VIII. Hall algebras

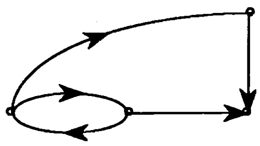
The results in §1 are outlined in [CP] §9.3D. The proof of Theorem (1.2) appears in [BGP] Theorem 3.1 and the proof of Theorem (1.4) appears in [Lu1] Prop. 5.7. The material in §2 is a combination of [Lu2] and [Lu] Part II. In particular, Theorem (2.7)(1) is proved in [Lu] 13.1.2, 12.3.2, and 9.2.7, Theorem (2.7)(2) is proved in [Lu] 13.1.5, 13.1.12e, 12.3.3, and 9.2.11, Theorem (2.7)(3) is proved in [Lu] 13.1.12d and 12.3.6. The statement about the symmetric form given in (2.6) is proved in [Lu] 12.2.2, 9.2.9 and the references given there. The proof of the isomorphism theorem in (2.8) is given in [Lu] 13.2.11 and in [Lu2] Th. 10.17. The material in §3 appears in [Lu1] §9. The isomorphism theorem in (3.4) is stated in [Lu1] 9.6.

1. Hall algebras

The Hall algebra is an algebra which has a basis labeled by representations of quivers and for which the structure constants with respect to this basis reflect the structure of these representations. The Hall algebra encodes a large amount of information about the representations of the quiver. Amazingly, this algebra is almost isomorphic to the nonnegative part of the quantum group.

(1.1) Quivers

A *quiver* is an oriented graph Γ, i.e. a set of vertices and directed edges. The following is an example of a quiver.



Every Dynkin diagram if type A, D or E can be made into a quiver by orienting the edges. Note that there are many possible ways of orienting the edges of a Dynkin diagram in order to make a quiver. For example the quivers

are both obtained by orienting the edges of the Dynkin diagram of type E_6.

(1.2) Representations of a quiver

A *representation* R of a quiver Γ over a field k is a labeling of the graph Γ such that

(1) Each vertex $i \in \Gamma$ is labeled by a vector space R_i over k,

(2) Each edge $i \to j$ in Γ is labeled by a (vector space) homomorphism $\phi_{ij}\colon R_i \to R_j$.

Define morphisms of representations of quivers in the natural way and make the category of representations of the quiver Γ. The *dimension* of a representation R is the vector $\dim(R) = (d_i)$ where, for each vertex $i \in \Gamma$, $d_i = \dim(R_i)$. An *irreducible* representation of Γ is a representation R of Γ such that the only subrepresentations of R are 0 and R.

A representation R of a quiver Γ is *indecomposable* if it cannot be written as $R = S \oplus T$ where S and T are nonzero representations of Γ.

Theorem. *Let Γ be a quiver.* [illegible]

(a) *There are a finite number of indecomposable representations of Γ if and only if Γ is an oriented Dynkin diagram of type A, D or E.*

(b) *If Γ is an oriented Dynkin diagram of type A, D or E then the indecomposable representations of Γ are in 1-1 correspondence with the positive roots for the Lie algebra $\mathfrak{g}$ corresponding to the Dynkin diagram.*

(1.3) Definition of the Hall algebra

Let Γ be a quiver and let $\mathbb{F}_q$ be a finite field with q elements. The *Hall algebra* or *Grothendieck ring* $R\Gamma$ of representations of Γ is the algebra over $\mathbb{C}$ with

(1) basis labeled by the isomorphism classes $[R]$ of representations of Γ over $\mathbb{F}_q$, and

(2) multiplication of two isomorphism classes $[R]$ and $[S]$ given by

$$[R] \cdot [S] = \sum_{[T]} C^T_{RS}[T] \qquad \text{where} \qquad C^T_{RS} = \mathrm{Card}\big(\{P \subseteq T \mid P \cong R,\ T/P \cong S\}\big).$$

[illegible]

(1.4) Connecting Hall algebras to the quantum group

Let Γ be a quiver which is obtained by orienting the edges of a Dynkin diagram of type A, D, or E, and let $\mathbb{F}_q$ be a finite field with q elements. Let us describe explicitly two types of indecomposable representations of Γ.

(1) Let i be a vertex of Γ. The representation

$$e_i \quad \text{given by} \quad V_j = \begin{cases} \mathbb{F}_q, & \text{if } j = i; \\ 0, & \text{if } j \neq i; \end{cases}$$

is an irreducible representation of Γ.

(2) Let $i \to j$ be an edge of Γ. The representation

$$e_{ij} \quad \text{given by} \quad V_\ell = \begin{cases} \mathbb{F}_q, & \text{if } \ell = i \text{ or } \ell = j; \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad \phi_{ij} = \mathrm{id}_{\mathbb{F}_q},$$

is an indecomposable (but not irreducible) representation of Γ.

The following relations hold in the Hall algebra $R\Gamma$,

$$e_{ij} = e_ie_j - e_je_i, \quad e_ie_{ij} = qe_{ij}e_i, \quad e_{ij}e_i = qe_je_{ij}, \quad \text{for each edge } i \to j \text{ in } \Gamma.$$

It is easier to prove the first relation by writing it in the form $e_ie_j = e_{ij} + e_je_i$. Combining the first two of these relations and the first and last of these relations respectively, gives the identities

$$e_i^2e_j - (q+1)e_ie_je_i + qe_je_i^2 = 0 \quad \text{and} \quad e_ie_j^2 - (q+1)e_je_ie_j + qe_j^2e_i = 0, \quad \text{respectively.}$$

We shall make the Hall algebra a bit bigger by adding the $K_i^{\pm 1}$s that are in the quantum group $U_q\mathfrak{g}$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ and let $U_q\mathfrak{g}$ be the rational version of the quantum group with $k = \mathbb{C}$ and $q \in \mathbb{C}$ the number of elements in the field $\mathbb{F}_q$. Let $U_q\mathfrak{h}$ be the subalgebra of $U_q\mathfrak{g}$ generated by $K_1^{\pm 1}, \ldots, K_r^{\pm 1}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots corresponding to the Lie algebra $\mathfrak{g}$ (see II (2.6)). Define

$$\widetilde{R\Gamma} = \text{algebra generated by } R\Gamma \text{ and } K_1^{\pm 1}, \ldots, K_r^{\pm 1} \text{ with the additional relations}$$

$$K_i[R]K_i^{-1} = q^{(\alpha_i, d(R))}[R], \quad \text{for all } 1 \le i \le r \text{ and representations } R \text{ of } \Gamma,$$

where $d(R) = \sum_{j=1}^{r} \dim(R_j)\alpha_j$, and the inner product in the exponent of q is the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ given in II (2.7).

Theorem. *Let Γ be a quiver which is obtained by orienting the edges of a Dynkin diagram of type A, D or E. Let $R\Gamma$ be the Hall algebra of representations of Γ over the finite field $\mathbb{F}_q$ with q elements and let $\widetilde{R\Gamma}$ be the extended Hall algebra defined above. Let $U_q\mathfrak{g}$ be the rational form of the quantum group with $k = \mathbb{C}$ which corresponds to the Dynkin diagram Γ and let*

$$U_q\mathfrak{b}^+ = \text{subalgebra of } U_q\mathfrak{g} \text{ generated by } K_1^{\pm 1}, \ldots, K_r^{\pm 1} \text{ and } E_1, \ldots E_r.$$

Choose elements $z_1, \cdots, z_r \in \mathbb{Z}$ such that $z_i - z_j = 1$ if $i \to j$ is an edge in Γ. Then the homomorphism of algebras determined by

$$\begin{array}{ccc} U_q\mathfrak{b}^+ & \longrightarrow & \widetilde{R\Gamma} \\ & & \\ K_i^{\pm 1} & \longmapsto & K_i^{\pm 1} \\ E_i & \longmapsto & K_i^{z_i}e_i \end{array}$$

is an isomorphism.

2. An algebra of perverse sheaves

In this section we shall construct an algebra $\mathcal{K}$ from a Dynkin diagram Γ. There is a strong relationship between this algebra and the quantum group $U_q\mathfrak{g}$ where $\mathfrak{g}$ is the simple complex Lie algebra corresponding to the Dynkin diagram Γ.

The algebra $\mathcal{K}$ is graded,

$$\mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu,$$

in the same way that the quantum group $U_q\mathfrak{n}^+$ is graded, see VII (1.2). The vector space $\mathcal{K}$ comes with natural shift maps $[n]$ which correspond to multiplication by q^n in the quantum group $U_q\mathfrak{b}^+$. The algebra $\mathcal{K}$ has a natural multiplication which comes from an induction functor and a natural "pseudo-comultiplication" which comes from a restriction functor. The multiplication and the pseudo-comultiplication turn out to be almost the same as the multiplication and the comultiplication on the quantum group $U_q\mathfrak{b}^+$. Lastly, the algebra $\mathcal{K}$ has a natural inner product $\{,\}$ that is related to the inner product $\langle,\rangle$ pairing $U_q\mathfrak{b}^-$ and $U_q b^+$, (see VII (2.1)).

In Theorem (2.8) we shall see that if we extend the algebra $\mathcal{K}$ a little bit, by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$ then we get an algebra $\widetilde{\mathcal{K}}$ such that

$$\widetilde{\mathcal{K}} \simeq U_q\mathfrak{b}^+.$$

This last fact is very similar to the case of the Hall algebra (1.4) where after extending the Hall algebra $R\Gamma$ by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$, we got an algebra $\widetilde{R\Gamma}$ which was also isomorphic to $U_q\mathfrak{b}^+$. We shall see in section 3 that this is not a coincidence, there is a concrete connection between $R\Gamma$ and the algebra $\mathcal{K}$. The advantage of working with the algebra $\mathcal{K}$ instead of the Hall algebra $R\Gamma$ is that $\mathcal{K}$ has more natural structure than $R\Gamma$, it has:

(a) a natural pseudo-comultiplication $r: \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$,

(b) a natural inner product $\{,\}: \mathcal{K} \times \mathcal{K} \to \mathbb{Z}((q))$,

(c) a natural involution $D: \mathcal{K} \to \mathcal{K}$,

(d) a natural basis coming from simple perverse sheaves.

The natural basis coming from simple perverse sheaves is called the *canonical basis.*

(2.1) Γ-graded vector spaces and the varieties E_V with G_V action

Let Γ be a quiver obtained by orienting the edges of a Dynkin diagram of type A, D or E. For convenience we label the vertices by $1, 2, \ldots, r$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ.

Let p be a positive prime integer and let $\overline{\mathbb{F}_p}$ be the algebraic closure of the finite field $\mathbb{F}_p$ with p elements. A *Γ-graded vector space* V over $\overline{\mathbb{F}_p}$ is a labeling of the graph Γ such that each vertex i is labeled by a vector space V_i over $\overline{\mathbb{F}_p}$. The *dimension* of a Γ-graded vector space V is the r-tuple of nonnegative integers $\dim(V) = (\dim(V_i))$. We shall identify dimensions of Γ-graded vector spaces with elements of

$$Q^+ = \sum_i \mathbb{N}\alpha_i \qquad \text{so that} \qquad \dim(V) = \sum_{i=1}^{r} \dim(V_i)\alpha_i,$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots for $\mathfrak{g}$ and $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Fix an element $\nu \in Q^+$ and a Γ-graded vector space V over $\overline{\mathbb{F}_p}$ such that $\dim(V) = \nu$. Define

$$G_V = \prod_i GL(V_i) \qquad \text{and} \qquad E_V = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j),$$

where the sum in the definition of E_V is over all edges of Γ. There is a natural action of G_V on E_V given by

$$g \cdot (\phi_{ij}) = (g_j \phi_{ij} g_i^{-1}), \quad \text{if } (\phi_{ij}) \in E_V \text{ and } g = (g_1, \cdots, g_r) \in G_V.$$

Let $x \in E_V$ and let W be a *Γ-graded subspace* of V, i.e. $W_i \subseteq V_i$ for all vertices i in Γ. The subspace W is *x-stable* if $xW_i \subseteq W_j$ for all edges $i \to j$ in Γ. We shall simply write $W \subseteq V$ if W is a Γ-graded subspace of V and $xW \subseteq W$ if W is x-stable.

(2.2) Definition of the categories $\mathcal{Q}_V$ and $\mathcal{Q}_T \otimes \mathcal{Q}_W$

The reader may skip this definition if it looks like too much to swallow. The only important thing at this stage is that $\mathcal{Q}_V$ is a category of objects and it is contained in a category called $D^b_c(E_V)$.

Let V be a Γ-graded vector space over $\overline{\mathbb{F}_p}$ and let E_V be the variety over $\overline{\mathbb{F}_p}$ defined in (2.1). Let $D^b_c(E_V)$ be the bounded derived category of $\overline{\mathbb{Q}}_l$-(constructible) sheaves on E_V, see IV (1.4). Recall that $D^b_c(E_V)$ comes endowed with shift functors IV (2.4),

$$\begin{array}{rccc} [n]\colon & D^b_c(E_V) & \longrightarrow & D^b_c(E_V) \\ & A & \longmapsto & A[n]. \end{array}$$

Define

$\mathcal{Q}_V$ = the full subcategory of $D^b_c(E_V)$ consisting of finite direct sums of simple perverse sheaves L such that some shift of L is a direct summand of $L_{\vec{\nu}}$ for some partition $\vec{\nu}$ of $\nu = \dim(V)$.

The complexes $L_{\vec{\nu}}$ are defined in (2.7). Let T and W be Γ-graded vector spaces over $\overline{\mathbb{F}_p}$. Define

$\mathcal{Q}_T \otimes \mathcal{Q}_W$ = the complexes $L \in D^b_c(E_T \times E_W)$ such that $L \cong \bigoplus_{i=1}^{s} A_i \otimes B_i$, for some $A_i \in \mathcal{Q}_T$, $B_i \in \mathcal{Q}_W$, and some positive integer s.

This is a subcategory of $D^b_c(E_T \times E_W)$.

(2.3) The Grothendieck group $\mathcal{K}$ associated to the categories $\mathcal{Q}_V$

Let $\nu \in Q^+$ and let V be a Γ-graded vector space of dimension ν. Let $\mathcal{Q}_V$ be as in (2.2). The important thing about $\mathcal{Q}_V$ at the moment is that it is a category related to E_V.

The *Grothendieck group* $\mathcal{K}(\mathcal{Q}_V)$ of the category $\mathcal{Q}_V$ is the $\mathbb{C}(q)$-module generated by the isomorphism classes of objects in $\mathcal{Q}_V$ with the addition operation given by the relations

$$[B_1 \oplus B_2] = [B_1] + [B_2], \quad \text{if } B_1, B_2 \in \mathcal{Q}_V,$$

and multiplication by q given by the relations

$$[B[n]] = q^n[B], \qquad \text{for } B \in \mathcal{Q}_V \text{ and } n \in \mathbb{Z},$$

where the map $B \to B[n]$ is the shift functor on $D^b_c(E_V)$, see IV (2.4). The structure of $\mathcal{K}(\mathcal{Q}_V)$ depends only on the element ν and so we shall often write $\mathcal{K}_\nu$ in place of $\mathcal{K}(\mathcal{Q}_V)$.

Define

$$\mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu.$$

The group $\mathcal{K}$ is graded in the same way that $\mathfrak{U}_q\mathfrak{n}^+$ is graded, see VII (1.2).

(2.4) Definition of the multiplication in $\mathcal{K}$

Let V be a Γ-graded vector space. Let T and W be Γ-graded vector spaces such that

$$W \subseteq V \qquad \text{and} \qquad V/W \cong T.$$

If $x \in E_V$ such that $xW \subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T \cong V/W$ induced by the action of x on V/W. Define

$$\mathcal{S} = \{x \in E_V \mid xW \subseteq W\},$$
$$P = \{g \in G_V \mid gW \subseteq W\}, \qquad U = \{g \in P \mid g_W = \mathrm{id}_W, g_T = \mathrm{id}_T\}.$$

The groups P and U are subgroups of G_V. The group P is the stabilizer of W in G_V, it is a parabolic subgroup of G_V. The group U is the unipotent radical of P.

Let $\mathcal{Q}_T \otimes \mathcal{Q}_W$ be the subcategory of $D^b_c(E_T \otimes E_W)$ which is defined in (2.2). The diagram

$$\begin{array}{ccccccc} E_T \times E_W & \xleftarrow{p_1} & G_V \times_U \mathcal{S} & \xrightarrow{p_2} & G_V \times_P \mathcal{S} & \xrightarrow{p_3} & E_V \\ (x_T, x_W) & \longleftarrow & (g,x) & \longmapsto & (g,x) & \longmapsto & gx \end{array}$$

induces the diagram

$$\mathcal{Q}_T \otimes \mathcal{Q}_W \longrightarrow D^b_c(E_T \times E_W) \xrightarrow{p_1^*} D^b_c(G \times_U \mathcal{S}) \xrightarrow{(p_2)_\flat} D^b_c(G \times_P \mathcal{S}) \xrightarrow{(p_3)_!} D^b_c(E_V)$$

where the first map is the inclusion map.

Theorem. *Let V be a Γ-graded vector space and let E_V be the variety with the G_V action which is defined in (2.1). Let W and T be Γ-graded vector spaces such that $W \subseteq V$ and $V/W \cong T$. Let $\mathcal{Q}_T \otimes \mathcal{Q}_W$ and $\mathcal{Q}_V$ be the categories of complexes of sheaves on $E_T \times E_W$ and E_V, respectively, which are defined in (2.2). There is a well defined functor*

$$\begin{array}{rccc} \mathrm{Ind}^V_{T,W}\colon & \mathcal{Q}_T \otimes \mathcal{Q}_W & \longrightarrow & \mathcal{Q}_V \\ & A & \longmapsto & \big((p_3)_!(p_2)_\flat p_1^* A\big)[\dim(p_1) - \dim(p_2)] \end{array}$$

where p_1, p_2, and p_3 are as defined in the diagram above, $\dim(p_1)$ is the dimension of the fibers of the map p_1, and $\dim(p_2)$ is the dimension of the fibers of the map p_2.

The *multiplication* in $\mathcal{K}$ is defined by the formula

$$[A] \cdot [B] = \big[\mathrm{Ind}^V_{T,W}(A \otimes B)\big], \qquad \text{for } A \in \mathcal{Q}_T \text{ and } B \in \mathcal{Q}_W.$$

With this multiplication $\mathcal{K}$ becomes an algebra. The strange shift by $[\dim(p_1) - \dim(p_2)]$ in the definition of $\mathrm{Ind}^V_{T,W}$ is there to make the multiplication in $\mathcal{K}$ match up with the multiplication in the nonnegative part of the quantum group $U_q\mathfrak{b}^+$, see Theorem (2.8) below.

(2.5) Definition of the pseudo-comultiplication $r\colon \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$

Let V be a Γ-graded vector space. Let T and W be Γ-graded vector spaces such that

$$W \subseteq V \qquad \text{and} \qquad V/W \cong T.$$

If $x \in E_V$ such that $xW \subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T \cong V/W$ induced by the action of x on V/W.

Define

$$\mathcal{S} = \{x \in E_V \mid xW \subseteq W\}$$

and let $\mathcal{Q}_V$ be the subcategory of $D^b_c(E_V)$ which is defined in (2.2). The diagram

$$\begin{array}{ccccc} E_V & \xleftarrow{\iota} & \mathcal{S} & \xrightarrow{\kappa} & E_T \times E_W \\ x & \longleftarrow\!\shortmid & x & \longmapsto & (x_T, x_W) \end{array}$$

induces the diagram

$$\mathcal{Q}_V \longrightarrow D^b_c(E_V) \xrightarrow{\iota^*} D^b_c(\mathcal{S}) \xrightarrow{\kappa_!} D^b_c(E_T \times E_W)$$

where the first map is the inclusion map.

Theorem. *Let V be a Γ-graded vector space and let E_V be the variety with the G_V action which is defined in (2.1). Let W and T be Γ-graded vector spaces such that $W \subseteq V$ and $V/W \cong T$. Let $\mathcal{Q}_T \otimes \mathcal{Q}_W$ and $\mathcal{Q}_V$ be the categories of complexes of sheaves on $E_T \times E_W$ and E_V, respectively, which are defined in (2.2). There is a well defined functor*

$$\begin{array}{rccc} \mathrm{Res}^V_{T,W}\colon & \mathcal{Q}_V & \longrightarrow & \mathcal{Q}_T \otimes \mathcal{Q}_W \\ & B & \longmapsto & (\kappa_! \iota^* B)[\dim(p_1) - \dim(p_2) - 2\dim(G_V/P)] \end{array}$$

where p_1, p_2, κ, and ι are as defined above, $\dim(p_1)$ is the dimension of the fibers of the map p_1, $\dim(p_2)$ is the dimension of the fibers of the map p_2, and P is the parabolic subgroup of G_V defined in (2.4).

The *pseudo-comultiplication* on $\mathcal{K}$ is the map $r\colon \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ defined by

$$r([A]) = \left[\mathrm{Res}^V_{T,W}(A)\right], \qquad \text{if } A \in \mathcal{Q}_V.$$

The strange shift by $[\dim(p_1) - \dim(p_2) - 2\dim(G_V/P)]$ in the definition of $\mathrm{Res}^V_{T,W}$ is there to make the pseudo-comultiplication in $\mathcal{K}$ match up with the comultiplication in the nonnegative part of the quantum group $U_q\mathfrak{b}^+$, see Theorem (2.8) below.

(2.6) The symmetric form on $\mathcal{K}$

Recall that we write $\mathcal{K}_\nu$ in place of $\mathcal{K}(\mathcal{Q}_V)$ since the structure of $\mathcal{K}(\mathcal{Q}_V)$ depends only on ν. For each $\nu \in Q^+$, define a bilinear form

$$\{,\}_\nu\colon \mathcal{K}_\nu \times \mathcal{K}_\nu \to \mathbb{C}(q) \qquad \text{by defining}$$

$$\{[B_1],[B_2]\}_\nu = \sum_j q^{-j} \dim\left(\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))\right),$$

for $B_1, B_2 \in \mathcal{Q}_V$. The vector spaces $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))$ are defined in (2.10) below. At this stage the important thing is that they depend only on B_1, B_2 and j.

Use the forms $\{,\}_\nu$, $\nu \in Q^+$, to define a bilinear form

$$\{,\}\colon \mathcal{K} \times \mathcal{K} \to \mathbb{Z}((q)) \qquad \text{on } \mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu \qquad \text{by setting}$$

$$\begin{aligned} \{\mathcal{K}_\mu, \mathcal{K}_\nu\} &= 0, && \text{if } \mu,\nu \in Q^+ \text{ such that } \mu \neq \nu, \text{ and} \\ \{x,y\} &= \{x,y\}_\nu, && \text{if } x,y \in \mathcal{K}_\nu. \end{aligned}$$

Theorem. *Let V be a Γ-graded vector space and let T and W be Γ-graded subspaces such that $W \subseteq V$ and $T \cong V/W$. Let $A \in \mathcal{Q}_T \otimes \mathcal{Q}_W$ and let $B \in \mathcal{Q}_V$. Then*

$$\{\, A \,,\, \mathrm{Res}^V_{T,W}(B) \,\} = \{\, \mathrm{Ind}^V_{T,W}(A) \,,\, B \,\}$$

The result in this theorem is an analogue of the property of the bilinear form $\langle , \rangle$ on the quantum group which is given in VII (2.1)(d).

(2.7) Definition of the elements $L_{\vec{\nu}} \in \mathcal{K}$

Let $\nu \in Q^+$ and let V be a Γ-graded subspace of dimension ν. A *partition* of ν is a sequence $\vec{\nu} = (\nu^1, \ldots, \nu^m)$ of elements of the root lattice Q such that

(1) each ν^j, $1 \le j \le m$, is a nonnegative integer multiple of a simple root, and

(2) $\nu^1 + \cdots + \nu^m = \nu$.

For example we might have $\vec{\nu} = (3\alpha_1, 2\alpha_3, 0, \alpha_1, 2\alpha_1)$ if $\nu = 6\alpha_1 + 2\alpha_3$. A *flag of type $\vec{\nu}$ in V* is a sequence

$$f = (V = V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V^{(m)} = 0)$$

of Γ-graded subspaces of V such that $\dim(V^{(\ell-1)}/V^{(\ell)}) = \nu^\ell$, for all $1 \le \ell \le m$.

Let $x \in E_V$. A flag f is *x-stable* if $xV^{(\ell)} \subseteq V^{(\ell)}$ for all $1 \le \ell \le m$. Define

$$\mathcal{F}_{\vec{\nu}} = \{(x, f) \mid x \in E_V,\ f \text{ is an } x\text{-stable flag of type } \vec{\nu} \text{ in } V\}.$$

The map

$$\begin{array}{ccc} \mathcal{F}_{\vec{\nu}} & \xrightarrow{\pi_{\vec{\nu}}} & E_V \\ (x,f) & \longmapsto & x \end{array} \qquad \text{induces a map} \qquad D^b_c(\mathcal{F}_{\vec{\nu}}) \xrightarrow{(\pi_{\vec{\nu}})_!} D^b_c(E_V).$$

Let $f(\vec{\nu}) = \dim(\mathcal{F}_{\vec{\nu}})$ and define

$$L_{\vec{\nu}} = \big((\pi_{\vec{\nu}})_! \mathbf{1}\big)[\dim(\mathcal{F}_{\vec{\nu}})], \qquad \text{i.e.}$$

$$\begin{array}{ccccc} D^b_c(\mathcal{F}_{\vec{\nu}}) & \xrightarrow{(\pi_{\vec{\nu}})_!} & D^b_c(E_V) & \xrightarrow{[\dim(\mathcal{F}_{\vec{\nu}})]} & D^b_c(E_V) \\ \mathbf{1} & \longmapsto & & \longmapsto & L_{\vec{\nu}} \end{array}$$

where $\mathbf{1}$ is the constant sheaf on $\mathcal{F}_{\vec{\nu}}$ and $[\dim(\mathcal{F}_{\vec{\nu}})]$ is a shift, see IV (2.4).

Theorem. *Let V be a Γ-graded vector space of dimension ν and let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$.*

(1) Let $\vec{\tau}$ and $\vec{\omega}$ be partitions of $\dim(T)$ *and* $\dim(W)$*, respectively. Then*

$$\mathrm{Ind}^V_{T,W}(L_{\vec{\tau}} \otimes L_{\vec{\omega}}) = L_{\vec{\tau}\vec{\omega}},$$

where, if $\vec{\tau} = (\tau^1, \tau^2, \ldots, \tau^s)$ *and* $\vec{\omega} = (\omega^1, \ldots, \omega^t)$, *then* $\vec{\tau}\vec{\omega} = (\tau^1, \ldots, \tau^s, \omega^1, \ldots, \omega^t)$.

(2) Let $\vec{\nu}$ *be a partition of* $\dim(V)$. *Then*

$$\mathrm{Res}^V_{T,W} L_{\vec{\nu}} \cong \bigoplus_{\vec{\tau},\vec{\omega}} (L_{\vec{\tau}} \otimes L_{\vec{\omega}})[M'(\vec{\tau},\vec{\omega})],$$

where the sum is over all $\vec{\tau}, \vec{\omega}$ *such that* $\vec{\tau}$ *is a partition of* $\dim(T)$, $\vec{\omega}$ *is a partition of* $\dim(W)$ *and* $\vec{\tau} + \vec{\omega} = \vec{\nu}$. *The positive integer* $M'(\vec{\tau},\vec{\omega})$ *is defined in (2.9) below.*

(3) Let $\nu = \alpha_i$ *be a simple root for* $\mathfrak{g}$ *and let* V *be a* Γ*-graded subspace such that* $\dim(V) = \alpha_i$. *Define* $L_i \in \mathcal{K}(\mathcal{Q}_V)$ *by* $L_i = L_{\vec{\nu}}$ *where* $\vec{\nu} = (\alpha_i)$. *Then*

$$\{\, [L_i]\, , \, [L_i]\, \} \;=\; \frac{1}{1-q^2}.$$

(2.8) The connection between $\mathcal{K}$ and the quantum group

We shall make the algebra

$$\mathcal{K} = \bigoplus_{\nu\in Q^+} \mathcal{K}_\nu$$

a bit bigger by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ and let $U_q\mathfrak{g}$ be the rational version of the quantum group with $k = \mathbb{C}(q)$ where q is an indeterminate. Let $U_q\mathfrak{h}$ be the subalgebra of $U_q\mathfrak{g}$ generated by $K_1^{\pm1}, \ldots, K_r^{\pm1}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots corresponding to the Lie algebra $\mathfrak{g}$. Define

$\widetilde{\mathcal{K}}$ = algebra generated by $\mathcal{K}$ and $K_1^{\pm1}, \ldots, K_r^{\pm1}$ with the additional relations

$$K_i x K_i^{-1} = q^{(\alpha_i,\nu)}x, \quad \text{for all } 1 \le i \le r \text{ and all } x \in \mathcal{K}_\nu,$$

where the inner product in the exponent of q is the inner product on $\mathfrak{h}^*_{\mathbb{R}}$ given in II (2.7).

Define a map $j^+ \colon \mathcal{K} \otimes \mathcal{K} \to \widetilde{\mathcal{K}} \otimes \widetilde{\mathcal{K}}$ by

$$j^+(x \otimes y) = xK_1^{\nu_1} \cdots K_r^{\nu_r} \otimes y, \quad \text{if } x \in \mathcal{K} \text{ and } y \in \mathcal{K}_\nu, \text{ where } \nu = \textstyle\sum_i \nu_i\alpha_i.$$

Use the map j^+ and the pseudo-comultiplication $r \colon \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ defined in (2.5) to define a *coproduct* on $\widetilde{\mathcal{K}}$ by

$$\begin{array}{rcll} \Delta\colon \quad \widetilde{\mathcal{K}} & \longrightarrow & \widetilde{\mathcal{K}} \otimes \widetilde{\mathcal{K}} & \\ & & & \\ K_i^{\pm1} & \longmapsto & K_i^{\pm1} \otimes K_i^{\pm1} & \text{for } 1 \le i \le r, \\ x & \longmapsto & j^+ r(x) & \text{for } x \in \mathcal{K}, \end{array}$$

where $r: \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is the pseudo-comultiplication defined in (2.5). Then $\widetilde{\mathcal{K}}$ is a Hopf algebra!

Theorem. *Let L_i be as defined in Theorem (2.7b). The algebra homomorphism determined by*

$$\begin{array}{rccc} \mathcal{I}: & \widetilde{\mathcal{K}} & \longrightarrow & U_q\mathfrak{b}^+ \\ \\ & L_i & \longmapsto & E_i \\ & K_i^{\pm 1} & \longmapsto & K_i^{\pm 1} \end{array}$$

is an isomorphism of Hopf algebras.

(2.9) Dictionary between $\mathcal{K}$ and $U_q\mathfrak{b}^+$

Let us make a small dictionary between the algebra $\mathcal{K}$ and the quantum group $U_q\mathfrak{b}^+$. Our intent is to describe, conceptually, the correspondence between the structures inherent in the algebra $\mathcal{K}$ and the structures in the quantum group $U_q\mathfrak{b}^+$. The map $\mathcal{I}$ is the isomorphism given in Theorem (2.8).

$\widetilde{\mathcal{K}}$	is isomorphic to	$U_q\mathfrak{b}^+$.
$\widetilde{\mathcal{K}}$ is the algebra generated by $\mathcal{K}$ and the $K_i^{\pm 1}$s.	Similarly,	$U_q\mathfrak{b}^+$ is the algebra generated by $U_q\mathfrak{n}^+$ and the $K_i^{\pm 1}$s.
$\mathcal{K}$ is graded, $\mathcal{K} = \bigoplus_{\nu\in Q^+} \mathcal{K}_\nu$.	Similarly,	$U_q\mathfrak{n}^+$ is graded, $U_\mathfrak{n}^+ = \bigoplus_{\nu\in Q^+} (U_q\mathfrak{n}^+)_\nu$.
The shift functor $[n]$ gives rise to multiplication by q^n in $\mathcal{K}$	which corresponds to	multiplication by q^n in $U_q\mathfrak{b}^+$.
The functor $\mathrm{Ind}_{T,W}^V$	corresponds to	the multiplication in $U_q\mathfrak{n}^+$.
The functor $\mathrm{Res}_{T,W}^V$	corresponds to	the comultiplication in $U_q\mathfrak{b}^+$.
The inner product $\{,\}$	corresponds to	the bilinear form $\langle,\rangle$ pairing $U_q\mathfrak{b}^-$ and $U_q\mathfrak{b}^+$.
A partition $\vec{\nu} = (\nu_1\alpha_{i_1}, \ldots, \nu_l\alpha_{i_l})$ indexes $L_{\vec{\nu}}$	which maps, under $\mathcal{I}$, to	$E_{i_1}^{(\nu_1)} \ldots E_{i_l}^{(\nu_l)}$ where $E_i^{(n)} = E_i^n/[n]!$.
The Verdier duality functor D	corresponds to	the $\mathbb{C}$-algebra involution $^-\colon U_q\mathfrak{n}^+ \to U_q\mathfrak{n}^+$ which sends $q \mapsto q^{-1}$ and $E_i \mapsto E_i$.
The simple perverse sheaves in the various $\mathcal{Q}_V$	map, under $\mathcal{I}$, to	a canonical basis in $U_q\mathfrak{n}^+$.

(2.10) Definition of the constant $M'(\tau, \omega)$ which was used in (2.7)

Let V be a Γ-graded vector space and let T and W be Γ-graded subspaces such that $W \subseteq V$ and $T \cong V/W$. If $x \in E_V$ such that $xW \subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T \cong V/W$ induced by the action of x on V/W. Let $\vec{\nu}$ be a partition of $\dim(V)$. If

$$f = (\ V = V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V^{(m)} = 0\ )$$

is a flag of type $\vec{\nu}$ in V then define

$$f_W = (\ (V\cap W) = (V^0 \cap W) \supseteq (V^{(1)}\cap W) \supseteq \cdots \supseteq (V^{(m)} \cap W) = 0\ ) \qquad \text{and}$$
$$f_T = (\ p(V) = p(V^{(0)}) \supseteq p(V^{(1)}) \supseteq \cdots \supseteq p(V^{(m)}) = 0\ ) \qquad \text{where } p\colon V \to V/W$$

is the canonical projection.

Let $\vec{\tau}$ be a partition of $\dim(T)$ and let $\vec{\omega}$ be a partition of $\dim(W)$, such that $\vec{\tau}+\vec{\omega}=\vec{\nu}$. Define

$$\tilde{F}(\vec{\tau},\vec{\omega})=\left\{(x,f)\;\middle|\; xW\subseteq W,\; \begin{matrix} f \text{ is an } x\text{-stable flag of type } \vec{\nu} \text{ in } V, \\ \text{and } f_W \text{ is a flag of type } \vec{\omega} \text{ in } W\end{matrix}\right\}.$$

Define a map

$$\begin{array}{rccc} \alpha\colon & \tilde{F}(\vec{\tau},\vec{\omega}) & \longrightarrow & \mathcal{F}_{\vec{\tau}}\times\mathcal{F}_{\vec{\omega}} \\ & (x,f) & \longmapsto & ((x_T,f_T),(x_W,f_W)) \end{array}$$

and define

$$M'(\tau,\omega)=\dim(p_1)-\dim(p_2)-2\dim(G_V/P)+\dim(\mathcal{F}_{\vec{\nu}})-\dim(\mathcal{F}_{\vec{\tau}})-\dim(\mathcal{F}_{\vec{\omega}})-2\dim(\alpha).$$

where p_1 and p_2 are the maps given in (2.4), P is the parabolic subgroup of G_V defined in (2.4), and $\dim(p_1)$, $\dim(p_2)$ and $\dim(\alpha)$ are the dimensions of the fibers of the maps p_1, p_2, and α, respectively.

(2.11) Definition of the vector spaces $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^*B_1\otimes t_\flat s^*B_2))$ from (2.6)

Let Ω be a smooth irreducible algebraic variety with a free action of G_V such that the $\overline{\mathbb{Q}}_l$-cohomology of Ω is zero in degrees $1,2,\ldots,m$ where m is a large integer. Consider the diagram

$$\begin{array}{ccccc} E_V & \xleftarrow{s} & \Omega\times E_V & \xrightarrow{t} & G\backslash(\Omega\times E_V) \\ x & \longleftarrow & (\omega,x) & \longmapsto & G_V(\omega,x) \end{array}\quad \text{and the diagram}\quad G_V\backslash(\Omega\times E_V)\xrightarrow{u}\{\text{point}\}.$$

These diagrams induce diagrams

$$\begin{array}{l} D^b_c(E_V)\xrightarrow{s^*}D^b_c(\Omega\times E_V)\xrightarrow{t_\flat}D^b_c(G\backslash(\Omega\times E_V)) \qquad \text{and} \\ D^b_c(G_V\backslash(\Omega\times E_V))\xrightarrow{u_!}D^b_c(\{\text{point}\}). \end{array}$$

With these notations one has that $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^*B_1\otimes t_\flat s^*B_2))$ is a sheaf on the space $\{\text{point}\}$, i.e. a $\overline{\mathbb{Q}}_l$-vector space.

(2.12) Some remarks on Part II of Lusztig's book

The construction of the algebra $\mathcal{K}$ and the relationship between it and the quantum group is detailed in Lusztig's book [Lu]. Lusztig works in much more generality there.

(1) Lusztig allows Γ to be an arbitrary quiver, rather than just a quiver gotten by orienting a Dynkin diagram of type A, D or E. It does not require any more theory than what we have already outlined in order to define the algebra $\mathcal{K}$ in this more general setting.

(2) Lusztig wants to construct algebras $\mathcal{K}$ which will be isomorphic to the nonnegative parts of the quantum groups corresponding to general Dynkin diagrams. In order to do this he must first consider only diagrams with single bonds and then 'fold' the

diagram by analyzing the action of an automorphism of the diagram. The addition of the folding automorphism into the theory is a nontrivial extension of what we have developed in these notes.

(3) We have ignored the effect of the orientation of the quiver. If one wants to compare the algebras $\mathcal{K}$ that are obtained by orienting the same quiver in different ways one must analyze a Fourier-Deligne transform between these two different algebras. The amazing thing is that, after one extends the algebras by adding the $K_i^{\pm 1}$s that are in the quantum group, the two different algebras (from the different orientations) become isomorphic!

3. The connection between representations of quivers and perverse sheaves

(3.1) Correspondence between orbits and isomorphism classes of representations of Γ

Let Γ be a quiver obtained by orienting the edges of a Dynkin diagram of type A, D or E. For convenience we label the vertices by $1, 2, \ldots, r$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ.

Let p be a positive prime integer and let $\overline{\mathbb{F}_p}$ be the algebraic closure of the finite field $\mathbb{F}_p$ with p elements. Fix an element $\nu \in Q^+$ (see VII (1.2)) and a Γ-graded vector space V over $\overline{\mathbb{F}_p}$ such that $\dim(V) = \nu$. Define

$$G_V = \prod_i GL(V_i) \qquad \text{and} \qquad E_V = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j),$$

where the sum in the definition of E_V is over all edges of Γ. The natural action of G_V on E_V is given by

$$g \cdot (\phi_{ij}) = (g_j \phi_{ij} g_i^{-1}), \quad \text{if } (\phi_{ij}) \in E_V \text{ and } g = (g_1, \cdots, g_r) \in G_V.$$

The group G_V is an algebraic group over $\overline{\mathbb{F}_p}$ and E_V is a variety over $\overline{\mathbb{F}_p}$ with a G_V action. Each element $(\phi_{ij}) \in E_V$ determines a representation of Γ of dimension $\dim(V)$. Each G_V-orbit in E_V determines an isomorphism class of representations of Γ. Let us make this correspondence precise.

An *orbit index for V* is a sequence of positive integers labeled by the positive roots

$$\vec{c} = (c_\alpha)_{\alpha \in R^+} \quad \text{such that} \quad \sum_{\alpha \in R^+} c_\alpha \alpha = \dim(V),$$

where R^+ is the set of positive roots for $\mathfrak{g}$. For each orbit index $\vec{c}$ for V define a representation of Γ by

$$R_{\vec{c}} = \bigoplus_{\alpha \in R^+} e_\alpha^{\oplus c_\alpha} \qquad \text{and let} \qquad \mathcal{O}_{\vec{c}} = \text{ the } G_V\text{-orbit in } E_V \text{ corresponding to } R_{\vec{c}},$$

where e_α is the indecomposable representation of Γ indexed by the positive root α, see Theorem (1.2b). Then we have a one-to-one correspondence

$$\begin{array}{ccc} G_V \text{ orbits in } E_V & \stackrel{1-1}{\longleftrightarrow} & \text{isomorphism classes of representations of } \Gamma \text{ of dimension } \nu \\ \mathcal{O}_{\vec{c}} & \longleftrightarrow & [R_{\vec{c}}] \end{array}$$

(3.2) Realizing the structure constants of the Hall algebra in terms of orbits

Let q be a power of the prime p. Since E_V is a variety over $\overline{\mathbb{F}_p}$ there is an action of the the qth power Frobenius map F on E_V, see [Ca] p. 503. If X is a subset of E_V then let X^F denote the set of points of X which are fixed under the action of the Frobenius map F.

Let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$. Recall the diagram

$$\begin{array}{ccccccc} E_T \times E_W & \stackrel{p_1}{\longleftarrow} & G_V \times_U \mathcal{S} & \stackrel{p_2}{\longrightarrow} & G_V \times_P \mathcal{S} & \stackrel{p_3}{\longrightarrow} & E_V \\ (x_T, x_W) & \longleftarrow\!\shortmid & (g,x) & \longmapsto & (g,x) & \longmapsto & gx \end{array}$$

given in (2.4). Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be orbit indices for T, W and V, respectively. Then we have

$$\begin{array}{ccccccc} E_T \times E_W & \stackrel{p_1}{\longleftarrow} & G_V \times_U \mathcal{S} & \stackrel{p_2}{\longrightarrow} & G_V \times_P \mathcal{S} & \stackrel{p_3}{\longrightarrow} & E_V \\ \mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}} & \leftrightarrow & p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}}) & \longmapsto & p_2(p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}})) & & \\ & & & & p_3^{-1}(\mathcal{O}_{\vec{c}}) & \longleftrightarrow & \mathcal{O}_{\vec{c}} \end{array}$$

Let $M = R_{\vec{a}}$, $N = R_{\vec{b}}$ and $P = R_{\vec{c}}$ be the representations of Γ given in (3.1). By a direct count, we have

$$C^P_{M,N} = \mathrm{Card}\left(\left(p_2(p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}})) \bigcap p_3^{-1}(\mathcal{O}_{\vec{c}}) \right)^F \right).$$

where $C^P_{M,N}$ are the structure coefficients of the Hall algebra $R\Gamma$ given in (1.3).

(3.3) Rewriting the Hall algebra in terms of functions constant on orbits

Let q be a power of the prime p. On any variety Y over $\overline{\mathbb{F}_p}$ there is an action of the the qth power Frobenius map F on E_V, see [Ca] p. 503. If X is a subset of Y then X^F denotes the set of points of X which are fixed under the action of the Frobenius map F.

Let l be a positive prime number, invertible in $\overline{\mathbb{F}_p}$. Let $\overline{\mathbb{Q}_l}$ be the algebraic closure of the field of l-adic numbers. Define

K_ν = the vector space of $\overline{\mathbb{Q}_l}$-valued functions on $(E_V)^F$ which are constant on the orbits $(\mathcal{O}_{\vec{c}})^F$ for all orbit indexes $\vec{c}$ for V.

Define

$$K = \bigoplus_{\nu \in Q^+} K_\nu,$$

where Q^+ is as in VII (1.2).

Define a multiplication on K as follows. Let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$. Recall the diagram

$$\begin{array}{ccccccc} E_T \times E_W & \xleftarrow{p_1} & G_V \times_U \dot{\mathcal{S}} & \xrightarrow{p_2} & G_V \times_P \mathcal{S} & \xrightarrow{p_3} & E_V \\ (x_T, x_W) & \longleftarrow & (g, x) & \longmapsto & (g, x) & \longmapsto & gx \end{array}$$

given in (2.4). Let $\tau = \dim(T)$ and $\omega = \dim(W)$. Given $f_1 \in K_\tau$ and $f_2 \in K_\omega$ define a function $f_1 * f_2$ as follows:

If $x \in (E_V)^F$ then

$$(f_1 * f_2)(x) = \sum_{x_T, x_W} C^V_{T,W} f_1(x_T) f_2(x_W),$$

where the sum is over all $x_T \in (E_T)^F$ and $x_W \in (E_W)^F$, and

$$C^V_{T,W} = \frac{\mathrm{Card}(\ \{(y,f) \in (G_V \times_P \mathcal{S})^F \mid p_1(y,f) = (x_T, x_W), p_3(p_2(y,f)) = x\}\ )}{\mathrm{Card}((G_T)^F)\mathrm{Card}((G_W)^F)}.$$

Let $\vec{c}$ be an orbit index and let $\chi_{\vec{c}}$ be the characteristic function of the orbit $\mathcal{O}_{\vec{c}}$, i.e.

$$\text{for } x \in (E_V)^F, \qquad \chi_{\vec{c}}(x) = \begin{cases} 1, & \text{if } x \in (\mathcal{O}_{\vec{c}})^F, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from the observation in (3.2) that the map

$$\begin{array}{ccc} K & \longrightarrow & R\Gamma \\ \chi_{\vec{c}} & \longmapsto & [R_{\vec{c}}] \end{array}$$

is an isomorphism of algebras, where $R\Gamma$ is the Hall algebra defined in (1.3).

(3.4) The isomorphism between $\mathcal{K}$ and K

Let $\vec{a}$ be an orbit index and let $\mathcal{O}_{\vec{a}}$ be the corresponding G_V-orbit in E_V as defined in (3.1). Let $F_{\vec{c}}$ be the constant sheaf $\overline{\mathbb{Q}_l}$ on the orbit $\mathcal{O}_{\vec{c}}$ extended by 0 on the complement. This sheaf can be viewed as the complex of sheaves A, for which $A^0 = F_{\vec{c}}$ and $A^i = 0$, for all $i \neq 0$. In this way $F_{\vec{c}}$ can be viewed as an element of $\mathcal{Q}_V$, see IV (1.4), and the isomorphism class $[F_{\vec{c}}]$ of $F_{\vec{c}}$ is an element of $\mathcal{K}$.

Theorem. *Let $\mathcal{K}$ be the algebra defined in §2 and let K be the algebra defined in (3.3). For each orbit index $\vec{c}$ let $\mathcal{O}_{\vec{c}}$ be the corresponding G_V orbit in E_V, as given in (3.1), and let $\chi_{\vec{c}}$ be the characteristic function of the orbit $\mathcal{O}_{\vec{c}}$. The map*

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & K \\ [F_{\vec{c}}] & \longmapsto & \chi_{\vec{c}} \end{array}$$

is an isomorphism of algebras.

This theorem is a consequence of an analogue of the Grothendieck trace formula. The Grothendieck trace formula, [Ca] p. 504, is the formula

$$|X^F| = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Tr}(F, H_c^i(X, \mathbb{Q}_l)),$$

which describes the number of points of X which are fixed under a Frobenius map F in terms of the trace of the action of the Frobenius map on the l-adic cohomology $H_c^i(X, \mathbb{Q}_l)$ of the variety X.

Theorems (3.4) and (2.8) together show that there is a natural connection between the algebra $\mathcal{K}$ and the Hall algebra $R\Gamma$ which was introduced in (1.3).

IX. Link invariants from quantum groups

The theorems of Alexander and Markov given in (1.4) and (1.5) are considered classical, they can be found in [Bi] Theorem 2.1 and Theorem 2.3, respectively. A sketch, with further references, of the proof of Theorem (1.7) can be found in [CP] 15.2. See [J] Prop. 6.2 for the proof of Theorem (1.2) and [Stb] Lemma 2.5 for the proof of Proposition (1.6).

(1.1) Knots, links and isotopy

A *knot* is an imbedded circle in $\mathbb{R}^3$. By circle we mean an S^1 and imbedded is in the sense of differential geometry. A *link* is a disjoint union of imbedded circles in $\mathbb{R}^3$. A link is *oriented* if each connected component is oriented. We shall identify a link with its "picture in the plane".

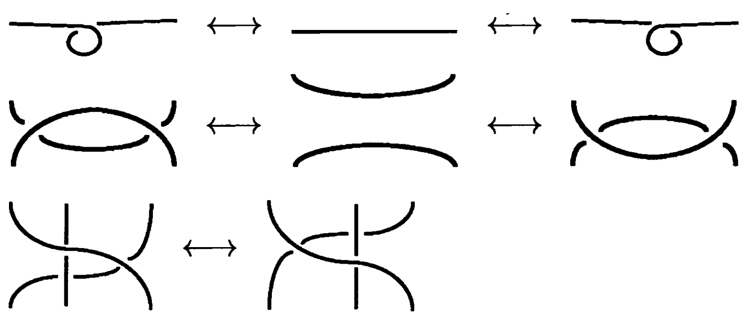
knot (unknot) knot (trefoil) link (Borromean rings)

The conceptual idea of when two links are the same is called ambient isotopy. More precisely, two oriented links L_1 and L_2 are *equivalent under ambient isotopy* if there is an orientation preserving diffeomorphism of $\mathbb{R}^3$ which takes L_1 to L_2. In terms of pictures in the plane L_1 and L_2 are equivalent under ambient isotopy if the picture for L_1 can be transformed into the picture for L_2 by a sequence of *Reidemeister moves*:

(R1)

(R2)

(R3)



These moves are applied locally to a region in the picture and all possible orientations of the strings are allowed. The equivalence relation on pictures in the plane gotten by only allowing moves (R2) and (R3) is called *regular isotopy.*

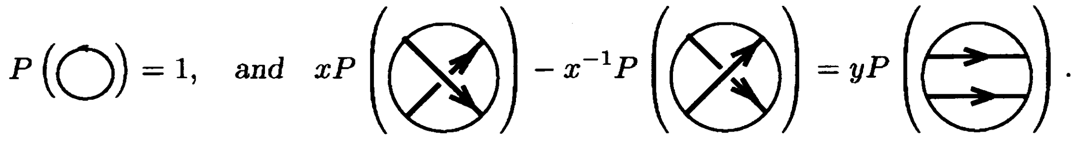
(1.2) Link invariants

Let S be a set. An *oriented link invariant* with values in S is a map

$$P : \mathcal{L} \longrightarrow S$$

from the set $\mathcal{L}$ of equivalence classes of oriented links under ambient isotopy to S.

Theorem. *There exists a unique oriented link invariant* $P : \mathcal{L} \longrightarrow \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ *such that*



The unusual notation in the second relation indicates changes to the link in a local region.

The link invariant defined in the above Theorem is the *HOMFLY polynomial.* Other famous link invariants can be obtained in a similar fashion by specializing x and y, as follows:

Jones polynomial	$x = t^{-1}$	and	$y = t^{1/2} - t^{-1/2}$,
Conway polynomial	$x = 1$	and	$y = y$,
Alexander polynomial	$x = 1$	and	$y = t^{1/2} - t^{-1/2}$.

(1.3) Braids

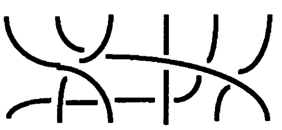
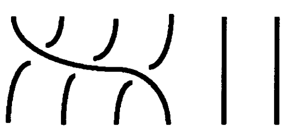
A *braid on m-strands* consists of two rows of m dots each, one above the other, and m strands in $\mathbb{R}^3$ such that

(1) each strand connects a dot in the top row to a dot in the bottom row,

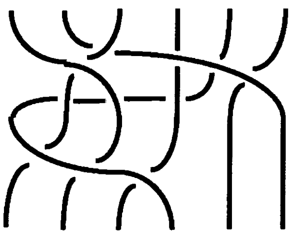
(2) the strands do not intersect,

(3) every dot is incident to exactly one strand.

Composition of two braids b_1, b_2 on m-strands is given by identifying the bottom points of b_1 with the top points of b_2. The following are braids on 6 strands,

$b_1 =$  , $b_2 =$  ,

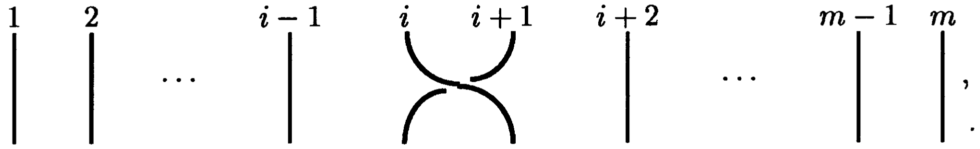
and the product $b_1 b_2$ is the braid

$b_1 b_2 =$  .

One should note that it is important to be careful in defining the word "strand" since the diagram

is not a legal braid.

The *braid group* $\mathcal{B}_m$ is the group of braids on m strands and it is a famous theorem of E. Artin that $\mathcal{B}_m$ has a presentation by generators

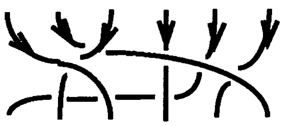
$g_i =$ 

for $1 \le i \le m-1$, and relations

$$\begin{aligned} g_i g_j &= g_j g_i, && \text{if } |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, && \text{for } 1 \le i \le m-2. \end{aligned}$$

(1.4) Every link is the closure of a braid

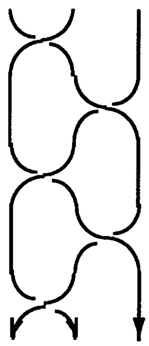
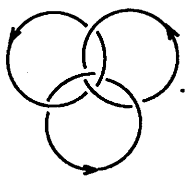
It will be convenient to "orient" the strands of a braid so that they "travel" from top to bottom.



The *closure* $(\hat{\beta}, m)$ of a braid $\beta \in \mathcal{B}_m$ on m-strands is the oriented link obtained by joining together (identifying) each dot in the top row to the corresponding dot in the bottom row. If

$\beta =$ , then $(\hat{\beta}, 3) =$ ,

and if

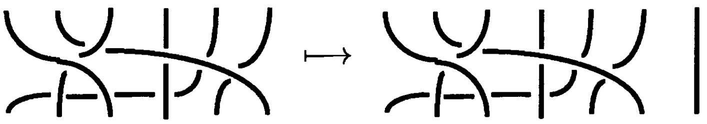
$\beta =$ , then $(\hat{\beta}, 3) =$ .

Theorem. *(Alexander) Every oriented link is the closure $(\hat{\beta}, m)$ of a braid $\beta \in \mathcal{B}_m$ for some m.*

(1.5) Markov equivalence

The braid group $\mathcal{B}_m$ can be embedded into the braid group $\mathcal{B}_{m+1}$ by adding a strand.

$$\mathcal{B}_6 \qquad \longrightarrow \qquad \mathcal{B}_7$$



Two braids $\beta_1 \in \mathcal{B}_m$ and $\beta_2 \in \mathcal{B}_n$ are *Markov equivalent* if they are equivalent under the equivalence relation on $\sqcup_m \mathcal{B}_m$ (disjoint union of $\mathcal{B}_m$) which is defined by the relations

$$\begin{aligned} &(\mathrm{M1}) \quad \beta' \sim \beta\beta'\beta^{-1}, && \text{for all } \beta, \beta' \in \mathcal{B}_k, \text{ and} \\ &(\mathrm{M2}) \quad \beta \sim \beta g_k \sim \beta g_k^{-1}, && \text{if } \beta \in \mathcal{B}_k; \end{aligned}$$

where in the relation (M2) the products βg_k and βg_k^{-1} are obtained by viewing β as an element of $\mathcal{B}_{k+1}$ under the imbedding $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$.

Theorem. *(Markov) Two braids $\beta_1 \in \mathcal{B}_m$ and $\beta_2 \in \mathcal{B}_n$ have equivalent closures $(\hat{\beta}_1, m)$ and $(\hat{\beta}_2, n)$ (under ambient isotopy) if and only if β_1 and β_2 are Markov equivalent.*

(1.6) Quantum dimensions and quantum traces

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group. Let $\tilde{\rho}$ be the element of $\mathfrak{h}$ such that $\alpha_i(\tilde{\rho}) = 1$ for all simple roots α_i, see II (2.6).

Let V be a finite dimensional $\mathfrak{U}_h\mathfrak{g}$ module. The *quantum dimension* of V is

$$\dim_q(V) = \mathrm{Tr}_V(e^{h\tilde{\rho}}).$$

If $z \in \mathrm{End}_{\mathfrak{U}_h\mathfrak{g}}(V)$ then the *quantum trace* of z is

$$\mathrm{tr}_q(z) = \mathrm{Tr}_V(e^{h\tilde{\rho}}z)\,.$$

Proposition. *Let $L(\lambda)$ be the irreducible $\mathfrak{U}_h\mathfrak{g}$-module of highest weight λ as given in VI (1.3) and VI (2.3). Then*

$$\dim_q(L(\lambda)) = \prod_{\alpha>0} \frac{1 - q^{(\lambda+\rho,\alpha)}}{1 - q^{(\rho,\alpha)}}, \qquad \textit{where} \quad q = e^h,$$

$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the half sum of the positive roots, and the inner product $(,)$ on $\mathfrak{h}_{\mathbb{R}}^*$ is as given in II (2.7).

(1.7) Quantum traces give us link invariants!

Recall that $\mathfrak{U}_h\mathfrak{g}$ is a quasitriangular Hopf algebra and that therefore the category of finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules is a braided SRMCwMFF. Let

$$\check{R}_{VV} : V \otimes V \longrightarrow V \otimes V$$

be the braiding isomorphism from $V \otimes V$ to $V \otimes V$. It follows from the identity I (3.5) that the map

$$\begin{array}{cccc} \Phi: & \mathcal{B}_m & \longrightarrow & \mathrm{End}_{\mathfrak{U}_h\mathfrak{g}}(V^{\otimes m}) \\ & g_i & \longmapsto & \check{R}_i = \mathrm{id}^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}^{\otimes m-(i+1)} \end{array}$$

is well defined and that $\Phi(\beta_1\beta_2) = \Phi(\beta_1)\Phi(\beta_2)$ for all braids $\beta_1, \beta_2 \in \mathcal{B}_m$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the correponding Drinfel'd-Jimbo quantum group. Let $L(\lambda)$ be an irreducible $\mathfrak{U}_h\mathfrak{g}$-module of highest weight λ (see VI (1.3) and VI (2.3)). Let ρ be the half sum of the positive roots and let $(,)$ be the inner product on $\mathfrak{h}_{\mathbb{R}}$ as given in II (2.7). For each braid β on m-strands define*

$$P(\hat{\beta}, m) = \left(\frac{1}{q^{<\lambda,\lambda+2\rho>}\dim_q(V)}\right)^m \mathrm{tr}_q(\Phi(\beta)),$$

where $q = e^h$. Then P is a well defined link invariant.

Remark. The above theorem gives the Jones polynomial when $\mathfrak{g} = \mathfrak{sl}_2$, the simple Lie algebra corresponding to the Dynkin diagram A_1, and $L(\lambda)$ is chosen to be the irreducible representation of $\mathfrak{U}_h\mathfrak{g}$ with highest weight $\lambda = \omega_1$.

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