,y], z\rangle + \langle y, [x,z]\rangle = 0, \quad \text{for all } x, y, z \in \mathfrak{g}.$$

(1.7) Characterizations of semisimple Lie algebras

Theorem. *A finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0 is semisimple if any of the following equivalent conditions holds:*

(1) $\mathfrak{g}$ is a direct sum of simple Lie subalgebras.
(2) The radical of $\mathfrak{g}$ is 0.
(3) Every finite dimensional $\mathfrak{g}$ module is completely decomposable and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
(4) The Killing form on $\mathfrak{g}$ is non-degenerate.

2. Finite dimensional complex simple Lie algebras

(2.1) Dynkin diagrams and Cartan matrices

A *Dynkin diagram* is one of the graphs in Table 1. A *Cartan matrix* is one of the matrices in Table 2. The (i,j) entry of a Cartan matrix is denoted $\alpha_j(H_i)$. Notice that every Cartan matrix satisfies the conditions,

(1) $\alpha_i(H_i) = 2$, for all $1 \le i \le r$,
(2) $\alpha_j(H_i)$ is a non positive integer, for all $i \ne j$,
(3) $\alpha_i(H_j) = 0$ if and only if $\alpha_j(H_i) = 0$.

If C is a Cartan matrix the vertices of the corresponding Dynkin diagram are labeled by $\alpha_i, 1 \le i \le r$, such that $\alpha_i(H_j)\alpha_j(H_i)$ is the number of lines connecting vertex α_i to vertex α_j. If $\alpha_j(H_i) > \alpha_i(H_j)$ then there is a $>$ sign on the edge connecting vertex α_j to vertex α_i, with the point towards α_i. With these conventions it is clear that the Cartan matrix contains exactly the same information as the Dynkin diagram; each can be constructed from the other.

(2.2) Classification of finite dimensional complex simple Lie algebras

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1\le i,j\le r}$. Let $\mathfrak{g}_C$ be the Lie algebra over $\mathbb{C}$ given by generators

$$X_1^-, X_2^-, \ldots, X_r^-, \quad H_1, H_2, \ldots, H_r, \quad X_1^+, X_2^+, \ldots, X_r^+,$$

and relations

$$
\begin{aligned}
&[H_i, H_j] = 0, && \text{for all } 1 \le i, j \le r, \\
&[H_i, X_j^+] = \alpha_j(H_i) X_j^+, && \\
&[H_i, X_j^-] = -\alpha_j(H_i) X_j^-, && \text{for all } 1 \le i, j \le r, \\
&[X_i^+, X_j^-] = \delta_{ij} H_i, && \text{for } 1 \le i, j \le r, \\
&\underbrace{[X_i^+, [X_i^+, \cdots [X_i^+}_{-\alpha_j(H_i)+1 \text{ brackets}}, X_j^+]] \cdots] = 0, && \\
&\underbrace{[X_i^-, [X_i^-, \cdots [X_i^-}_{-\alpha_j(H_i)+1 \text{ brackets}}, X_j^-]] \cdots] = 0, && \text{for } i \ne j.
\end{aligned}
$$

Theorem. *Let C be a Cartan matrix and let $\mathfrak{g}_C$ be the Lie algebra defined above.*

(1) The Lie algebra $\mathfrak{g}_C$ is a finite dimensional complex simple Lie algebra.

(2) Every finite dimensional complex simple Lie algebra is isomorphic to $\mathfrak{g}_C$ for some Cartan matrix C.

(3) If C, C' are Cartan matrices then

$$\mathfrak{g}_C \simeq \mathfrak{g}_{C'} \quad \text{if and only if} \quad C = C'.$$

(2.3) Triangular decomposition

Fix a Cartan matrix $C = (\alpha_i(H_j))_{1 \le i,j \le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. Define

$$
\begin{aligned}
\mathfrak{n}^- &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^-. \\
\mathfrak{h} &= \mathbb{C}\text{-span } \{H_1, H_2, \ldots, H_r\}, \\
\mathfrak{n}^+ &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+,
\end{aligned}
$$

The elements $X_1^-, X_2^- \ldots, X_r^-, H_1, \ldots, H_r, X_1^+, X_2^+, \ldots, X_r^+$ are linearly independent in $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

The Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* of $\mathfrak{g}$ and the Lie subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a *Borel subalgebra* of $\mathfrak{g}$. The *rank* of $\mathfrak{g}$ is $r = \dim \mathfrak{h}$.

(2.4) Weights and weight spaces

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1\le i,j\le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. Let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ and define the *fundamental weights* $\omega_1, \ldots, \omega_r \in \mathfrak{h}^*$ by

$$\omega_i(H_j) = \delta_{ij}, \quad \text{for } 1 \le i, j \le r.$$

Let V be a $\mathfrak{g}$-module and let $\mu = \sum_{i=1}^r \mu_i\omega_i \in \mathfrak{h}^*$. The subspace

$$\begin{aligned} V_\mu &= \{v \in V \mid hv = \mu(h)v, \quad \text{for} \quad h \in \mathfrak{h}\} \\ &= \{v \in V \mid H_i v = \mu_i v, \quad \text{for} \quad 1 \le i \le r\} \end{aligned}$$

is the *μ-weight space* of V. Vectors $v \in V_\mu$ are *weight vectors* of V of *weight* μ, $\mathrm{wt}(v) = \mu$. The *weights* of the $\mathfrak{g}$-module V are the elements $\mu \in \mathfrak{h}^*$ such that $V_\mu \ne 0$. If μ is a weight of V, the *multiplicity* of μ in V is $\dim(V_\mu)$. A *highest weight vector* in a $\mathfrak{g}$-module V is a weight vector $v \in V$ such that $\mathfrak{n}^+ v = 0$ or, equivalently, a weight vector $v \in V$ such that $X_i^+ v = 0$, for $1 \le i \le r$.

The set of *dominant integral weights* P^+ and the *weight lattice* P are the subsets of $\mathfrak{h}^*$ given by

$$P^+ = \sum_{i=1}^r \mathbb{N}\omega_i \quad \text{and} \quad P = \sum_{i=1}^r \mathbb{Z}\omega_i, \quad \text{respectively,}$$

where $\mathbb{N} = \mathbb{Z}_{\ge 0}$.

(2.5) Classification of simple $\mathfrak{g}$-modules

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Every finite dimensional $\mathfrak{g}$-module V is a direct sum of its weight spaces and all weights of V are elements of P,*

$$V = \bigoplus_{\mu\in P} V_\mu.$$

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra.*

(1) Every finite dimensional irreducible $\mathfrak{g}$-module V contains a unique, up to constant multiples, highest weight vector $v^+ \in V$ and $wt(v^+) \in P^+$.
(2) Conversely, if $\lambda \in P^+$, then there is a unique (up to isomorphism) finite dimensional irreducible $\mathfrak{g}$-module, V^λ, with highest weight vector of weight λ.

(2.6) Roots and the root lattice

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1\le i,j\le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ (see (1.5)) makes $\mathfrak{g}$ into a finite dimensional $\mathfrak{g}$-module. An element $\alpha \in P$, $\alpha \neq 0$ is a *root* if the weight space $\mathfrak{g}_\alpha \neq 0$. A root is *positive*, $\alpha > 0$, if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}^+$ and *negative*, $\alpha < 0$, if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}^-$. We have

$$\dim \mathfrak{g}_\alpha = 1 \quad \text{for all roots } \alpha,$$

$$\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha, \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The roots α_i, $1 \le i \le r$, given by $\mathfrak{g}_{\alpha_i} = \mathbb{C}X_i^+$ are the *simple roots*. The Cartan matrix is the transition matrix between the simple roots and the fundamental weights,

$$\alpha_i = \sum_{j=1}^r \alpha_i(H_j)\omega_j, \quad \text{for } 1 \le i \le r.$$

The *root lattice* is the lattice $Q \subseteq P \subseteq \mathfrak{h}^*$ given by $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$.

(2.7) The inner product on $\mathfrak{h}_{\mathbb{R}}^*$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. There exist unique positive integers $d_1, d_2, \cdots, d_r$ such that $gcd(d_1, \cdots, d_r) = 1$ and the matrix $(d_i\alpha_j(H_i))_{1\le i,j\le r}$ is symmetric. The integers $d_1, d_2, \cdots, d_r$ are given explicitly by

$$\begin{array}{ll}
A_r,\ D_r,\ E_6,\ E_7,\ E_8: & d_i = 1 \text{ for all } 1 \le i \le r, \\
B_r: & d_i = 1 \text{ for } 1 \le i \le r-1, \quad \text{and} \quad d_r = 2, \\
C_r: & d_i = 2, \text{ for } 1 \le i \le r-1, \quad \text{and} \quad d_r = 1, \\
F_4: & d_1 = d_2 = 1, \quad \text{and} \quad d_3 = d_4 = 2, \\
G_2: & d_1 = 3, \quad \text{and} \quad d_2 = 1.
\end{array}$$

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$. Define

$$\mathfrak{h}_{\mathbb{R}}^* = \sum_{i=1}^r \mathbb{R}\alpha_i,$$

so that $\mathfrak{h}_{\mathbb{R}}^*$ is a real vector space of dimension r. Define an symmetric inner product on $\mathfrak{h}_{\mathbb{R}}^*$ by

$$(\alpha_i, \alpha_j) = d_i\alpha_i(H_j), \qquad \text{for } \le i, j \le r,$$

where the values $\alpha_j(H_i)$ are the entries of the Cartan matrix corresponding to $\mathfrak{g}$.

(2.8) The Weyl group corresponding to $\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let R be the set of roots of $\mathfrak{g}$ and let $\alpha_1, \ldots, \alpha_r$ be the simple roots. For each root $\alpha \in R$ define a linear transformation of $\mathfrak{h}^*_{\mathbb{R}}$ by

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \text{where} \quad \alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

The Weyl group corresponding to $\mathfrak{g}$ is the group of linear transformations of $\mathfrak{h}^*_{\mathbb{R}}$ generated by the reflections s_α, $\alpha \in R$,

$$W = < s_\alpha \mid \alpha \in R > .$$

The *simple reflections* in W are the elements $s_i = s_{\alpha_i}$, $1 \le i \le r$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let W be the Weyl group corresponding to $\mathfrak{g}$.*

(a) The Weyl group W is a finite group.

(b) The Weyl group W can be presented by generators $s_1, \ldots, s_r$ and relations

$$\begin{aligned} s_i^2 &= 1, && 1 \le i \le r, \\ \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} && \text{for } i \ne j, \end{aligned}$$

where

$$m_{ij} = \begin{cases} 2, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 0, \\ 3, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 1, \\ 4, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 2, \\ 6, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 3. \end{cases}$$

Let $w \in W$. A *reduced decomposition for* w is an expression

$$w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$$

of w as a product of generators which is as short as possible. The length $\ell(w)$ of this expression is the *length of* w.

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let W be the Weyl group corresponding to $\mathfrak{g}$.*

(a) There is a unique longest element w_0 in W.

(b) Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition for the longest element of W. Then the elements

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N}),$$

are the positive roots of $\mathfrak{g}$.

3. Enveloping algebras

(3.1) Motivation for the enveloping algebra

A Lie algebra $\mathfrak{g}$ is not an algebra, at least as defined in I (1.1), because the bracket is not associative. We would like to find an algebra, or even better a Hopf algebra, $\mathfrak{U}\mathfrak{g}$, for which the category of modules for $\mathfrak{U}\mathfrak{g}$ is the same as the category of modules for $\mathfrak{g}$. In other words we want $\mathfrak{U}\mathfrak{g}$ to carry all the information that $\mathfrak{g}$ does and to be a Hopf algebra.

(3.2) Definition of the enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra over k. Let $T(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}$ be the tensor algebra of $\mathfrak{g}$ and let J be the ideal of $T(\mathfrak{g})$ generated by the tensors

$$x \otimes y - y \otimes x - [x,y], \quad \text{where} \quad x, y \in \mathfrak{g}.$$

The *enveloping algebra of* $\mathfrak{g}$, $\mathfrak{U}\mathfrak{g}$, is the associative algebra

$$\mathfrak{U}\mathfrak{g} = \frac{T(\mathfrak{g})}{J}.$$

There is a canonical map

$$\begin{array}{rccc} \alpha_0\colon & \mathfrak{g} & \longrightarrow & \mathfrak{U}\mathfrak{g} \\ & x & \longmapsto & x + J\,. \end{array}$$

The algebra $\mathfrak{U}\mathfrak{g}$ can be given by the following universal property:

> Let $\alpha : \mathfrak{g} \to A$ be a mapping of $\mathfrak{g}$ into an associative algebra A over k such that
>
> $$\alpha([x,y]) = \alpha(x)\alpha(y) - \alpha(y)\alpha(x),$$
>
> for all $x, y \in \mathfrak{g}$, and let 1 and 1_A denote the identities in $\mathfrak{U}\mathfrak{g}$ and A respectively. Then there exists a unique algebra homomorphism $\tau : \mathfrak{U}\mathfrak{g} \to A$ such that $\tau(1) = 1_A$ and $\alpha = \tau \circ \alpha_0$, i.e. the following diagram commutes.
>
> $$\begin{tikzcd} \mathfrak{g} \arrow[r, "\alpha_0"] \arrow[dr, "\alpha"'] & \mathfrak{U}\mathfrak{g} \arrow[d, "\tau"] \\ & A \end{tikzcd}$$

(3.3) A functorial way of realising the enveloping algebra

If A is an algebra over k, as defined in I (1.1), then define a bracket on A by

$$[x,y] = xy - yx, \quad \text{for all } x, y \in A.$$

This defines a Lie algebra structure on A and we denote the resulting Lie algebra by $L(A)$ to distinguish it from A. L is a functor from the category of algebras to the category of

Lie algebras. $\mathfrak{U}$ is a functor from the category of Lie algebras to the category of algebras. In fact $\mathfrak{U}$ is the left adjoint of the functor L since

$$\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{U}\mathfrak{g}, A) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, L(A))$$

for all Lie algebras $\mathfrak{g}$ and all algebras A.

(3.4) The enveloping algebra is a Hopf algebra

The enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ is a Hopf algebra if we define

(a) a comultiplication, $\Delta: \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$, by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \text{for all } x \in \mathfrak{g},$$

(b) a counit, $\epsilon: \mathfrak{U}\mathfrak{g} \to k$, by

$$\epsilon(x) = 0, \quad \text{for all } x \in \mathfrak{g},$$

(c) and an antipode, $S: \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g}$, by

$$S(x) = -x, \quad \text{for all } x \in \mathfrak{g}.$$

(3.5) Modules for the enveloping algebra and the Lie algebra are the same!

Every $\mathfrak{g}$-module M is a $\mathfrak{U}\mathfrak{g}$-module and vice versa, since there is a unique extension of the action of $\mathfrak{g}$ on M to a $\mathfrak{U}\mathfrak{g}$-action on M.

(3.6) The Lie algebra can be recovered from its enveloping algebra!

An element x of a Hopf algebra A is *primitive* if

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

It can be shown that if char $k = 0$ then the subspace $\mathfrak{g}$ of $\mathfrak{U}\mathfrak{g}$ is the set of primitive elements of $\mathfrak{U}\mathfrak{g}$. Thus, if char $k = 0$, we can "determine" the Lie algebra $\mathfrak{g}$ from the algebra $\mathfrak{U}\mathfrak{g}$ and the Hopf algebra structure on it.

(3.7) A basis for the enveloping algebra

The following statement is the *Poincaré-Birkhoff-Witt* theorem.

Suppose that $\mathfrak{g}$ has a totally ordered basis $(x_i)_{i\in\Lambda}$. Then the elements

$$x_{i_1} x_{i_2} \cdots x_{i_n}$$

in the enveloping algebra $\mathfrak{U}\mathfrak{g}$, where $i_1 \leq i_2 \leq \cdots \leq i_n$ is an arbitrary increasing finite sequence of elements of Λ, form a basis a $\mathfrak{U}\mathfrak{g}$.

5. The enveloping algebra of a complex simple Lie algebra

(5.1) A presentation by generators and relations
Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. Then the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ can be presented as the algebra over $\mathbb{C}$ generated by

$$X_1^-, X_2^-, \ldots, X_r^-, \quad H_1, H_2, \ldots, H_r, \quad X_1^+, X_2^+, \ldots, X_r^+,$$

with relations

$$[H_i, H_j] = 0, \qquad \text{for all } 1 \le i,j \le r,$$

$$[H_i, X_j^+] = \alpha_j(H_i)X_j^+,$$
$$\qquad \text{for all } 1 \le i,j \le r,$$
$$[H_i, X_j^-] = -\alpha_j(H_i)X_j^-,$$

$$[X_i^+, X_j^-] = \delta_{ij}H_i, \qquad \text{for } 1 \le i,j \le r,$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \binom{1-\alpha_j(H_i)}{s} (X_i^\pm)^s X_j^\pm (X_i^\pm)^t = 0, \qquad \text{for } i \ne j,$$

where, if $a, b \in \mathfrak{U}\mathfrak{g}$, we use the notation $[a,b] = ab - ba$. Note that since

$$\underbrace{[a,[a,\cdots[a,b]]}_{\ell \text{ brackets}}\cdots] = \sum_{s+t=\ell} (-1)^s \binom{\ell}{s} a^s b a^t,$$

for any two elements $a, b \in \mathfrak{U}\mathfrak{g}$ and any positive integer ℓ, the relations for $\mathfrak{U}\mathfrak{g}$ are exactly the same as the relations for $\mathfrak{g}$ given in (2.2).

(5.2) Triangular decomposition
Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2). Recall from (2.3) that $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where

$$\begin{aligned} \mathfrak{n}^- &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^-. \\ \mathfrak{h} &= \mathbb{C}\text{-span } \{H_1, H_2, \ldots, H_r\}, \\ \mathfrak{n}^+ &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+ \end{aligned}$$

It follows from this and the Poincaré-Birkhoff-Witt theorem that

$$\mathfrak{U}\mathfrak{g} \cong \mathfrak{U}\mathfrak{n}^- \otimes \mathfrak{U}\mathfrak{h} \otimes \mathfrak{U}\mathfrak{n}^+, \qquad \text{as vector spaces.}$$

(5.3) Grading on $\mathfrak{U}\mathfrak{n}^+$ and $\mathfrak{U}\mathfrak{n}^-$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2). Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$ and let

$$Q^+ = \sum_i \mathbb{N}\alpha_i, \qquad \text{where } \mathbb{N} = \mathbb{Z}_{\geq 0}.$$

For each element $\nu = \sum_{i=1}^r \nu_i \alpha_i \in Q^+$ define

$$(\mathfrak{U}\mathfrak{n}^+)_\nu = \text{span-}\{X_{i_1}^+ \cdots X_{i_p}^+ \mid X_{i_1}^+ \cdots X_{i_p}^+ \text{ has } \nu_j\text{-factors of type } X_j^+\}$$
$$(\mathfrak{U}\mathfrak{n}^-)_\nu = \text{span-}\{X_{i_1}^- \cdots X_{i_p}^- \mid X_{i_1}^- \cdots X_{i_p}^- \text{ has } \nu_j\text{-factors of type } X_j^-\}.$$

Then

$$\mathfrak{U}\mathfrak{n}^- = \bigoplus_{\nu \in Q^+} (\mathfrak{U}\mathfrak{n}^-)_\nu, \quad \text{and} \quad \mathfrak{U}\mathfrak{n}^+ = \bigoplus_{\nu \in Q^+} (\mathfrak{U}\mathfrak{n}^+)_\nu,$$

as vector spaces.

(5.4) Poincaré-Birkhoff-Witt bases of $\mathfrak{U}\mathfrak{n}^-$, $\mathfrak{U}\mathfrak{h}$, and $\mathfrak{U}\mathfrak{n}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2), let $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{h}$ be as in (2.3) and recall the root spaces $\mathfrak{g}_\alpha$ from (2.6). Let W be the Weyl group corresponding to $\mathfrak{g}$. Fix a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \cdots s_{i_N}$, and define

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \qquad \ldots, \qquad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The elements $\beta_1, \ldots, \beta_N$ are the positive roots $\mathfrak{g}$ and the elements $-\beta_1, \ldots, -\beta_N$ are the negative roots of $\mathfrak{g}$.

For each root α, fix an element $X_\alpha \in \mathfrak{g}_\alpha$.

Since $\mathfrak{g}_\alpha$ is 1-dimensional X_α is uniquely defined, up to multiplication by a constant. Since

$$\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha, \quad \mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \quad \mathfrak{h} = \text{span-}\{H_1, H_2, \ldots H_r\} \quad \text{and} \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

it follows that

$$\{X_{\beta_1}, \ldots, X_{\beta_N}\} \qquad \text{is a basis of } \mathfrak{n}^+,$$

$$\{X_{-\beta_1}, \ldots, X_{-\beta_N}\} \qquad \text{is a basis of } \mathfrak{n}^-, \text{ and}$$

$$\{H_1, H_2, \ldots, H_r\} \qquad \text{is a basis of } \mathfrak{h}.$$

Then, by the Poincaré-Birkhoff-Witt theorem,

$$\{X_{\beta_1}^{p_1} X_{\beta_2}^{p_2} \cdots X_{\beta_N}^{p_N} \mid p_1, \ldots, p_N \in \mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{n}^+,$$

$$\{X_{-\beta_1}^{n_N} X_{-\beta_2}^{n_2} \cdots X_{-\beta_1}^{n_1} \mid n_1, \ldots, n_N \in \mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{n}^-, \text{ and}$$

$$\{H_1^{s_1} H_2^{s_2} \cdots H_1^{s_r} \mid s_1, \ldots, s_N \in \mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{h}.$$

(5.5) The Casimir element in $\mathfrak{U}\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and let $\langle , \rangle$ be the Killing form on $\mathfrak{g}$ (see (1.6)). Let $\{b_i\}$ be a basis of $\mathfrak{g}$ and let $\{b^i\}$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form. Let c be the element of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ given by

$$c = \sum_i b_i b^i.$$

Then

$$c \text{ is in the center of } \mathfrak{U}\mathfrak{g}.$$

Any central element of $\mathfrak{U}\mathfrak{g}$ must act on each finite dimensional simple module by a constant. For each dominant integral weight λ let V^λ be the finite dimensional simple $\mathfrak{U}\mathfrak{g}$-module indexed by λ (see (2.5)). Let ρ be the element of $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\rho = \tfrac{1}{2} \sum_{\alpha > 0} \alpha,$$

where the sum is over all positive roots for $\mathfrak{g}$. Then the element

$$c \text{ acts on } V^\lambda \text{ by the constant } (\lambda + \rho, \lambda + \rho) - (\rho, \rho),$$

where inner product on $\mathfrak{h}_{\mathbb{R}}^*$ is as given in (2.7).

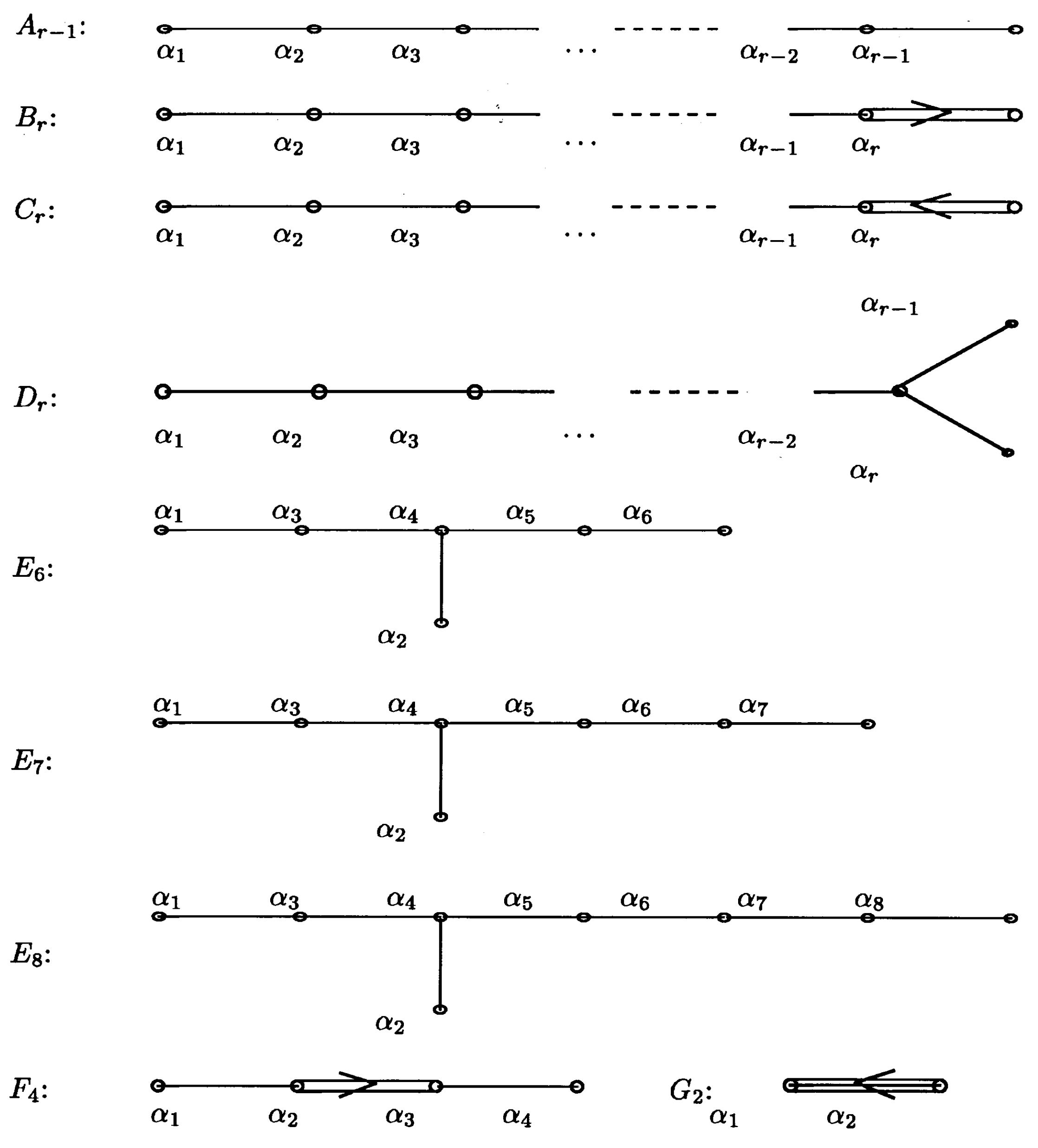


Table 1. Dynkin diagrams corresponding to finite dimensional complex simple Lie algebras

$$A_{r-1}: \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix} \qquad B_r: \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -2 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix}$$

$$C_r: \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -2 & 2 \end{pmatrix} \qquad D_r: \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad E_7: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4: \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad G_2: \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Table 2. Cartan matrices corresponding to finite dimensional complex simple Lie algebras

III. Deformations of Hopf algebras

The basic material on completions given in §1 can be found in many books, in particular, [AM] Chapt 10. The book [SS] has a comprehensive treatment of deformation theory. Theorem (2.6) is stated and proved in [SS] Prop. 11.3.1.

1. h-adic completions

(1.1) Motivation for h-adic completions

We will be working with algebras over $\mathbb{C}[[h]]$, the ring of formal power series in a variable h with coefficients in $\mathbb{C}$. A typical element of $\mathbb{C}[[h]]$ which is not in $\mathbb{C}[h]$ is the element

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots.$$

The ring $\mathbb{C}[[h]]$ is just $\mathbb{C}[h]$ extended a little bit so that some nice elements that we want to write down, like e^h, are in $\mathbb{C}[[h]]$.

An algebra over $\mathbb{C}[[h]]$ is a vector space over $\mathbb{C}[[h]]$, i.e. a free $\mathbb{C}[[h]]$-module, which has a multiplication and an identity which satisfy the conditions in I (1.1). If A is an algebra over $\mathbb{C}$ then we can extend coefficients and get a new algebra $A \otimes_{\mathbb{C}} \mathbb{C}[[h]]$ which is over $\mathbb{C}[[h]]$. But sometimes this new algebra is not quite big enough so we need to extend it a little bit and work with the h-adic completion $A[[h]]$ which contains all the nice elements that we want to write down.

Continuing in this vein we will want to consider the tensor product $A[[h]] \otimes A[[h]]$. Again, this algebra is not quite big enough and we extend it to get a slightly bigger object $A[[h]]\hat{\otimes}A[[h]]$ so that all the elements we want are available.

(1.2) The algebra $A[[h]]$, an example of an h-adic completion

If A is an algebra over k then the set

$$A[[h]] = \{a_0 + a_1 h + a_2 h^2 + \cdots \mid a_i \in A\}$$

of formal power series with coefficients in A is the completion of the $k[[h]]$-module $k[[h]]\otimes_k A$ in the h-adic topology. The $k[[h]]$-linear extension of the multiplication in A gives $A[[h]]$ the structure of a $k[[h]]$-algebra. The ring $A[[h]]$ is, in general, larger than $k[[h]] \otimes_k A$. For each element $a = \sum_{j\geq 0} a_j h^j \in A[[h]]$ the element

$$e^{ha} = \sum_{\ell\geq 0} \frac{(ha)^\ell}{\ell!} = 1 + a_0 h + (a_0^2 + 2a_1)\left(\frac{h^2}{2}\right) + (a_0^3 + 3(a_0a_1 + a_1a_0) + 6a_2)\left(\frac{h^3}{3!}\right) + \cdots$$

is a well defined element of $A[[h]]$.

(1.3) Definition of the h-adic topology

Let k be a field and let h be an indeterminate. The ring $k[[h]]$ is a local ring with unique maximal ideal (h). Let M be a $k[[h]]$-module. The sets

$$m + h^n M, \quad m \in M, n \in \mathbb{N},$$

form a basis for a topology on M called the *h-adic topology*. Define a map $d: M \times M \to \mathbb{R}$ by

$$d(x,y) = e^{-v(x-y)}, \quad \text{for all } x, y \in M,$$

where e is a real number $e > 1$ and $v(x)$ is the largest nonnegative integer n such that $x \in h^n M$. Then d is a metric on M which generates the h-adic topology.

(1.4) Definition of an h-adic completion

Let M be a $k[[h]]$-module. The completion of the metric space M is a metric space $\hat{M}$ which contains M in a natural way and which has a natural $k[[h]]$-module structure. The completion $\hat{M}$ of M is defined in the usual way, as a set of equivalence classes of Cauchy sequences of elements of M. Let us review this construction.

A sequence of elements $\{p_n\}$ in M is a *Cauchy sequence* in the h-adic topology if for every positive integer $\ell > 0$ there exists a positive integer N such that

$$p_n - p_m \in h^\ell M, \quad \text{for all } m, n > N,$$

i.e. $p_n - p_m$ is "divisible" by h^ℓ for all $n, m > N$. Two Cauchy sequences $P = \{p_n\}$ and $Q = \{q_n\}$ are *equivalent* if the sequence $\{p_n - q_n\}$ converges to 0, i.e.

$$P \sim Q \text{ if for every } \ell \text{ there exists an } N \text{ such that } p_n - q_n \in h^\ell M \text{ for all } n > N.$$

The set of all equivalence classes of Cauchy sequences in M is the *completion* $\hat{M}$ of M.

The completion $\hat{M}$ is a $k[[h]]$-module where the operations are determined by

$$P + Q = \{p_n + q_n\}, \quad \text{and} \quad aP = \{ap_n\},$$

where $P = \{p_n\}$ and $Q = \{q_n\}$ are Cauchy sequences with elements in M and $a \in k[[h]]$. Define a map

$$\begin{array}{rccc} \phi: & M & \longrightarrow & \hat{M} \\ & m & \longmapsto & [(m, m, m, \ldots)], \end{array}$$

i.e. $\phi(m)$ is the equivalence class of the sequence $\{p_n\}$ such that $p_n = m$ for all n. This map is injective and thus we can view M as a submodule of $\hat{M}$.

2. Deformations