(2.1) Motivation for deformations

We are going to make the quantum group by deforming the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of a complex simple Lie algebra $\mathfrak{g}$ *as a Hopf algebra*. This last condition is important because the enveloping algebra $\mathfrak{U}\mathfrak{g}$ does not have any deformations as an algebra.

(2.2) Deformation as a Hopf algebra

Assume that $(A, m, \iota, \Delta, \epsilon, S)$ is a Hopf algebra over k. Let $A[[h]]\hat{\otimes}A[[h]]$ denote the completion of $A[[h]] \otimes_{k[[h]]} A[[h]]$ in the h-adic topology. A *deformation of A as a Hopf algebra* is a tuple $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ where

$$\begin{gathered} m_h\colon A[[h]]\hat{\otimes}A[[h]] \longrightarrow A[[h]], \qquad \Delta_h\colon A[[h]] \longrightarrow A[[h]]\hat{\otimes}A[[h]], \\ \iota_h\colon k[[h]] \longrightarrow A[[h]], \quad \epsilon_h\colon A[[h]] \longrightarrow k[[h]], \quad \text{and} \quad S_h\colon A[[h]] \longrightarrow A[[h]], \end{gathered}$$

are $k[[h]]$-linear maps which are continuous in the h-adic topology, satisfy axioms (1) - (7) in the definition of a Hopf algebra, and can be written in the form

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots \\ \Delta_h &= \Delta + \Delta_1 h + \Delta_2 h^2 + \cdots \\ \iota_h &= \iota + \iota_1 h + \iota_2 h^2 + \cdots \\ \epsilon_h &= \epsilon + \epsilon_1 h + \epsilon_2 h^2 + \cdots \\ S_h &= S + S_1 h + S_2 h^2 + \cdots \end{aligned}$$

where, for each positive integer i,

$$\begin{gathered} m_i\colon A \otimes A \longrightarrow A, \qquad \Delta_i\colon A \longrightarrow A \otimes A, \\ \iota_i\colon k \longrightarrow A, \qquad \epsilon_i\colon A \longrightarrow k, \quad \text{and} \quad S_i\colon A \longrightarrow A, \end{gathered}$$

are k-linear maps which are extended first $k[[h]]$-linearly and then to the h-adic completion. We shall abuse language (only slightly) and call $(A[[h]], m_h, \iota_h, \epsilon_h, \Delta_h, S_h)$ a Hopf algebra over $k[[h]]$.

(2.3) Definition of equivalent deformations

Two Hopf algebra deformations $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ and $(A[[h]], m'_h, \iota'_h, \Delta'_h, S'_h)$ of a Hopf algebra $(A, m, \iota, \Delta, \epsilon, S)$ are *equivalent* if there is an isomorphism

$$f_h : (A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h) \longrightarrow (A[[h]], m'_h, \iota'_h, \Delta'_h, \epsilon'_h, S'_h)$$

of h-adically complete Hopf algebras over $k[[h]]$ which can be written in the form

$$f_h \;=\; \mathrm{id}_A + f_1 h + f_2 h^2 + \cdots$$

such that, for each positive integer i, $f_i : A \to A$ is a k-linear map which is extended $k[[h]]$-linearly to $k[[h]] \otimes_k A$ and then to the h-adic completion $A[[h]]$.

(2.4) Definition of the trivial deformation as a Hopf algebra

Let $(A, m, \iota, \Delta, \epsilon, S)$ be a Hopf algebra. The *trivial deformation of A as a Hopf algebra* is the Hopf algebra $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ over $k[[h]]$ such that $m_h = m$, $\iota_h = \iota$, $\Delta_h = \Delta$, $\epsilon_h = \epsilon$ and $S_h = S$ (extended to $A[[h]]$).

(2.5) Deformation as an algebra

Assume that (A, m, ι) is an algebra over k. Let $A[[h]]\hat{\otimes}A[[h]]$ denote the completion of $A[[h]] \otimes_{k[[h]]} A[[h]]$ in the h-adic topology. A *deformation of A as an algebra* is a tuple $(A[[h]], m_h, \iota_h)$ where

$$m_h: A[[h]]\hat{\otimes}A[[h]] \longrightarrow A[[h]], \qquad \iota_h: k[[h]] \longrightarrow A[[h]],$$

are $k[[h]]$-linear maps which are continuous in the h-adic topology, satisfy the axioms the definition of an algebra (see I (1.1)) and can be written in the form

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots \\ \iota_h &= \iota + \iota_1 h + \iota_2 h^2 + \cdots \end{aligned}$$

where, for each positive integer i,

$$m_i: A \otimes A \longrightarrow A, \qquad \iota_i: k \longrightarrow A,$$

are k-linear maps which are extended first $k[[h]]$-linearly and then to the h-adic completion. We shall abuse language (only slightly) and call $(A[[h]], m_h, \iota_h)$ an algebra over $k[[h]]$.

This definition is exactly like the definition of a deformation as a Hopf algebra in (2.2) above except that we only need to start with an algebra and we only require the result to be an algebra. We can define *equivalence of deformations as algebras* in exactly the same way that we defined them for deformations as Hopf algebras except that we only require the isomorphism f_h to be an algebra isomorphism instead of a Hopf algebra isomorphism.

(2.6) The trivial deformation as an algebra

Let (A, m, ι) be an algebra. The *trivial deformation* of A as an algebra is the algebra $(A[[h]], m_h, \iota_h)$ over $k[[h]]$ such that $m_h = m$ and $\iota_h = \iota$ (extended to $A[[h]]$). The deformation of the quantum group given in V (1.3) is even more incredible if one keeps the following theorem in mind.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$. Then $\mathfrak{U}\mathfrak{g}$ has no deformations as an algebra (up to equivalence of deformations).*

In other words, all deformations of $\mathfrak{U}\mathfrak{g}$ as an algebra are equivalent to the trivial deformation of $\mathfrak{U}\mathfrak{g}$.

IV. Perverse sheaves

To any reader that has not met sheaves before: I suggest that you don't read this section, only refer to it a few times while you are reading Chapter VIII of these notes. The most important thing, from the point of view of these notes, is to understand the basic structures given in Chapter VIII; anyone who is going to study these topics in more depth can always come back and learn these definitions later.

A large part of the material in this section is basic material about derived categories. This material can usually be found in texts which treat homological algebra. Everything in this section, except the definition and properties of perverse sheaves given in §3 can be found in [KS] Chapt. I-III. The definition of a perverse sheaf is in [BBD] 4.0 and the proof of Theorem (3.1) is in [BBD] Theorem 1.3.6. The Theorems in (3.2) are proved in [BBD] 2.1.9-2.1.11 and Theorem 4.3.1, respectively. We shall not review the definition of sheaves, it can be found in many textbooks, see [KS] Chapt. II.

1. The category $D_c^b(X)$

(1.1) Complexes of sheaves

Let X be an algebraic variety. A *complex of sheaves on* X is a sequence of sheaves A^i on X and morphisms of sheaves $d_i\colon A^i \to A^{i+1}$,

$$A = \left(\cdots \xrightarrow{d_{-2}} A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \cdots\right) \qquad \text{such that} \qquad d_{i+1}d_i = 0.$$

The morphisms $d_i\colon A^i \to A^{i+1}$ are called the *differentials* of the complex A. Let A and B be complexes of sheaves. A *morphism* $f\colon A \to B$ is a set of maps $f_n\colon A^n \to B^n$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{-2}} & A^{-1} & \xrightarrow{d_{-1}} & A^0 & \xrightarrow{d_0} & A^1 & \xrightarrow{d_1} & \cdots \\
 & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
\cdots & \xrightarrow{d_{-2}} & B^{-1} & \xrightarrow{d_{-1}} & B^0 & \xrightarrow{d_0} & B^1 & \xrightarrow{d_1} & \cdots
\end{array}$$

commutes.

The ith cohomology sheaf of a complex A is the sheaf

$$\mathcal{H}^i(A) = \frac{\ker(A^i \to A^{i+1})}{\operatorname{im}(A^{i-1} \to A_i)}$$

We have a well defined complex of sheaves $\mathcal{H}(A)$ given by

$$\cdots \xrightarrow{d_{-2}} \mathcal{H}^{-1}(A) \xrightarrow{d_{-1}} \mathcal{H}^0(A) \xrightarrow{d_0} \mathcal{H}^1(A) \xrightarrow{d_1} \cdots$$

A *quasi-isomorphism* $f: A \xrightarrow{\sim} B$ is a morphism $f: A \to B$ such that the induced morphism $\mathcal{H}(f): \mathcal{H}(A) \to \mathcal{H}(B)$ is an isomorphism. Note that every isomorphism is a quasi-isomorphism but *not* the other way around (even though the notation may be confusing).

(1.2) The category $K(X)$ and derived functors

Let X be an algebraic variety. Let A and B be complexes of sheaves on X. Two morphisms $f: A \to B$ and $g: A \to B$ are *homotopic* if there is a collection of morphisms $k_i: A^i \to B^{i-1}$ such that

$$f_n - g_n = k_{n+1}d_n + d_{n-1}k_n.$$

The motivation for this definition is that if f and g are homotopic then $\mathcal{H}(f) = \mathcal{H}(g)$.

Define $K(X)$ to be the category given by

Objects: Complexes of sheaves on X.

Morphisms: A $K(X)$-morphism from a complex A to a complex B is an homotopy equivalence class of morphisms from A to B.

This just means that, in the category $K(X)$, we identify homotopic morphisms.

Let A be a complex of sheaves on X. An *injective resolution* of A is a quasi-isomorphism $A \xrightarrow{\sim} J$ such that J^i is injective (an injective object in the category of sheaves on X) for all i. Let $Sh(X)$ denote the category of sheaves on X and let $F: Sh(X) \to Sh(X)$ be a functor. The *right derived functor* of F is the functor $RF: K(X) \to K(X)$ given by

$$RF(A) = F(J) \;=\; (\cdots \xrightarrow{F(d_{-2})} F(J^{-1}) \xrightarrow{F(d_{-1})} F(J^0) \xrightarrow{F(d_0)} F(J^1) \xrightarrow{F(d_1)} \cdots)$$

where J is an injective resolution of A. The *ith derived functor of* F is the functor $R^iF: K(X) \to Sh(X)$ given by

$$R^iF(A) \;=\; \mathcal{H}^i(F(J)),$$

where J is an injective resolution of A. In other words $R^iF(A)$ is the ith cohomology sheaf of the complex $RF(A)$.

(1.3) Bounded complexes and constructible complexes

A complex of sheaves A is *bounded* if there exists a positive integer n such that $A^m = 0$ and $A^{-m} = 0$ for all $m > n$.

An *algebraic stratification* of an algebraic variety X is a finite partition $X = \bigsqcup_\alpha X_\alpha$ of X into *strata* such that

(a) For each α, the stratum X_α is a smooth locally closed algebraic subvariety in X,

(b) The closure of each stratum is a union of strata, and

(c) The Whitney condition holds (see Verdier [Ver]).

Let l be a prime number and let $\overline{\mathbb{Q}_l}$ be the algebraic closure of the field $\mathbb{Q}_l$ of l-adic numbers. A sheaf F on X is $\overline{\mathbb{Q}_l}$*-constructible* if there is an algebraic stratification $X = \bigsqcup_\alpha X_\alpha$ such that, for each α, the restriction of F to X_α is a locally constant sheaf of finite dimensional vector spaces over $\overline{\mathbb{Q}_l}$. A complex $A \in K(X)$ is $\overline{\mathbb{Q}_l}$*-constructible* if $\mathcal{H}^i(A)$ is $\overline{\mathbb{Q}_l}$-constructible for all i.

(1.4) Definition of the category $D_c^b(X)$

Let X be a variety. Let A and B be complexes of sheaves on X. Define an equivalence relation on diagrams

$$A \xleftarrow{\sim} C \longrightarrow B$$

in $K(X)$ which have A and B as end points by saying that the diagram $A \xleftarrow{\sim} C \longrightarrow B$, is equivalent to the diagram $A \xleftarrow{\sim} C' \longrightarrow B$, if there exists a commutative diagram

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr] & \\
A \arrow[r, "\sim"] & D \arrow[u] \arrow[d] \arrow[r] & B \\
& C' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The notation $C \xleftarrow{\sim} A$ denotes that the map is a quasi-isomorphism. The *bounded derived category of* $\overline{\mathbb{Q}_l}$*-constructible sheaves* on X is the category $D_c^b(X)$ given by

Objects: Bounded, $\overline{\mathbb{Q}_l}$-constructible complexes of sheaves on X.

Morphisms: A morphism from A to B is an equivalence class of diagrams $A \xleftarrow{\sim} C \longrightarrow B$.

This definition of morphisms is a formal mechanism that inverts all quasi-isomorphisms. It ensures (in a coherent way) that "inverses" of quasi-isomorphisms are morphisms, i.e. that $A \xleftarrow{\sim} B$ is a morphism from A to B.

Given two morphisms $A \xleftarrow{\sim} D \longrightarrow B$ and $B \xleftarrow{\sim} E \longrightarrow C$ in $D_c^b(X)$ one can show that there always exists a commutative diagram

$$\begin{tikzcd}
& & F \arrow[dl] \arrow[dr] & & \\
& D \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

and one defines the composition of the two morphisms $A \xleftarrow{\sim} D \longrightarrow B$ and $B \xleftarrow{\sim} E \longrightarrow C$ to be the morphism defined by the diagram $A \xleftarrow{\sim} F \longrightarrow C$.

2. Functors

(2.1) The direct image with compact support functor $f_!$

A map $g: X \to Y$ between locally compact algebraic varieties is *compact* if the inverse image of every compact subset of Y is a compact subset of X.

Let $f: X \to Y$ be a morphism of locally compact algebraic varieties. Let F be a sheaf on X. The *support*, supp s, of a section s of F on an open set V is the complement in V of the union of open sets $U \subseteq V$ such that $s|_U = 0$.

The *direct image with compact support sheaf* $f_!F$, is the sheaf on Y defined by setting

$$\Gamma(U; f_!F) = \{s \in \Gamma(f^{-1}(U); F) \mid f : \text{supp } s \to U \text{ is compact}\},$$

for every open set U in Y. (For a sheaf F on X and an open set U in X, $\Gamma(U; F) = F(U)$.) This defines a functor $f_! : Sh(X) \to Sh(Y)$, where $Sh(X)$ denotes the category of sheaves on X.

Let $f: X \to Y$ be a morphism of locally compact algebraic varieties. The *direct image with compact support functor* $f_!: D^b_c(X) \to D^b_c(Y)$ is given by

$$f_! = Rf_!,$$

so that $f_!$ is the right derived functor of the functor $f_!: Sh(X) \to Sh(Y)$.

(2.2) The inverse image functor f^*

Let $f: X \to Y$ be a morphism of algebraic varieties. Let F be a sheaf on Y. The *inverse image* sheaf f^*F is the sheaf on X associated to the presheaf

$$V \longmapsto \lim_{U \supseteq f(V)} F(U), \qquad \text{for all } V \text{ open in } X,$$

where the limit is over all open sets U in Y which contain $f(V)$. This defines a functor $f^*: Sh(Y) \to Sh(X)$, where $Sh(X)$ denotes the category of sheaves on X. It is very common to denote this functor by f^{-1} but we shall follow [BBD] and [Lu] and use the notation f^*.

The *inverse image functor* $f^*: D^b_c(Y) \to D^b_c(X)$ is given by

$$f^* = Rf^*,$$

so that f^* is the right derived functor of the functor $f^*: Sh(Y) \to Sh(X)$.

(2.3) The functor $f_\flat$

Let $f: X \to Y$ be a morphism of algebraic varieties. Let $A \in D^b_c(X)$. Then $f_\flat A$ is the unique (up to isomorphism) complex on Y such that

$$A \cong f^*(f_\flat A).$$

Actually, I have cheated here: We can only be sure that the complex $f_\flat A$ is well defined if f is a locally trivial principal G-bundle, A is a semisimple G-equivariant complex on X and we require $f_\flat A$ to be a semisimple complex on Y, see [Lu] 8.1.7 and 8.1.8 for definitions and details.

(2.4) The shift functor $[n]$

Let A be a complex of sheaves on X. For each integer n define a new complex $A[n]$, with differentials $d[n]_i$, by

$$(A[n])^i = A^{n+i}, \qquad \text{and} \qquad (d[n])_i = (-1)^n d_{n+i}.$$

The *shift functor* is the functor

$$\begin{array}{ccc} D^b_c(X) & \xrightarrow{[n]} & D^b_c(X) \\ A & \longrightarrow & A[n]. \end{array}$$

(2.5) The Verdier duality functor D

This definition is too involved for us to take the energy to repeat it here, we shall refer the reader to [KS] §3.1. The main thing that we will need to know is that this functor exists.

3. Perverse sheaves

(3.1) Definition of perverse sheaves

Let X be an algebraic variety. The *support*, supp F, of a sheaf F on X is the complement of the union of open sets $U \subseteq X$ such that $F\big|_U = 0$.

A complex $A \in D^b_c(X)$ is a *perverse sheaf* if

(a) dim supp $\mathcal{H}^i(A) = 0$ for $i \geq 0$ and dim supp $\mathcal{H}^i(A) \leq -i$ for $i < 0$, and

(b) dim supp $\mathcal{H}^i(D(A)) = 0$ for $i \geq 0$ and dim supp $\mathcal{H}^i(D(A)) \leq -i$ for $i < 0$,

where $D(A)$ is the Verdier dual of A.

An abelian category is a category which has a direct sum operation and for which every morphism has a kernel and a cokernel. See [KS] I §1.2 for a precise definition.

Theorem. *The full subcategory of $D^b_c(X)$ whose objects are perverse sheaves on X is an abelian category.*

(3.2) Intersection cohomology complexes

Theorem. *Let $Y \subseteq X$ be a smooth locally closed subvariety of complex dimension $d > 0$ and let $\mathcal{L}$ be a locally constant sheaf on Y. There is a unique complex $IC(Y, \mathcal{L})$ in $D^b_c(X)$ such that*

(1) $\mathcal{H}^i(IC(Y, \mathcal{L})) = 0$, if $i < -d$,

(2) $\mathcal{H}^{-d}(IC(Y,\mathcal{L}))\mid_Y = \mathcal{L}$,
(3) dim supp $\mathcal{H}^i(IC(Y,\mathcal{L})) \leq -i$, if $i > -d$,
(4) dim supp $\mathcal{H}^i(D(IC(Y,\mathcal{L}))) \leq -i$, if $i > -d$,

The complexes $IC(Y,\mathcal{L})$ are the *intersection cohomology* complexes and an explicit construction of these complexes is given in [BBD] Prop. 2.1.11.

Theorem. *The simple objects of the category of perverse sheaves are the intersection complexes $IC(Y,\mathcal{L})$ as $\mathcal{L}$ runs through the irreducible locally constant sheaves on various smooth locally closed subvarieties $Y \subseteq X$.*

V. Quantum groups

The definition of the quantum group and the uniqueness theorem, Theorem (1.4), are stated in [D1] §6 Example 6.2. Theorem (1.4) appears with proof in [SS] Theorem 11.4.1. The statements in (3.3) and (3.4) can be found in [CP] 9.2.1 and 9.3.1 and the treatment there gives references for where to find the proofs.

1. Definition, uniqueness, and existence

(1.1) Making the Cartan matrix symmetric

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. There exist unique positive integers $d_1, d_2, \cdots, d_r$ such that $gcd(d_1, \cdots, d_r) = 1$ and the matrix $(d_i\alpha_j(H_i))_{1\le i,j\le r}$ is symmetric. The integers $d_1, d_2, \cdots, d_r$ are given explicitly by

$$\begin{array}{ll}
A_r,\ D_r,\ E_6,\ E_7,\ E_8: & d_i = 1 \text{ for all } 1\le i\le r,\\
B_r: & d_i = 1 \text{ for } 1\le i\le r-1, \quad \text{and} \quad d_r = 2,\\
C_r: & d_i = 2, \text{ for } 1\le i\le r-1, \quad \text{and} \quad d_r = 1,\\
F_4: & d_1 = d_2 = 1, \quad \text{and} \quad d_3 = d_4 = 2,\\
G_2: & d_1 = 3, \quad \text{and} \quad d_2 = 1.
\end{array}$$

(1.2) The Poisson homomorphism δ

Let $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ be the $\mathbb{C}$-linear map given by

$$\delta(H_i) = 0, \quad \delta(X_i^{\pm}) = d_i(X_i^{\pm}\otimes H_i - H_i\otimes X_i^{\pm}), \quad 1\le i\le r.$$

There is a unique extension of the map $\delta : \mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ to a $\mathbb{C}$-linear map $\delta : \mathfrak{U}\mathfrak{g}\to\mathfrak{U}\mathfrak{g}\otimes\mathfrak{U}\mathfrak{g}$ such that

$$\delta(xy) = \Delta(x)\delta(y) + \delta(x)\Delta(y), \quad \text{for all } x, y\in\mathfrak{U}\mathfrak{g}.$$

(1.3) The definition of the quantum group

A *Drinfel'd-Jimbo quantum group* $\mathfrak{U}_h\mathfrak{g}$ *corresponding* to $\mathfrak{g}$ is a deformation of $\mathfrak{U}\mathfrak{g}$ as a Hopf algebra over $\mathbb{C}$ such that

(1) Poisson condition :

$$\frac{\Delta_h(a) - \Delta_h^{\mathrm{op}}(a)}{h} \pmod h = \delta(a \bmod h), \quad \text{for all } a\in\mathfrak{U}_h\mathfrak{g}.$$

(If $\Delta_h(a) = \sum_a a_{(1)}\otimes a_{(2)}$ then $\Delta_h^{\mathrm{op}}(a) = \sum_a a_{(2)}\otimes a_{(1)}$.)

(2) Cartan subalgebra condition:

There is a subalgebra $\mathfrak{U}_h\mathfrak{h} \subseteq \mathfrak{U}_h\mathfrak{g}$ such that

(a) $\mathfrak{U}_h\mathfrak{h}$ is cocommutative, i.e. $\Delta_h(a) = \Delta_h^{\mathrm{op}}(a)$, for all $a \in \mathfrak{U}_h\mathfrak{h}$,

(b) The mapping $\mathfrak{U}_h\mathfrak{h}/h\mathfrak{U}_h\mathfrak{h} \to \mathfrak{U}\mathfrak{g}$ is injective with image $\mathfrak{U}\mathfrak{h}$.

(3) Cartan involution condition:

There is a mapping $\theta : \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ such that

(a) $\theta^2 = \mathrm{id}_{\mathfrak{U}_h\mathfrak{g}}$,

(b) $\theta(\mathfrak{U}_h\mathfrak{h}) = \mathfrak{U}_h\mathfrak{h}$,

(c) θ is an algebra homomorphism and a coalgebra antihomomorphism, i.e.

$$\begin{aligned} \theta(ab) &= \theta(a)\theta(b), \quad \text{for all } a, b \in \mathfrak{U}_h\mathfrak{g}, \text{ and} \\ \Delta_h(\theta(a)) &= (\theta \otimes \theta)\Delta_h^{\mathrm{op}}(a), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}, \end{aligned}$$

(d) θ mod h is the Cartan involution.

(1.4) Uniqueness of the quantum group

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. The Drinfel'd-Jimbo quantum group $\mathfrak{U}_h\mathfrak{g}$ corresponding to $\mathfrak{g}$ is unique (up to equivalence of deformations).*

(1.5) Definition of q-integers and q-factorials

For any symbol q define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \text{and}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad \text{for all positive integers } m \geq n,$$

(1.6) Presentation of the quantum group by generators and relations

Note the similarities (and the differences) between the following presentation of the quantum group by generators and relations and the presentation of the enveloping algebra of $\mathfrak{g}$ given in II (2.2).

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1 \leq i,j \leq r}$ be the corresponding Cartan matrix. The Drinfel'd-Jimbo quantum group $\mathfrak{U}_h\mathfrak{g}$ corresponding to $\mathfrak{g}$ can be presented as the algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by*

$$X_1^-, X_2^-, \ldots, X_r^-, \quad H_1, H_2, \ldots, H_r, \quad X_1^+, X_2^+, \ldots, X_r^+,$$

with relations

$$
\begin{aligned}
&[H_i, H_j] = 0, && \text{for all } 1 \le i, j \le r,\\
&[H_i, X_j^+] = \alpha_j(H_i) X_j^+, && \\
& && \text{for all } 1 \le i, j \le r,\\
&[H_i, X_j^-] = -\alpha_j(H_i) X_j^-, && \\
&[X_i^+, X_j^-] = \delta_{ij}\frac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}}, && \text{for } 1 \le i, j \le r,
\end{aligned}
$$

$$
\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_i h}} (X_i^{\pm})^s X_j^{\pm} (X_i^{\pm})^t = 0, \qquad \text{for } i \ne j,
$$

and with Hopf algebra structure given by

$$
\begin{gathered}
\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,\\
\Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+, \qquad \Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-,\\
S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ e^{-d_i h H_i}, \quad S_h(X_i^-) = -e^{d_i h H_i} X_i^-,\\
\epsilon_h(H_i) = \epsilon_h(X_i^+) = \epsilon_h(X_i^-) = 0,
\end{gathered}
$$

Cartan subalgebra $\mathfrak{U}\mathfrak{h}[[h]] \subseteq \mathfrak{U}_h\mathfrak{g}$, *and Cartan involution* $\theta\colon \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathfrak{U}_h\mathfrak{g}$ *determined by*

$$
\theta(X_i^+) = -X_i^-, \qquad \theta(X_i^-) = -X_i^+, \qquad \theta(H_i) = -H_i.
$$

2. The rational form of the quantum group

The rational form of the quantum group is an algebra which is similar to the algebra $\mathfrak{U}_h\mathfrak{g}$ except that it is over an arbitrary field k. There are two reasons for introducing this algebra.

(1) In the case when $k = \mathbb{C}(q)$ is the field this new algebra $U_q\mathfrak{g}$ has "integral forms" which can be used to specialize q to special values.

(2) In the case when $k = \mathbb{C}$ and q is a power of a prime then part of this algebra appears naturally as a Hall algebra of representations of quivers or, equivalently, as a Grothendieck ring of G-equivariant perverse sheaves on certain varieties E_V.

(2.1) Definition of the rational form of the quantum group

Many authors use the following form $U_q\mathfrak{g}$ of the quantum group as *the definition* of the quantum group.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. Let k be a field and let $q \in k$ be an nonzero element of k. The *rational form of the Drinfel'd-Jimbo quantum group* $U_q\mathfrak{g}$ corresponding to $\mathfrak{g}$ is the algebra $U_q\mathfrak{g}$ over k generated by

$$F_1, F_2, \ldots, F_r, \quad K_1, K_2, \ldots, K_r, \quad K_1^{-1}, K_2^{-1}, \ldots, K_r^{-1}, \quad E_1, E_2, \ldots, E_r,$$

with relations

$$\begin{aligned}
&K_iK_j = K_jK_i, && \text{for all } 1 \le i,j \le r,\\
&K_iK_i^{-1} = K_i^{-1}K_i = 1, && \text{for all } 1 \le i \le r,\\
&K_iE_jK_i^{-1} = q^{d_i\alpha_j(H_i)}E_j, && \\
& && \text{for all } 1 \le i,j \le r,\\
&K_iF_jK_i^{-1} = q^{-d_i\alpha_j(H_i)}F_j, && \\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, && \text{for } 1 \le i,j \le r,
\end{aligned}$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} E_i^sE_jE_i^t = 0, \qquad \text{for } i \ne j,$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} F_i^sF_jF_i^t = 0, \qquad \text{for } i \ne j,$$

and with Hopf algebra structure given by

$$\begin{gathered}
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,\\
S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i,\\
\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad \epsilon(F_i) = 0.
\end{gathered}$$

It is very common to take q to be an indeterminate and to let $k = \mathbb{C}(q)$ be the field of rational functions in q.

(2.2) Relating the rational form and the original form of the quantum group

The relations in the rational form of the quantum group are obtained from the relations in the presentation of $\mathfrak{U}_h\mathfrak{g}$ by making the following replacements:

$$e^h \longrightarrow q, \quad e^{hd_iH_i} \longrightarrow K_i, \quad X_i^- \longrightarrow F_i, \quad X_i^+ \longrightarrow E_i.$$

The ring $U_q\mathfrak{g}$ is an algebra over k and $q \in k$ while the ring $\mathfrak{U}_h\mathfrak{g}$ is an algebra over $\mathbb{C}[[h]]$ where h is an indeterminate. They have many similar properties. Most of the theorems about the structure of the algebra $\mathfrak{U}_h\mathfrak{g}$ have analogues for the case of the algebra $U_q\mathfrak{g}$. The category of modules for $U_q\mathfrak{g}$ is very similar to the category of module for the enveloping algebra $\mathfrak{U}\mathfrak{g}$. One should note, however, in contrast to Chapt. VI Theorem (1.1) which says that $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}[[h]]$, it is *not* true that $U_q\mathfrak{g}$ is isomorphic to $\mathfrak{U}\mathfrak{g}$, even if $k = \mathbb{C}$ and $q \in k$. This fact complicates many of the proofs when one is trying to generalize results from the classical case of $\mathfrak{U}\mathfrak{g}$ to the quantum case $U_q\mathfrak{g}$.

3. Integral forms of the quantum group

There are two different commonly used integral forms of a $\mathbb{C}(q)$-algebra $U_q\mathfrak{g}$, the "non-restricted integral form" $U_\mathcal{A}\mathfrak{g}$ and the "restricted integral form" $U_\mathcal{A}^{\mathrm{res}}\mathfrak{g}$. Let us begin by defining integral forms precisely.

(3.1) Definition of integral forms

Let q be an indeterminate and let U_q be an algebra over $\mathbb{C}(q)$, the field of rational functions in q. An *integral form* of U_q is a $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_\mathcal{A}$ of U_q such that the map

$$U_\mathcal{A} \otimes_\mathcal{A} \mathbb{C}(q) \longrightarrow U_q$$

is an isomorphism of $\mathbb{C}(q)$ algebras. In other words, upon extending scalars from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{C}(q)$ the algebra $U_\mathcal{A}$ turns into U_q.

(3.2) Motivation for integral forms

The purpose of defining integral forms of algebras is that we can use them to specialize the variable q to certain elements of $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, etc. Let $U_\mathcal{A}$ be an integral form of an algebra U_q over $\mathbb{C}(q)$ and let $\eta \in \mathbb{C}$, $\eta \neq 0$. The *specialization at* $q = \eta$ (over $\mathbb{C}$) of $U_\mathcal{A}$ is the algebra over $\mathbb{C}$ given by

$$U_\eta = U_\mathcal{A} \otimes_\mathcal{A} \mathbb{C}, \quad \text{where the equation} \quad qc = \eta c$$

describes how $\mathbb{C}$ is an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$-module. Similarly, we can define specializations of $U_\mathcal{A}$ over any field. With this last definition in mind we see that one could regard an integral form of U_q as an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_\mathcal{A}$ such that U_q is the specialization of $U_\mathcal{A}$ over $\mathbb{C}(q)$ at $q = q$.

(3.3) Definition of the non-restricted integral form of the quantum group

Let q be an indeterminate and let $k = \mathbb{C}(q)$ be the field of rational functions in q. Let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group. For each $1 \leq i \leq r$, define elements

$$[K_i; 0]_{q^{d_i}} = \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}.$$

The *non-restricted integral form of* $U_q\mathfrak{g}$ is the $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}\mathfrak{g}$ of $U_q\mathfrak{g}$ generated by the elements

$$F_1, F_2, \ldots, F_r, \quad K_1^{\pm 1}, K_2^{\pm 1}, \ldots, K_r^{\pm 1}, \quad [K_1; 0], [K_2; 0], \ldots, [K_r; 0], \quad E_1, E_2, \ldots, E_r.$$

The Hopf algebra structure on $U_q\mathfrak{g}$ restricts to a well defined Hopf algebra structure on $U_{\mathcal{A}}\mathfrak{g}$.

(3.4) Definition of the restricted integral form of the quantum group

Let q be an indeterminate and let $k = \mathbb{C}(q)$ be the field of rational functions in q. Let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group. The *restricted integral form of* $U_q\mathfrak{g}$ is the $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$ of $U_q\mathfrak{g}$ generated by the elements $K_1^{\pm 1}, K_2^{\pm 1}, \ldots, K_r^{\pm 1}$, and the elements

$$F_i^{(\ell)} = \frac{F_i^\ell}{[\ell]_{q^{d_i}}!}, \quad \text{and} \quad E_i^{(\ell)} = \frac{E_i^\ell}{[\ell]_{q^{d_i}}!}, \quad \text{for all } 1 \le i \le r \text{ and all } \ell \ge 1.$$

(The notation for the q-factorials is as in (1.5).) The Hopf algebra structure on $U_q\mathfrak{g}$ restricts to a well defined Hopf algebra structure on $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$. It is nontrivial to prove that $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$ is an integral form of $U_q\mathfrak{g}$.

VI. Modules for quantum groups

The isomorphism theorem in (1.1) is found (with proof) in [D2] p. 330-331. The proof of this theorem uses several cohomological facts,

$$H^2(\mathfrak{g}, \mathfrak{U}\mathfrak{g}) = 0, \quad H^1(\mathfrak{h}, \mathfrak{U}\mathfrak{g}/(\mathfrak{U}\mathfrak{g})^{\mathfrak{h}}) = 0, \quad \text{and} \quad H^1(\mathfrak{g}, \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}) = 0.$$

The correspondence theorem in (1.3) is also found in [D2] p.331. All of the results in section 2 can be found, with detailed proofs, in [Ja] Chapt. 5.

1. Finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules

(1.1) As algebras, $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}[[h]]$

The algebra $\mathfrak{U}\mathfrak{g}[[h]]$ is just the enveloping algebra of the Lie algebra $\mathfrak{g}$ except over the ring $\mathbb{C}[[h]]$ (and then h-adically completed) instead of over the field $\mathbb{C}$. It acts exactly like the algebra $\mathfrak{U}\mathfrak{g}$, the only difference is that we have extended coefficients.

The following theorem says that the algebra $\mathfrak{U}_h\mathfrak{g}$ and the algebra $\mathfrak{U}\mathfrak{g}[[h]]$ are exactly the same! In fact we have already seen that this must be so, since $\mathfrak{U}\mathfrak{g}$ has no deformations as an algebra (Chapt. III Theorem (2.6)). One might ask: If $\mathfrak{U}_h\mathfrak{g}$ and $\mathfrak{U}\mathfrak{g}[[h]]$ are the same then what is big deal about quantum groups? The answer is: They are the same as algebras but they are *not* the same when you look at them as Hopf algebras.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathfrak{g}$. Then there is an isomorphism of algebras*

$$\varphi : \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathfrak{U}\mathfrak{g}[[h]], \qquad \textit{such that}$$

$$\begin{aligned} &(a)\ \varphi = \mathrm{id}_{\mathfrak{U}\mathfrak{g}} \pmod h, \quad \textit{and} \\ &(b)\ \varphi\big|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}, \end{aligned}$$

where, in the second condition, $\mathfrak{h} = \mathbb{C}$-span$\{H_1, \ldots, H_r\} \subseteq \mathfrak{U}_h\mathfrak{g}$.

(1.2) Definition of weight spaces in a $\mathfrak{U}_h\mathfrak{g}$ module

A *finite dimensional $\mathfrak{U}_h\mathfrak{g}$-module* is a $\mathfrak{U}_h\mathfrak{g}$-module that is a finitely generated free module as a $\mathbb{C}[[h]]$-module. If M is a finite dimensional $\mathfrak{U}_h\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$ define the *μ-weight space of M* to be the subspace

$$M_\mu = \{m \in M \mid am = \mu(a)m, \quad \text{for all } a \in \mathfrak{h}\}.$$

The *dimension* of the weight space M_μ is the number of elements in a basis for it, as a $\mathbb{C}[[h]]$-module.

(1.3) Classification of modules for $\mathfrak{U}_h\mathfrak{g}$

Theorem (1.1) says that $\mathfrak{U}_h\mathfrak{g}$ and $\mathfrak{U}\mathfrak{g}[[h]]$ are the same as algebras. Since the category of finite dimensional modules for an algebra depends only on its algebra structure it follows immediately that the category of finite dimensional modules for $\mathfrak{U}_h\mathfrak{g}$ is the same as the category of modules for $\mathfrak{U}\mathfrak{g}[[h]]$.

Theorem. *There is a one to one correspondence between the isomorphism classes of finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules and the isomorphism classes of finite dimensional $\mathfrak{U}\mathfrak{g}$-modules given by*

$$\begin{array}{ccc} \mathfrak{U}_h\mathfrak{g}\text{-modules} & \xleftrightarrow{1-1} & \mathfrak{U}\mathfrak{g}\text{-modules} \\ & & \\ M & \longleftrightarrow & M/hM \\ V[[h]] & \longleftrightarrow & V \end{array}$$

where the $\mathfrak{U}_h\mathfrak{g}$ module structure on $V[[h]]$ is defined by the composition

$$\mathfrak{U}_h\mathfrak{g} \xrightarrow{\sim} \mathfrak{U}\mathfrak{g}[[h]] \longrightarrow \mathrm{End}(V[[h]]).$$

It follows from condition (b) of Theorem (1.1) that, under the correspondence in the Theorem above, weight spaces of $\mathfrak{U}_h\mathfrak{g}$-modules are taken to weight spaces of $\mathfrak{U}\mathfrak{g}$-modules and their dimension remains the same. Furthermore, irreducible finite dimensional $\mathfrak{U}\mathfrak{g}$-modules correspond taken to indecomposable $\mathfrak{U}_h\mathfrak{g}$-modules and vice versa. (Note that $hV[[h]]$ is always a $\mathfrak{U}_h\mathfrak{g}$-submodule of the $\mathfrak{U}_h\mathfrak{g}$-module $V[[h]]$.)

The previous theorem combined with Chapt. II Theorem (2.5) gives the following corollary.

Corollary. *Let $P^+ = \sum_{i=1}^r \mathbb{N}\omega_i$ be the set of dominant integral weights for $\mathfrak{g}$, as in (2.4). For every $\lambda \in P^+$ there is a unique (up to isomorphism) finite dimensional indecomposable $\mathfrak{U}_h\mathfrak{g}$-module $L(\lambda)$ corresponding to λ.*

2. Finite dimensional $U_q\mathfrak{g}$-modules

The category of finite dimensional modules for the rational form $U_q\mathfrak{g}$ of the quantum group is slightly different from the category of finite dimensional modules for $\mathfrak{U}_h\mathfrak{g}$. The construction of the finite dimensional irreducible modules for $U_q\mathfrak{g}$ is similar to the construction of these modules in the case of the Lie algebra $\mathfrak{g}$. Let us describe how this is done.

(2.1) Construction of the Verma module $M(\lambda)$ and the simple module $L(\lambda)$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group over a field k and with $q \in k$. We shall assume that char $k \neq 2, 3$ and q is not a root of unity in k.

Let $\lambda \in P$ be an element of the weight lattice for $\mathfrak{g}$. The *Verma module* $M(\lambda)$ is be the $U_q\mathfrak{g}$-module generated by a single vector v_λ where the action of $U_q\mathfrak{g}$ satisfies the relations

$$E_i v_\lambda = 0, \quad \text{and} \quad K_i v_\lambda = q^{(\lambda, \alpha_i)} v_\lambda, \quad \text{for all } 1 \leq i \leq r.$$

The map

$$\begin{array}{ccc} U_q\mathfrak{n}^- & \longrightarrow & M(\lambda) \\ y & \longmapsto & y v_\lambda \end{array}$$

is a vector space isomorphism.

The module $M(\lambda)$ has a unique maximal proper submodule. For each $\lambda \in P$ define

$$L(\lambda) = \frac{M(\lambda)}{N}$$

where N is the maximal proper submodule of the Verma module $M(\lambda)$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group over a field k with $q \in k$. Assume that* char $k \neq 2, 3$ *and that q is not a root of unity in k. Let $\lambda \in P$ be an element of the weight lattice of $\mathfrak{g}$ and let $L(\lambda)$ be the $U_q\mathfrak{g}$-module defined above.*

(a) The module $L(\lambda)$ is a simple $U_q\mathfrak{g}$-module.

(b) The module $L(\lambda)$ is finite dimensional if and only if λ is a dominant integral weight.

(2.2) Twisting $L(\lambda)$ to get $L(\lambda, \sigma)$

Let Q be the root lattice corresponding to $\mathfrak{g}$ as given in II (2.6) and let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism. The homomorphism σ induces an automorphism $\sigma: U_q\mathfrak{g} \to U_q\mathfrak{g}$ of $U_q\mathfrak{g}$ defined by

$$\begin{array}{cccc} \sigma: & U_q\mathfrak{g} & \longrightarrow & U_q\mathfrak{g} \\ & E_i & \longmapsto & \sigma(\alpha_i) E_i \\ & F_i & \longmapsto & F_i \\ & K_i^{\pm 1} & \longmapsto & \sigma(\pm\alpha_i) K_i^{\pm 1}, \end{array}$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots for $\mathfrak{g}$. Let $\lambda \in P$ be an element of the weight lattice and let $L(\lambda)$ be the irreducible $U_q\mathfrak{g}$-module defined in (2.1). Define a $U_q\mathfrak{g}$-module $L(\lambda, \sigma)$ by defining

(a) $L(\lambda, \sigma) = L(\lambda)$ as vector spaces,

(b) $U_q\mathfrak{g}$ acts on $L(\lambda, \sigma)$ by the formulas

$$u \star m = \sigma(u) m, \quad \text{for all } u \in U_q\mathfrak{g},\ m \in L(\lambda),$$

where σ is the automorphism of $U_q\mathfrak{g}$ defined above.

(2.3) Classification of finite dimensional irreducible modules for $U_q\mathfrak{g}$

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the rational form of the quantum group over a field k. Assume that* char $k \neq 2, 3$ *and* $q \in k$ *is not a root of unity in k. Let P^+ be the set of dominant integral weights for $\mathfrak{g}$ and let Q be the root lattice for $\mathfrak{g}$ (see II (2.6)).*

(a) *Let $\lambda \in P^+$ and let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism. The modules $L(\lambda, \sigma)$ defined in (2.2) are all finite dimensional irreducible $U_q\mathfrak{g}$-modules.*
(b) *Every finite dimensional $U_q\mathfrak{g}$-module is isomorphic to $L(\lambda, \sigma)$ for some $\lambda \in P^+$ and some group homomorphism $\sigma: Q \to \{\pm 1\}$.*

(2.4) Weight spaces for $U_q\mathfrak{g}$-modules

Retain the notations and assumptions from (2.3) and let $(,)$ be the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ defined in II (2.7). Let M be a finite dimensional $U_q\mathfrak{g}$-module. Let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism and let $\lambda \in P$. The (λ, σ)-*weight space* of M is the vector space

$$M_{(\lambda,\sigma)} = \{m \in M \mid K_i m = \sigma(\alpha_i) q^{(\lambda, \alpha_i)} m \text{ for all } 1 \leq i \leq r.\}$$

The following proposition is analogous to Chapt. II Proposition (2.4).

Proposition. *Every finite dimensional $U_q\mathfrak{g}$-module is a direct sum of its weight spaces.*

The following theorem says that the dimensions of the weight spaces of irreducible $U_q\mathfrak{g}$-modules coincide with the dimensions of the weight space of corresponding irreducible modules for the Lie algebra $\mathfrak{g}$.

Theorem. *Let $\lambda \in P^+$ be a dominant integral weight and let σ be a group homomorphism $\sigma: Q \to \{\pm 1\}$. Let V^λ be the simple $\mathfrak{g}$-module indexed by the λ and let $L(\lambda, \sigma)$ be the irreducible $U_{\mathfrak{g}}$-module indexed by the pair (λ, σ). Then, for all $\mu \in P$ and all group homomorphisms $\tau: Q \to \{\pm 1\}$,*

$$\dim_k\big(L(\lambda,\sigma)_{\mu,\sigma}\big) = \dim_{\mathbb{C}}\big((V^\lambda)_\mu\big) \qquad \text{and} \qquad \dim_k\big(L(\lambda,\sigma)_{\mu,\tau}\big) = 0, \quad \text{if } \sigma \neq \tau.$$