VII. Properties of quantum groups

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathfrak{g}$ that was defined in V (1.3). We shall often use the presentation of $\mathfrak{U}_h\mathfrak{g}$ given in V (1.6). In this chapter we shall describe some of the structure which quantum groups have. In many cases this structure is similar to the structure of the enveloping algebra $\mathfrak{U}\mathfrak{g}$.

The proofs of the triangular decomposition and the grading on the quantum group given in §1 can be found in [Ja] 4.7 and 4.21. The proof of the statements in (2.1) and (2.3), concerning the pairing $\langle , \rangle$, can be found in [Ja] 6.12, 6.18, 8.28, and 6.22. The statement in (2.2) follows from Chapt I, Prop. (5.5). The theorem giving the existence and uniqueness of the $\mathcal{R}$-matrix is stated in [D2] p.329 and the uniqueness is proved there. The existence follows from (7.4); see also [Lu] Theorem 4.1.2. The properties of the $\mathcal{R}$-matrix stated in (3.3) are proved in [D2] Prop. 3.1 and Prop. 4.2. Proofs of the statements in the section on the Casimir element can be found in [D2] Prop 2.1, Prop 3.2 and Prop. 5.1.

Theorem (5.2a) is proved in [Ja] 8.15-8.17 and [Lu] 39.2.2. Theorem (5.2b) is a non-trivial, but very natural, extension of well known results which appear, for example, in [Ja] Chapt. 8. The proof is a combination of the methods used in [CP] 8.2B and [Ja] 8.4 and a calculation similar to that in the proof of [Ja] Lemma 8.3. The properties of the element T_{w_0} given in (5.3) are proved in the following places: The formula for $\sigma(T_{w_0})T_{w_0}$ is proved in [CP] 8.2.4; The formula for $T_{w_0}^{-1}$ is proved by a method similar to [Ja] 8.4; The formula for $\Delta_h(T_{w_0})$ is proved in [CP] 8.3.11 and the remainder of the formulas are proved in [CP] 8.2.3.

The construction of the Poincaré-Birkhoff-Witt basis of $\mathfrak{U}_h\mathfrak{g}$ given in section 6 appears in detail in [Ja] 8.18-8.30. The statement that $\mathfrak{U}_h\mathfrak{g}$ is almost a quantum double, Theorem (7.3), appears in [D1] §13, and an outline of the proof can be found in [CP] 8.3. The proof of Theorem (8.4) can be gleaned from a combination of [Ja] 6.11 and 6.18. Both of the books [Lu] and [Jo] also contain this fact.

1. Triangular decomposition and grading

(1.1) Triangular decomposition of $\mathfrak{U}_h\mathfrak{g}$

The triangular decomposition of the quantum group $\mathfrak{U}_h\mathfrak{g}$ is analogous to the triangular decomposition of the Lie algebra $\mathfrak{g}$ and the triangular decomposition of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ given in II (2.3) and II (4.2).

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be* *the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Define*

$$\mathfrak{U}_h\mathfrak{n}^- = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^-,$$

$$\mathfrak{U}_h\mathfrak{h} = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } H_1, H_2, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{n}^+ = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+.$$

The map

$$\begin{array}{ccc} \mathfrak{U}_h\mathfrak{n}^- \otimes \mathfrak{U}_h\mathfrak{h} \otimes \mathfrak{U}_h\mathfrak{n}^+ & \longrightarrow & \mathfrak{U}_h\mathfrak{g} \\ u^- \otimes u^0 \otimes u^+ & \longmapsto & u^-u^0u^+ \end{array}$$

is an isomorphism of vector spaces.

(1.2) The grading on $\mathfrak{U}_h\mathfrak{n}^+$ and $\mathfrak{U}_h\mathfrak{n}^-$

The gradings on the positive part $\mathfrak{U}_h\mathfrak{n}^+$ and on the negative part $\mathfrak{U}_h\mathfrak{n}^-$ of the quantum group $\mathfrak{U}_h\mathfrak{g}$ are exactly analogous to the gradings on the postive part $\mathfrak{U}\mathfrak{n}^+$ and the negative part $\mathfrak{U}\mathfrak{n}^-$ of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ which are given in II (4.3).

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$ and let*

$$Q^+ = \sum_i \mathbb{N}\alpha_i, \qquad \text{where } \mathbb{N} = \mathbb{Z}_{\geq 0}.$$

For each element $\nu = \sum_{i=1}^r \nu_i\alpha_i \in Q^+$ define

$$(\mathfrak{U}_h\mathfrak{n}^+)_\nu = \text{span-}\{X_{i_1}^+ \cdots X_{i_p}^+ \mid X_{i_1}^+ \cdots X_{i_p}^+ \text{ has } \nu_j\text{-factors of type } X_j^+\}$$

$$(\mathfrak{U}_h\mathfrak{n}^-)_\nu = \text{span-}\{X_{i_1}^- \cdots X_{i_p}^- \mid X_{i_1}^- \cdots X_{i_p}^- \text{ has } \nu_j\text{-factors of type } X_j^-\}.$$

Then

$$\mathfrak{U}_h\mathfrak{n}^- = \bigoplus_{\nu\in Q^+} (\mathfrak{U}_h\mathfrak{n}^-)_\nu \quad \text{and} \quad \mathfrak{U}_h\mathfrak{n}^+ = \bigoplus_{\nu\in Q^+} (\mathfrak{U}_h\mathfrak{n}^+)_\nu,$$

as vector spaces.

2. The inner product $\langle , \rangle$

In some sense the nonnegative part $\mathfrak{U}_h\mathfrak{b}^+$ of the quantum group is the dual of the nonpositive part $\mathfrak{U}_h\mathfrak{b}^-$ of the quantum group. This is reflected in the fact that there is a nondegenerate bilinear pairing between the two. Later we shall see that this pairing can be extended to a pairing on all of $\mathfrak{U}_h\mathfrak{g}$. The extended pairing is an analogue of the Killing form on $\mathfrak{g}$ in two ways:

(1) it is an ad-invariant form on $\mathfrak{U}_h\mathfrak{g}$, and

(2) upon restriction to $\mathfrak{g}$ it coincides $\pmod h$ with the Killing form.

(2.1) The pairing between $\mathfrak{U}_h\mathfrak{b}^-$ and $\mathfrak{U}_h\mathfrak{b}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Define

$$\mathfrak{U}_h\mathfrak{b}^- = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^- \text{ and } H_1, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{b}^+ = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+ \text{ and } H_1, \ldots, H_r.$$

Theorem.

(1) There is a unique $\mathbb{C}[[h]]$-bilinear pairing

$$\langle , \rangle : \mathfrak{U}_h\mathfrak{b}^- \times \mathfrak{U}_h\mathfrak{b}^+ \longrightarrow \mathbb{C}[[h]] \quad \textit{which satisfies}$$

(a) $\langle 1, 1\rangle = 1,$

(b) $\langle H_i, H_j\rangle = \dfrac{\alpha_j(H_i)}{d_j},$

(c) $\langle X_i^-, X_j^+\rangle = \delta_{ij}\dfrac{1}{e^{d_ih} - e^{-d_ih}},$

(d) $\langle ab, c\rangle = \langle a\otimes b, \Delta_h(c)\rangle,$ *for all* $a, b \in \mathfrak{U}_h\mathfrak{b}^-$ *and* $c \in \mathfrak{U}_h\mathfrak{b}^+,$

(e) $\langle a, bc\rangle = \langle \Delta_h^{\mathrm{op}}(a), b\otimes c\rangle,$ *for all* $a \in \mathfrak{U}_h\mathfrak{b}^-$ *and* $b, c \in \mathfrak{U}_h\mathfrak{b}^+.$

(2) The pairing $\langle , \rangle$ is nondegenerate.

(3) The pairing $\langle , \rangle$ respects the gradings on $\mathfrak{U}_h\mathfrak{n}^+$ and $\mathfrak{U}_h\mathfrak{n}^-$ in the following sense:

(a) Let $\mu, \nu \in Q^+$.

$$\textit{If } \mu \neq \nu \textit{ then } \big\langle (\mathfrak{U}_h\mathfrak{b}^-)_\mu, (\mathfrak{U}_h\mathfrak{b}^+)_\nu \big\rangle = 0.$$

(b) Let $\nu \in Q^+$. The restriction of the pairing $\langle , \rangle$ to $(\mathfrak{U}_h\mathfrak{n}^-)_\nu \times (\mathfrak{U}_h\mathfrak{n}^+)_\nu$ is a nondegenerate pairing

$$\langle , \rangle \colon (\mathfrak{U}_h\mathfrak{n}^-)_\nu \times (\mathfrak{U}_h\mathfrak{n}^+)_\nu \to \mathbb{C}[[h]].$$

If θ is the Cartan involution of $\mathfrak{U}_h\mathfrak{g}$ as given in V (1.6) and S_h is the antipode of $\mathfrak{U}_h\mathfrak{g}$ then

$$\langle \theta(u^+), \theta(u^-)\rangle = \langle u^-, u^+\rangle \quad \text{and} \quad \langle S_h(u^-), S_h(u^+)\rangle = \langle u^-, u^+\rangle,$$

for all $u^- \in \mathfrak{U}_h\mathfrak{b}^-$ and $u^+ \in \mathfrak{U}_h\mathfrak{b}^+$.

(2.2) Extending the pairing to an ad-invariant pairing on $\mathfrak{U}_h\mathfrak{g}$

The triangular decomposition (1.1) of $\mathfrak{U}_h\mathfrak{g}$ says that $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}_h\mathfrak{n}^- \otimes \mathfrak{U}_h\mathfrak{h} \otimes \mathfrak{U}_h\mathfrak{n}^+$ and that

every element $u \in \mathfrak{U}_h\mathfrak{g}$ can be written in the form $u^-u^0u^+$,

where $u^- \in \mathfrak{U}_h\mathfrak{n}^-$, $u^0 \in \mathfrak{U}_h\mathfrak{h}$, and $u^+ \in \mathfrak{U}_h\mathfrak{n}^+$. We can use this to extend the pairing defined in (2.1) to a pairing

$$\langle , \rangle : \mathfrak{U}_h\mathfrak{g} \times \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathbb{C}[[h]] \quad \text{defined by the formula}$$

$$\langle u_1^- u_1^0 u_1^+, u_2^- u_2^0 u_2^+ \rangle = \langle u_1^-, S_h(u_2^0 u_2^+)\rangle \langle u_2^-, S_h^{-1}(u_1^0 u_1^+)\rangle,$$

for all $u_1^-, u_2^- \in \mathfrak{U}_h\mathfrak{n}^-$, $u_1^0, u_2^0 \in \mathfrak{U}_h\mathfrak{h}$, and $u_1^+, u_2^+ \in \mathfrak{U}_h\mathfrak{n}^+$, where S_h is the antipode of $\mathfrak{U}_h\mathfrak{g}$. Then

$$\langle \mathrm{ad}_u(v_1), v_2\rangle = \langle v_1, \mathrm{ad}_{S_h(u)}(v_2)\rangle, \quad \text{for all } u, v_1, v_2 \in \mathfrak{U}_h\mathfrak{g},$$

This formula says that the extended pairing $\langle , \rangle$ is an ad-invariant pairing as defined in I (5.5). The pairing $\langle , \rangle : \mathfrak{U}_h\mathfrak{g} \times \mathfrak{U}_h\mathfrak{g} \to \mathbb{C}[[h]]$ is *not* symmetric, see I (5.5).

(2.3) Duality between matrix coefficients for representations and $U_q\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the rational form of the quantum group over a field k, where char $k \neq 2, 3$ and $q \in k$ is not a root of unity. Let Q be the root lattice for $\mathfrak{g}$.

Theorem. *Let M be a finite dimensional $U_q\mathfrak{g}$ module such that all weights λ of M satisfy $2\lambda \in Q$. Then, for each pair $n^* \in M^*$ and $m \in M$ there is a unique element $u \in U_q\mathfrak{g}$ such that*

$$n^*(vm) = \langle v, u\rangle, \quad \text{for all } v \in U_q\mathfrak{g},$$

where $\langle , \rangle$ is the bilinear form on $U_q\mathfrak{g}$ given by (2.1) after making the substitutions in V (2.2).

The function

$$c_{m,n^*} : U_q\mathfrak{g} \to \mathbb{C}(q) \quad \text{defined by} \quad c_{m,n^*}(v) = \langle n^*, vm\rangle$$

is the *(m, n^*)-matrix coefficient of v acting on M*. The above theorem gives a duality between matrix coefficient functions and $U_q\mathfrak{g}$. It also says that every element of $U_q\mathfrak{g}$ is determined by how it acts on finite dimensional $U_q\mathfrak{g}$-modules.

3. The universal $\mathcal{R}$-matrix

(3.1) Motivation for the $\mathcal{R}$-matrix

The following theorem states that there is an element $\mathcal{R}$ such that the pair $(\mathfrak{U}_h\mathfrak{g}, \mathcal{R})$ is a quasitriangular Hopf algebra. In particular, this implies that the category of finite dimensional modules for the quantum group $\mathfrak{U}_h\mathfrak{g}$ is a braided SRMCwMFF.

(3.2) Existence and uniqueness of $\mathcal{R}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Recall the Killing form on $\mathfrak{g}$ from II (1.6).

Let $\{\tilde{H}_i\}$ be an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form and define

$$t_0 = \sum_{i=1}^{r} \tilde{H}_i \otimes \tilde{H}_i.$$

If $\nu \in Q^+$ (see (1.2)) and $\nu = \sum_{i=1}^r \nu_i\alpha_i$ where $\alpha_1, \ldots, \alpha_r$ are the simple roots, define n_ν to be the smallest number of positive roots $\alpha > 0$ whose sum is equal to ν.

The element $\mathcal{R}$ is not quite an element of $\mathfrak{U}_h\mathfrak{g} \otimes \mathfrak{U}_h\mathfrak{g}$ so we have to make the tensor product just a tiny bit bigger. To do this we let $\mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ denote the h-adic completion of the tensor product $\mathfrak{U}_h\mathfrak{g} \otimes \mathfrak{U}_h\mathfrak{g}$, see III §1.

Theorem. *There exists a unique invertible element $\mathcal{R} \in \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ such that*

$$\mathcal{R}\Delta_h(a)\mathcal{R}^{-1} = \Delta_h^{\mathrm{op}}(a), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}, \text{ and}$$

$$\mathcal{R} \text{ has the form } \quad \mathcal{R} = \sum_{\nu \in Q^+} \exp\left\{h\left(t_0 + \tfrac{1}{2}(H_\nu \otimes 1 - 1 \otimes H_\nu)\right)\right\} P_\nu, \qquad \text{where}$$

$P_\nu \in (\mathfrak{U}_h\mathfrak{n}^-)_\nu \otimes (\mathfrak{U}_h\mathfrak{n}^+)_\nu$,
$H_\nu = \sum_{i=1}^r \nu_i H_i$, *if* $\nu = \sum_i \nu_i\alpha_i$,
P_ν is a polynomial in $X_i^+ \otimes 1$ and $1 \otimes X_i^-$, $1 \le i \le r$, with coefficients in $\mathbb{C}[[h]]$, such that
the smallest power of h in P_ν with nonzero coefficient is h^{n_ν}.

(3.3) Properties of the $\mathcal{R}$-matrix

Recall V (1.6) that $\mathfrak{U}_h\mathfrak{g}$ is a Hopf algebra with comultiplication Δ_h, counit ϵ_h, and antipode S_h and that $\mathfrak{U}_h\mathfrak{g}$ comes with a Cartan involution θ. The following formulas describe the relationship between the $\mathcal{R}$-matrix and the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$. If $\mathcal{R} = \sum a_i \otimes b_i$ then let

$$\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1, \quad \mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \quad \text{and} \quad \mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i,$$

$$\text{and let} \qquad \mathcal{R}_{21} = \sum b_i \otimes a_i.$$

Let $\sigma: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ be the $\mathbb{C}$-algebra automorphism of $\mathfrak{U}_h\mathfrak{g}$ given by $\sigma(h) = -h$, $\sigma(X_i^{\pm}) = X_i^{\pm}$, and $\sigma(H_i) = H_i$. With these notations we have

$$(\Delta_h \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (\mathrm{id} \otimes \Delta_h)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$(\epsilon_h \otimes \mathrm{id})(\mathcal{R}) = 1 = (\mathrm{id} \otimes \epsilon_h)(\mathcal{R}),$$

$$(S_h \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S_h^{-1})(\mathcal{R}) = \mathcal{R}^{-1} \quad \text{and} \quad (S_h \otimes S_h)(\mathcal{R}) = \mathcal{R},$$

$$(\theta \otimes \theta)(\mathcal{R}) = \mathcal{R}_{21} \quad \text{and} \quad (\sigma \otimes \sigma)(\mathcal{R}) = \mathcal{R}^{-1}.$$

4. An analogue of the Casimir element

(4.1) Definition of the element u

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). The antipode $S_h: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ is an antiautomorphism of $\mathfrak{U}_h\mathfrak{g}$, see I (2.1). This means that the map $S_h^2: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ is an automorphism of $\mathfrak{U}_\mathfrak{g}$. The following theorem says that this automorphism is inner!

Theorem. *Let $\mathcal{R} \in \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ be the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ as defined in (3.2). Suppose that $\mathcal{R} = \sum a_i \otimes b_i$ and define $u = \sum S(b_i)a_i$. Then u is invertible and*

$$uau^{-1} = S_h^2(a), \quad \textit{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

(4.2) Properties of the element u.

The relationship of the element u to the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$ is given by the formulas

$$\Delta_h(u) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(u \otimes u), \quad S_h(u) = u, \quad \text{and} \quad \epsilon_h(u) = 1,$$

where $\mathcal{R}_{12} = \mathcal{R} = \sum a_i \otimes b_i$ is the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ given in (3.2), and $\mathcal{R}_{21} = \sum b_i \otimes a_i$. The inverse of the element u is given by

$$u^{-1} = \sum S_h^{-1}(d_j)c_j, \quad \text{where} \quad \mathcal{R}^{-1} = \sum c_j \otimes d_j.$$

(4.3) Why the element u is an analogue of the Casimir element

Let $\tilde{\rho}$ be the element of $\mathfrak{h}$ such that $\alpha_i(\tilde{\rho}) = 1$ for all simple roots α_i of $\mathfrak{g}$. An easy check on the generators of $\mathfrak{U}_h\mathfrak{g}$ shows that

$$e^{h\tilde{\rho}}ae^{-h\tilde{\rho}} = S_h^2(a), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

It follows that

$$\text{the element } e^{-h\tilde{\rho}}u = ue^{-h\tilde{\rho}} \text{ is a central element in } \mathfrak{U}_h\mathfrak{g}.$$

Any central element of $\mathfrak{U}_h\mathfrak{g}$ must act on each finite dimensional simple $\mathfrak{U}_h\mathfrak{g}$-module by a constant. For each dominant integral weight λ let $L(\lambda)$ be the finite dimensional simple $\mathfrak{U}_h\mathfrak{g}$-module indexed by λ (see VI (1.3)). As in II (4.5), let ρ be the element of $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\rho = \tfrac{1}{2}\sum_{\alpha>0}\alpha,$$

where the sum is over all positive roots for $\mathfrak{g}$. Then the element

$$e^{-h\tilde{\rho}}u \text{ acts on } L(\lambda) \text{ by the constant } q^{-(\lambda+\rho,\lambda+\rho)+(\rho,\rho)},$$

where $q = e^h$ and the inner product in the exponent of q is the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ given in II (2.7). Note the analogy with II (4.5). It is also interesting to note that

$$(e^{-h\tilde{\rho}}u)^2 = uS_h(u).$$

5. The element T_{w_0}

(5.1) The automorphism $\phi \circ \theta \circ S_h$

Let W be the Weyl group corresponding to $\mathfrak{g}$ and let w_0 be the longest element of W (see II (2.8)). Let $s_1, \ldots, s_r$ be the simple reflections in W. For each $1 \le i \le r$ there is a unique $1 \le j \le r$ such that $w_0 s_i w_0^{-1} = s_j$. The map given by

$$\phi(X_i^{\pm}) = X_j^{\pm}, \quad \text{and} \quad \phi(H_i) = H_j, \quad \text{where} \quad w_0 s_i w_0^{-1} = s_j, \quad \text{for } 1 \le i \le r,$$

extends to an automorphism of $\mathfrak{U}_h\mathfrak{g}$. Let $\tilde{\theta}$ be the anti-automorphism of $\mathfrak{U}_h\mathfrak{g}$ defined by $\tilde{\theta}(X_i^{\pm}) = X_i^{\mp}$ and $\tilde{\theta}(H_i) = H_i$. This is an analogue of the Cartan involution. Let S_h be the antipode of $\mathfrak{U}_h\mathfrak{g}$ as given in V (1.6). These are both anti-automorphisms of $\mathfrak{U}_h\mathfrak{g}$. The composition

$$(S_h \circ \tilde{\theta} \circ \phi) \colon \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$$

is an automorphism of $\mathfrak{U}_h\mathfrak{g}$. The following result says that this automorphism is inner.

(5.2) Definition of the element T_{w_0}

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let $q = e^h$ and for each $1 \le i \le r$ let

$$E_i^{(r)} = \frac{(X_i^+)^r}{[r]_{q^{d_i}}!} \quad F_i^{(r)} = \frac{(X_i^+)^r}{[r]_{q^{d_i}}!} \quad \text{and} \quad K_i = e^{hd_iH_i},$$

where the notation for q-factorials is as in V (1.5). For each $1 \le i \le r$, define

$$T_i = \sum_{a,b,c \ge 0} (-1)^b q^{b-ac+(c+a-b)(c-a)} E_i^{(a)} F_i^{(b)} E_i^{(c)} K_i^{c-a},$$

where the sum is over all nonnegative integers a,b, and c.

Theorem.

(a) The elements T_i satisfy the relations

$$\underbrace{T_iT_jT_iT_j\cdots}_{m_{ij}\ \text{factors}} = \underbrace{T_jT_iT_jT_i\cdots}_{m_{ij}\ \text{factors}} \qquad \textit{for } i \ne j,$$

where the m_{ij} are as given in II (2.8).

(b) Let $w_0 = s_{i_1}\cdots s_{i_N}$ be a reduced word for the longest element of the Weyl group W, see II (2.8). Define

$$T_{w_0} = T_{i_1}\cdots T_{i_N}.$$

Then T_{w_0} is invertible and

$$T_{w_0} a T_{w_0}^{-1} = (S_h \circ \tilde{\theta} \circ \phi)(a), \qquad \textit{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

(5.3) Properties of the element T_{w_0}

Let $u \in \mathfrak{U}_h\mathfrak{g}$ be the analogue of the Casimir element for $\mathfrak{U}_h\mathfrak{g}$ as given in §4 and let σ be the $\mathbb{C}$-algebra automorphism of $\mathfrak{U}_h\mathfrak{g}$ given in (3.3). Let $\tilde{\sigma}$ be the $\mathbb{C}$-linear automorphism of $\mathfrak{U}_h\mathfrak{g}$ given by $\tilde{\sigma}(h) = -h$, $\tilde{\sigma}(X_i^{\pm}) = X_i^{\mp}$, and $\tilde{\sigma}(H_i) = -H_i$. Then

$$\sigma(T_{w_0})T_{w_0} = u \quad \text{and} \quad T_{w_0}^{-1} = \tilde{\sigma}(T_{w_0}).$$

The relationship between the element T_{w_0} and the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$ is given by the formulas

$$\Delta_h(T_{w_0}) = \mathcal{R}_{12}^{-1}(T_{w_0} \otimes T_{w_0}) = (T_{w_0} \otimes T_{w_0})\mathcal{R}_{21}^{-1}, \quad S_h(T_{w_0}) = T_{w_0}e^{h\tilde{\rho}}, \quad \text{and} \quad \epsilon_h(T_{w_0}) = 1,$$

where $\mathcal{R}_{12} = \mathcal{R} = \sum a_i \otimes b_i$ is the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ given in (3.2), and $\mathcal{R}_{21} = \sum b_i \otimes a_i$.

6. The Poincaré-Birkhoff-Witt basis of $\mathfrak{U}_h\mathfrak{g}$

(6.1) Root vectors in $\mathfrak{U}_h\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let T_i be the elements of $\mathfrak{U}_h\mathfrak{g}$ given in (5.2). Define an automorphism $\tau_i: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ by

$$\tau_i(u) = T_i u T_i^{-1}, \qquad \text{for all } u \in \mathfrak{U}_h\mathfrak{g},$$

Let W be the Weyl group corresponding to $\mathfrak{g}$. Fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$ of the longest element $w_0 \in W$, see II (2.8). Define

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The elements $\beta_1, \ldots, \beta_N$ are the positive roots $\mathfrak{g}$. Define elements of $\mathfrak{U}_h\mathfrak{g}$ by

$$X_{\beta_1}^{\pm} = X_{i_1}^{\pm}, \quad X_{\beta_2}^{\pm} = \tau_{i_1}(X_{i_2}^{\pm}), \quad \ldots, \quad X_{\beta_N}^{\pm} = \tau_{i_1}\tau_{i_2} \cdots \tau_{i_{N-1}}(X_{i_N}^{\pm}).$$

These elements *depend* on the choice of the reduced decomposition. They are analogues of the elements X_β and $X_{-\beta}$ in $\mathfrak{U}\mathfrak{g}$ which are given in II (4.4).

(6.2) Poincaré-Birkhoff-Witt bases of $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, and $\mathfrak{U}_h\mathfrak{n}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, and $\mathfrak{U}_h\mathfrak{n}^+$ be the subalgebras of $\mathfrak{U}_h\mathfrak{g}$ defined in (1.1). The following bases of $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, $\mathfrak{U}_h\mathfrak{n}^+$, and $\mathfrak{U}_h\mathfrak{g}$ are analogues of the Poincaré-Birkhoff-Witt bases of $\mathfrak{U}\mathfrak{n}^-$, $\mathfrak{U}\mathfrak{h}$, and $\mathfrak{U}\mathfrak{n}^+$ which are given in II (4.4).

Theorem. *Let $X_{\beta_1}^{\pm}, \ldots, X_{\beta_N}^{\pm}$ be the elements of $\mathfrak{U}_h\mathfrak{g}$ defined in (6.1). Then*

$$\{(X_{\beta_1}^{+})^{p_1}(X_{\beta_2}^{+})^{p_2} \cdots (X_{\beta_N}^{+})^{p_N} \mid p_1, \ldots, p_N \in \mathbb{Z}_{\geq 0}\} \quad \text{is a basis of } \mathfrak{U}_h\mathfrak{n}^+,$$

$$\{(X_{\beta_1}^{-})^{n_1}(X_{\beta_2}^{-})^{n_2} \cdots (X_{\beta_1}^{-})^{n_N} \mid n_1, \ldots, n_N \in \mathbb{Z}_{\geq 0}\} \quad \text{is a basis of } \mathfrak{U}_h\mathfrak{n}^-,$$

$$\{H_1^{s_1} H_2^{s_2} \cdots H_1^{s_r} \mid s_1, \ldots, s_N \in \mathbb{Z}_{\geq 0}\} \quad \text{is a basis of } \mathfrak{U}_h\mathfrak{h}.$$

(6.3) The PBW-bases of $\mathfrak{U}_h n^-$ and $\mathfrak{U}_h\mathfrak{n}^+$ are dual bases with respect to $\langle , \rangle$ (almost)

Recall the pairing between $\mathfrak{U}_h\mathfrak{b}^-$ and $\mathfrak{U}_h\mathfrak{b}^+$ given in (2.1).

Theorem. *Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of the longest element of the Weyl group and let β_j and $X^{\pm}_{\beta_j}$, $1 \le j \le N$, be the elements defined in (6.1). Let $p_1, \ldots, p_N, n_1, \ldots, n_N \in \mathbb{Z}_{\ge 0}$. Then*

$$\langle (X^-_{\beta_1})^{n_1}(X^-_{\beta_2})^{n_2}\cdots(X^-_{\beta_N})^{n_N}, (X^+_{\beta_1})^{p_1}(X^+_{\beta_2})^{p_2}\cdots(X^+_{\beta_N})^{p_N}\rangle = \prod_{j=1}^N \delta_{n_j,p_j}\langle (X^-_{i_j})^{n_j}, (X^+_{i_j})^{n_j}\rangle,$$

where δ_{n_j,p_j} is the Kronecker delta.

Furthermore, we have that, for each $1 \le i \le r$,

$$\langle (X_i^-)^n, (X_i^+)^n\rangle = (-1)^n q^{-d_i n(n-1)/2}\frac{[n]_{q^{d_i}}!}{(q^{d_i}-q^{-d_i})^n}, \qquad \text{where } q = e^h.$$

7. The quantum group is a quantum double (almost)

(7.1) The identification of $(\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}}$ with $\mathfrak{U}_h\mathfrak{b}^-$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Define

$$\mathfrak{U}_h\mathfrak{b}^- = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^- \text{ and } H_1, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{b}^+ = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+ \text{ and } H_1, \ldots, H_r,$$

except let us distinguish the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^+$ from the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^-$ by writing H_i^+ and H_i^- respectively, instead of just H_i in both cases.

The nondegeneracy of the pairing $\langle , \rangle$ between $\mathfrak{U}_h\mathfrak{b}^+$ and $\mathfrak{U}_h\mathfrak{b}^-$ (see (2.1)) shows that $\mathfrak{U}_h\mathfrak{b}^-$ is essentially the dual of $U_h\mathfrak{b}^+$. Furthermore, it follows from the conditions

$$\langle x_1x_2, y\rangle = \langle x_1\otimes x_2, \Delta_h(y)\rangle \quad \text{and} \quad \langle x, y_1y_2\rangle = \langle \Delta^{\mathrm{op}}(x), y_1\otimes y_2\rangle$$

that the multiplication in $\mathfrak{U}_h\mathfrak{b}^-$ is the adjoint of the comultiplication in $\mathfrak{U}_h\mathfrak{b}^+$ and the opposite of the comultiplication in $\mathfrak{U}_h\mathfrak{b}^-$ is the adjoint of the multiplication in $\mathfrak{U}_h\mathfrak{b}^+$. Thus (here we are fudging a bit since $\mathfrak{U}_h\mathfrak{b}^+$ is infinite dimensional),

$$\mathfrak{U}_h\mathfrak{b}^- \simeq (\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}} \qquad \text{as Hopf algebras,}$$

where $(\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}}$ is the Hopf algebra defined in I (5.2).

(7.2) Recalling the quantum double

Recall, from I (5.3), that the quantum double $D(A)$ of a finite dimensional Hopf algebra A is the new Hopf algebra

$$D(A) = \{\, a\alpha \mid a \in A,\ \alpha \in A^{*\mathrm{coop}} \} \cong A \otimes A^{*\mathrm{coop}}$$

with multiplication determined by the formulas

$$\alpha a = \sum_{\alpha,a} \langle \alpha_{(1)}, S^{-1}(a_{(1)})\rangle \langle \alpha_{(3)}, a_{(3)}\rangle a_{(2)}\alpha_{(2)}, \quad \text{and}$$

$$a\alpha = \sum_{\alpha,a} \langle \alpha_{(1)}, a_{(1)}\rangle \langle \alpha_{(3)}, S^{-1}(a_{(3)})\rangle \alpha_{(2)}a_{(2)},$$

where, if Δ is the comultiplication in A and $A^{*\mathrm{coop}}$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad \text{and} \quad (\Delta \otimes \mathrm{id}) \circ \Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}.$$

The comultiplication $D(A)$ is determined by the formula

$$\Delta(a\alpha) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)},$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}$.

(7.3) The relation between $D(\mathfrak{U}_h\mathfrak{b}^+)$ and $\mathfrak{U}_h\mathfrak{g}$

With the definition of the quantum double in mind it is natural that we should define the quantum double of $\mathfrak{U}_h\mathfrak{b}^+$ to be the algebra

$$D(\mathfrak{U}_h\mathfrak{b}^+) = (\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}} \otimes \mathfrak{U}_h\mathfrak{b}^+ \cong \mathfrak{U}_h\mathfrak{b}^- \otimes \mathfrak{U}_h\mathfrak{b}^+$$

with multiplication and comultiplication given by the formulas in (7.2). The following theorem says that the quantum group $\mathfrak{U}_h\mathfrak{g}$ is almost the quantum double of $\mathfrak{U}_h\mathfrak{b}^+$, in other words, $\mathfrak{U}_h\mathfrak{g}$ is almost completely determined by pasting two copies of $\mathfrak{U}_h\mathfrak{b}^+$ together.

Theorem. *Let $(B_{ij}) = C^{-1}$ be the inverse of the Cartan matrix corresponding to $\mathfrak{g}$ and, for each $1 \le i \le r$, define*

$$H_i^* = \sum_{j=1}^{r} B_{ij}H_j \ \in \mathfrak{U}_h\mathfrak{g}.$$

(a) There is a surjective homomorphism $\phi\colon D(\mathfrak{U}_h\mathfrak{b}^+) \longrightarrow \mathfrak{U}_h\mathfrak{g}$ determined by

$$\begin{array}{rccc} \phi\colon & D(\mathfrak{U}_h\mathfrak{b}^+) & \longrightarrow & \mathfrak{U}_h\mathfrak{g} \\ & X_i^+ & \longmapsto & X_i^+ \\ & H_i^+ & \longmapsto & H_i \\ & X_i^- & \longmapsto & X_i^- \\ & H_i^- & \longmapsto & H_i^* \end{array} \qquad \text{and thus} \qquad \frac{D(\mathfrak{U}_h\mathfrak{b}^+)}{\ker\phi} \cong \mathfrak{U}_h\mathfrak{g}.$$

(Recall (7.1) that we distinguish the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^+$ from the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^-$ by writing H_i^+ and H_i^- respectively, instead of just H_i in both cases.)
(b) The ideal $\ker\phi$ *is the ideal generated by the relations*

$$H_i^- - \left(\sum_{j=1}^r B_{ij}H_j^+\right), \qquad \text{where } 1 \le i \le r.$$

(7.4) Using the $\mathcal{R}$-matrix of $D(\mathfrak{U}_h\mathfrak{b}^+)$ to get the $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$
Recall (7.2) that the double $D(\mathfrak{U}_h\mathfrak{b}^+)$ comes with a natural universal $\mathcal{R}$-matrix given by

$$\tilde{\mathcal{R}} = \sum_i b_i \otimes b^i,$$

where the sum is over a basis $\{b_i\}$ of $\mathfrak{U}_h\mathfrak{b}^+$ and $\{b^i\}$ is the dual basis in $\mathfrak{U}_h\mathfrak{b}^-$ with respect to the form $\langle,\rangle$ given in (2.1). We have used the notation $\tilde{\mathcal{R}}$ here to distinguish it from the element $\mathcal{R}$ in Theorem (3.2). The element $\tilde{\mathcal{R}}$ is not exactly in the tensor product $D(\mathfrak{U}_h\mathfrak{b}^+) \otimes D(\mathfrak{U}_h\mathfrak{b}^+)$ but if we make the tensor product just a tiny bit bigger by taking the h-adic completion $D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+)$ of $D(\mathfrak{U}_h\mathfrak{b}^+) \otimes D(\mathfrak{U}_h\mathfrak{b}^+)$ then we do have

$$\tilde{\mathcal{R}} \in D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+).$$

The image of $\tilde{\mathcal{R}}$ under the homomorphism

$$\begin{array}{rccc} \phi\otimes\phi\colon & D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+) & \longrightarrow & \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g} \\ & \tilde{\mathcal{R}} & \longmapsto & \mathcal{R} \end{array}$$

coincides with the element $\mathcal{R}$ given in Theorem (3.2). This means that we actually get the element $\mathcal{R}$ in Theorem (3.2) for free by realising the quantum group as a quantum double (almost).

8. The quantum Serre relations occur naturally

In this section we will see that the most complicated of the defining relations in the quantum group can be obtained in quite a natural way. More specifically, the ideal generated by them is the radical of a certain bilinear form.

(8.1) Definition of the algebras $U_h\mathfrak{b}^+$ and $U_h\mathfrak{b}^-$
Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix.

Let $\mathbf{U}_h\mathfrak{b}^+$ be the associative algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by

$$H_1, H_2, \cdots, H_r, \quad X_1^+, X_2^+, \cdots, X_r^+,$$

with relations

$$[H_i, H_j] = 0, \quad \text{and} \quad [H_i, X_j^+] = \alpha_j(H_i)X_j^+, \quad \text{for all } 1 \le i, j \le r,$$

and define an algebra homomorphism $\Delta_h\colon \mathbf{U}_h\mathfrak{b}^+ \to \mathbf{U}_h\mathfrak{b}^+\hat{\otimes}\mathbf{U}_h\mathfrak{b}^+$ by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \text{and} \quad \Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+,$$

where $\mathbf{U}_h\mathfrak{b}^+\hat{\otimes}\mathbf{U}_h\mathfrak{b}^+$ denotes the h-adic completion of the tensor product $\mathbf{U}_h\mathfrak{b}^+ \otimes_{\mathbb{C}[[h]]} \mathbf{U}_h\mathfrak{b}^+$.

Let $\mathbf{U}_h\mathfrak{b}^-$ be the associative algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by

$$X_1^-, X_2^-, \cdots, X_r^-, \quad H_1, H_2, \cdots, H_r,$$

with relations

$$[H_i, H_j] = 0, \quad \text{and} \quad [H_i, X_j^-] = -\alpha_j(H_i)X_j^-, \quad \text{for all } 1 \le i, j \le r,$$

and define an algebra homomorphism $\Delta_h\colon \mathbf{U}_h\mathfrak{b}^- \to \mathbf{U}_h\mathfrak{b}^-\hat{\otimes}\mathbf{U}_h\mathfrak{b}^-$ by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \text{and} \quad \Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-,$$

where $\mathbf{U}_h\mathfrak{b}^-\hat{\otimes}\mathbf{U}_h\mathfrak{b}^-$ denotes the h-adic completion of the tensor product $\mathbf{U}_h\mathfrak{b}^- \otimes_{\mathbb{C}[[h]]} \mathbf{U}_h\mathfrak{b}^-$.

(8.2) The difference between the algebras $\mathbf{U}_h\mathfrak{b}^\pm$ and the algebras $\mathfrak{U}_h\mathfrak{b}^\pm$

The algebras $\mathbf{U}_h\mathfrak{b}^+$ are much larger than the algebras $\mathfrak{U}_h\mathfrak{b}^+$ used in (2.1) since they have fewer relations between the $X_i^\pm$ generators.

(8.3) A pairing between $\mathbf{U}_h\mathfrak{b}^+$ and $\mathbf{U}_h\mathfrak{b}^-$

In exactly the same way that we had a pairing between $\mathfrak{U}_h\mathfrak{b}^+$ and $\mathfrak{U}_h\mathfrak{b}^-$ in (2.1), there is a unique $\mathbb{C}[[h]]$-bilinear pairing

$$\langle , \rangle : \mathbf{U}_h\mathfrak{b}^- \times \mathbf{U}_h\mathfrak{b}^+ \longrightarrow \mathbb{C}[[h]] \quad \text{which satisfies}$$

(a) $\langle 1, 1\rangle = 1$,

(b) $\langle H_i, H_j\rangle = \dfrac{\alpha_j(H_i)}{d_j}$,

(c) $\langle X_i^-, X_j^+\rangle = \delta_{ij}\dfrac{h}{e^{d_i h} - e^{-d_i h}}$,

(d) $\langle ab, c\rangle = \langle a \otimes b, \Delta_h(c)\rangle$, for all $a, b \in \mathbf{U}_h\mathfrak{b}^-$ and $c \in \mathbf{U}_h\mathfrak{b}^+$,

(e) $\langle a, bc\rangle = \langle \Delta_h^{\mathrm{op}}(a), b \otimes c\rangle$, for all $a \in \mathbf{U}_h\mathfrak{b}^-$ and $b, c \in \mathbf{U}_h\mathfrak{b}^+$.

(8.4) The radical of $\langle , \rangle$ is generated by the quantum Serre relations

Let $\mathfrak{r}^-$ and $\mathfrak{r}^+$ be the left and right radicals, respectively, of the form $\langle , \rangle$ defined in (8.3), i.e.

$$\begin{aligned} \mathfrak{r}^- &= \{a \in \mathbf{U}_h\mathfrak{b}^- \mid \langle a, b\rangle = 0 \text{ for all } b \in \mathbf{U}_h\mathfrak{b}^+\}, \quad \text{and} \\ \mathfrak{r}^+ &= \{b \in \mathbf{U}_h\mathfrak{b}^+ \mid \langle a, b\rangle = 0 \text{ for all } a \in \mathbf{U}_h\mathfrak{b}^-\}. \end{aligned}$$

Theorem. *The sets $\mathfrak{r}^-$ and $\mathfrak{r}^+$ are the ideals of $\mathbf{U}_h\mathfrak{b}^-$ and $\mathbf{U}_h\mathfrak{b}^+$ generated by the elements*

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_i h}} (X_i^-)^s X_j^- (X_i^-)^t, \qquad \text{for } i \neq j,$$

and

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_i h}} (X_i^+)^s X_j^+ (X_i^+)^t, \qquad \text{for } i \neq j,$$

respectively.

It follows from this theorem that the quantum group $\mathfrak{U}_h\mathfrak{g}$ is determined by the algebras $\mathbf{U}_h\mathfrak{b}^+$, $\mathbf{U}_h\mathfrak{b}^-$ and the form $\langle , \rangle$. A construction of the quantum group along these lines would be very similar to the standard construction of Kac-Moody Lie algebras (see [K] §1.3).