SCHUBERT CALCULUS

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1. Introduction. In 1874, H. Schubert published his celebrated treatise, "*Kalkül der Abzählenden Geometrie*" (Calculus of Enumerative Geometry [22]). It dealt with finding the number of points, lines, planes, etc., satisfying certain geometric conditions, an important problem about a hundred years ago. In the book, Schubert drew much from the vast literature on the subject and introduced some far-reaching ideas of his own.

As was often the case in early algebraic geometry, the methods of enumerative geometry were intuitive and rested on a weak foundation. However, the beauty of the subject inspired many mathematicians to develop rigorously the foundational material, such as topological and algebraic intersection theories. This work is of far greater importance than the original enumerative problems.

In a brief article, we can only hope to highlight a rigorous development of the early ideas, but we shall try to illustrate each discussion with an example of lines in 3-space.

Here is a typical enumerative problem: How many lines in 3-space, in general, intersect four given lines? Schubert would specialize the given four lines so as to make the first intersect the second and the third intersect the fourth. In this special case there are obviously two lines intersecting the four: the line joining the two points of intersection and the line of intersection of the two planes—one determined by the first two lines and the other by the second two. Now Schubert's "principle of conservation of number" asserts that there must be two solutions in the general case as well. This principle, which grew out of Poncelet's principle of continuity, is Schubert's most important contribution to the subject.

Our first step will be to make the concept of specializing a line more precise. This we do in section two, where we show more generally that all the d-planes in n-space can in a natural way be made into a manifold. Then we may interpret specialization as moving in a continuous way.

Next, we must analyze the condition that a line L intersect a given line A. This condition means that any two points which determine L and any two points which determine A are dependent and the latter requirement can be conveniently expressed

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in terms of determinants. Section three is devoted to expressing the more general condition that a d-plane in n-space intersect in a prescribed way a given nested sequence (or flag) of linear spaces.

In section four, we interpret and justify the "principle of conservation of number" in the way it was first rigorously done, with the aid of the cohomology theory of manifolds. Then, having defined all our terms, we present the three main theorems of the symbolic formalism, known as Schubert calculus, for solving enumerative problems. We indicate the several different approaches to proving these theorems and give appropriate references in section five.

In section five, we also mention some generalizations, applications, open questions and references pertaining to the material in the other sections. We make no claims of completeness; the choices were made partly out of personal taste. However, we hope that these things will be of interest to some readers and perhaps inspire them to pursue matters further.

2. The Grassmann manifold. The space of n-tuples $(a(1), \cdots, a(n))$ of complex numbers is commonly called **affine n-space** and denoted by A^n.

If we try to make sense of the "principle of conservation of number" for configurations in affine space we encounter some difficulties. For example, in section one we found that there are two lines in 3-space which intersect four general lines by specializing the four. However, if we specialize them so that the first intersects the second and the third intersects the fourth but so that the plane of the first two lines is parallel to the plane of the second two, then there will be only one solution. If we specialize the four so that the first intersects the second but the third is parallel to the fourth and the plane of the first two is parallel to the plane of the second two, then there will be no solution. Thus we may obtain 0, 1, or 2 solutions by specializing appropriately. Of course the missing solutions lie "at infinity" and we ought to work in projective space.

A point P of **projective n-space** P^n is defined by an $(n+1)$-tuple $(p(0), \cdots, p(n))$ of complex numbers not all zero. The $p(i)$ are called the **coordinates** of P. Another $(n+1)$-tuple $(q(0), \cdots, q(n))$ also defines P if and only if there is a number c satisfying $p(i) = cq(i)$ for $i = 0, \cdots, n$.

Identifying a point $(a(1), \cdots, a(n))$ of A^n with the point $(1, a(1), \cdots, a(n))$ of P^n, we may think of P^r as A^n completed by the points $(0, b(1), \cdots, b(n))$ "at infinity" in P^n. Then, for example, it is not hard to see that two parallel planes, which do not intersect in A^3, will intersect in a line lying "at infinity" in P^3 and that the solutions missing above do lie "at infinity" in this sense.

A **linear space** L in P^n is defined as the set of points $P = (p(0), \cdots, p(n))$ of P^n whose coordinates $p(j)$ satisfy a system of linear equations $\sum_{j=0}^{n} b_{\alpha j} p(j) = 0$ with $\alpha = 1, \cdots, (n-d)$. We say that L is **d-dimensional** if these $(n-d)$ equations are independent, that is if the $(n-d) \times (n+1)$ matrix of coefficients $[b_{\alpha j}]$ has a nonzero $(n-d) \times (n-d)$-minor. By linear algebra, there are then $(d+1)$ points

$P_i = (p_i(0), \cdots, p_i(n))$ in L with $i = 0, \cdots, d$ which span L. Of course, we call L a **line** if $d = 1$, a **plane** if $d = 2$ and a **hyperplane** if $d = (n - 1)$. We also call a d-dimensional linear space a **d-plane** for short.

The rest of this section is devoted to representing in a natural way the d-planes in $\boldsymbol{P}^n$ by the points of a certain manifold $G_{d,n}$ lying in a projective space $\boldsymbol{P}^N$ where we put once and for all

$$N = \binom{n+1}{d+1} - 1.$$

For convenience, let us make the following convention. For any $(d + 1) \times (n + 1)$-matrix $[p_i(j)]$ with $i = 0, \cdots, d$ and $j = 0, \cdots, n$, and any sequence of $(d + 1)$ integers $j_0 \cdots j_d$ with $0 \leqq j_\beta \leqq n$, let us denote by $p(j_0 \cdots j_d)$ the determinant of the $(d + 1) \times (d + 1)$-matrix $[p_i(j_\beta)]$ with $i, \beta = 0, \cdots, d$. Of course, we have the usual formulas:

$$\text{(A)} \qquad \begin{aligned} &p(j_0 \cdots j_d) = 0 \text{ if any two of the } j_\beta \text{ are equal;} \\ &p(j_0 \cdots j_d) = -p(j_0 \cdots j_{\beta-1} j_{\beta+1} j_\beta j_{\beta+2} \cdots j_d) \text{ for } \beta = 0, \cdots, d-1. \end{aligned}$$

A function p on the set of all sequences $j_0 \cdots j_d$ with $0 \leqq j_\beta \leqq n$ which satisfies the formulas (A) is called an **alternating function**. It is evident that an alternating function is determined by its values on the subset of sequences $j_0 \cdots j_d$ with $0 \leqq j_0 < \cdots < j_d \leqq n$ and that any function on this subset extends to an alternating function on the whole set. Note that the number of sequences $j_0 \cdots j_d$ with $0 \leqq j_0 < \cdots < j_d \leqq n$ is exactly $(N + 1)$.

Fix a d-plane L in $\boldsymbol{P}^n$. Pick $(d + 1)$ points $P_i = (p_i(0), \cdots, p_i(n))$ with $i = 0, \cdots, d$ which span L, and form the $(d + 1) \times (n + 1)$-matrix $[p_i(j)]$. By linear algebra at least one of the $(N + 1)$ determinants $p(j_0 \cdots j_d)$ with $0 \leqq j_0 < \cdots < j_d \leqq n$ must be nonzero. So, when ordered lexicographically, these determinants define a point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $\boldsymbol{P}^N$.

Let $Q_i = (q_i(0), \cdots, q_i(n))$ for $i = 0, \cdots, d$ be another $(d + 1)$ points spanning L. Then linear algebra yields a nonsingular $(d + 1) \times (d + 1)$-matrix C which carries the P_i into the Q_i; in other words, we have $[q_i(j)] = C \cdot [p_i(j)]$ where the dot denotes matrix multiplication. Clearly we then have $q(j_0 \cdots j_d) = \det(C) p(j_0 \cdots j_d)$, where $\det(C)$ denotes the determinant of C. So the points Q_i give rise to the same point of $\boldsymbol{P}^N$ as the points P. Therefore L canonically gives rise to a point of $\boldsymbol{P}^N$. The coordinates $p(j_0 \cdots j_d)$ of this point are called the **Plücker coordinates** of L.

Not every point of $\boldsymbol{P}^N$ arises from some d-plane in $\boldsymbol{P}^n$. In fact, we shall now prove that the Plücker coordinates $p(j_0 \cdots j_d)$ of a d-plane L in $\boldsymbol{P}^n$ satisfy the following quadratic relations:

$$\text{(QR)} \qquad \sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0 \cdots j_{d-1} k_\lambda) p(k_0 \cdots \check{k}_\lambda \cdots k_{d+1}) = 0,$$

where $j_0 \cdots j_{d-1}$ and $k_0 \cdots k_{d+1}$ are any sequences of integers with $0 \leqq j_\beta, k_\gamma \leqq n$. Here

$\check{k}_\lambda$ means that the integer k_λ has been removed from the sequence and the $p(j_0 \cdots j_d)$ are to be interpreted according to the formulas (A).

Explicitly, we want to establish the relation among determinants,

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda \begin{vmatrix} \vdots & \vdots & \vdots \\ p_i(j_0) \cdots & p_i(j_{d-1}) & p_i(k_\lambda) \\ \vdots & \vdots & \vdots \end{vmatrix} \begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & \check{p}_d(k_\lambda) & \cdots \end{vmatrix} = 0.$$

Expanding the first determinants along their last column, we obtain the relation,

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda \left\{ \sum_{i=0}^{d} (-1)^{d+i} \begin{vmatrix} \vdots & \vdots \\ \check{p}_i(j_0) \cdots & \check{p}_i(j_{d-1}) \\ \vdots & \vdots \end{vmatrix} p_i(k_\lambda) \right\} \begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & \check{p}_d(k_\lambda) & \cdots \end{vmatrix} = 0.$$

Rearranging the terms, we obtain the relation,

$$\sum_{i=0}^{d} (-1)^{d+i} \begin{vmatrix} \vdots & \vdots \\ \check{p}_i(j_0) \cdots & \check{p}_i(j_{d-1}) \\ \vdots & \vdots \end{vmatrix} \left\{ \sum_{\lambda=0}^{d+1} (-1)^\lambda p_i(k_\lambda) \begin{vmatrix} \cdots & \check{p}_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & \check{p}_d(k_\lambda) & \cdots \end{vmatrix} \right\} = 0.$$

Now this relation can be obtained by expanding the second determinants in the following relation along the first row:

$$\sum_{i=0}^{d} (-1)^{d+1} \begin{vmatrix} \vdots & \vdots \\ \check{p}_i(j_0) \cdots & \check{p}_i(j_{d-1}) \\ \vdots & \vdots \end{vmatrix} \begin{vmatrix} \cdots & p_i(k_\lambda) & \cdots \\ \cdots & p_0(k_\lambda) & \cdots \\ & \vdots & \\ \cdots & p_d(k_\lambda) & \cdots \end{vmatrix} = 0.$$

However, these second determinants are zero because two rows are equal. Thus the quadratic relations (QR) are satisfied by the Plücker coordinates of a d-plane in P^n.

Conversely, any point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of P^N whose coordinates satisfy the quadratic relations (QR) arises from a unique d-plane L in P^n. To prove this assertion, we shall simply "solve" the quadratic relations. First, we assume that $p(k_0 \cdots k_d)$ is not zero and show that the $(N+1)$ coordinates $p(j_0 \cdots j_d)$ are already determined by the $[(d+1)(n-d)+1]$ coordinates of the form $p(k_0 \cdots \check{k}_\lambda \cdots k_d j_\alpha)$, that is by the coordinates $p(i_0 \cdots i_d)$ with at most one of $i_0, \cdots, i_d$ not among $k_0, \cdots, k_d$.

Let $j_0 \cdots j_d$ be a sequence of integers of which exactly m are not among the integers $k_0, \cdots, k_d$ and let j_β be one of these m. The quadratic relation (QR) corresponding to the sequences $j_0 \cdots \check{j}_\beta \cdots j_d$ and $k_0 \cdots k_d j_\beta$ obviously yields the equation,

$$p(j_0 \cdots \check{j}_\beta \cdots j_d j_\beta) p(k_0 \cdots k_d) = \sum_{\lambda=0}^{d} (-1)^\lambda p(j_0 \cdots \check{j}_\beta \cdots j_d k_\lambda) p(k_0 \cdots \check{k}_\lambda \cdots k_d j_\beta).$$

Now if k_λ is among $j_0, \cdots, j_d$, then $p(j_0 \cdots \check{j}_\beta \cdots j_d k_\lambda)$ is zero; if k_λ is not among $j_0, \cdots, j_d$ then exactly $(m-1)$ of $j_0, \cdots, \check{j}_\beta, \cdots, j_d, k_\lambda$ are not among $k_0, \cdots, k_d$. Thus if we have $m \geqq 2$, we can express $p(j_0 \cdots j_d)p(k_0 \cdots k_d)$ in terms of the coordinates $p(i_0 \cdots i_d)$ with at most $(m-1)$ of $i_0, \cdots, i_d$ not among $k_0, \cdots, k_d$. Continuing this process of multiplying by $p(k_0 \cdots k_d)$ and of using a quadratic relation, we find we can express $p(j_0 \cdots j_d)p(k_0 \cdots k_d)^{m-1}$ as a polynomial in the coordinates $p(i_0 \cdots i_d)$ with at most one of $i_0, \cdots, i_d$ not among $k_0, \cdots, k_d$. Since we assumed $p(k_0 \cdots k_d) \neq 0$, we have proved our assertion that these $[(d+1)(n-d)+1]$ coordinates determine the others.

Without loss of generality, we assume $p(k_0 \cdots k_d) = 1$. We are going to construct a d-plane L in $\boldsymbol{P}^n$ whose Plücker coordinates are equal to the coordinates $p(j_0 \cdots j_d)$ of the given point in $\boldsymbol{P}^N$. For $i = 0, \cdots, d$ and $j = 0, \cdots, n$ put

$$p_i(j) = p(k_0 \cdots k_{i-1} j k_{i+1} \cdots k_d).$$

The vectors $(p_i(0), \cdots, p_i(n))$ for $i = 0, \cdots, d$ are linearly independent because we have $p_i(k_\gamma) = 0$ for $i \neq \gamma$ and $p_i(k_i) = 1$. So, these vectors span a d-plane L in $\boldsymbol{P}^n$. Now the Plücker coordinate $p'(j_0 \cdots j_d)$ of L is defined as the determinant of the matrix $[p_i(j_\beta)]$ with $i, \beta = 0, \cdots, d$. So, if we have $j_\beta = k_\beta$ for $\beta \neq \lambda$, this matrix coincides with the identity matrix outside the λ-th column. Hence we have

$$p'(j_0 \cdots j_d) = p_\lambda(j_\lambda) = p(j_0 \cdots j_d),$$

whenever at most one j_λ of $j_0, \cdots, j_d$ is not among $k_0, \cdots, k_d$. Since we proved above that these coordinates determine the rest, we have $p'(j_0 \cdots j_d) = p(j_0 \cdots j_d)$ for all sequences $j_0 \cdots j_d$. Thus the point $(\cdots, p(j_0 \cdots j_d), \cdots)$ arises from the d-plane L.

Finally, let L' be another d-plane in $\boldsymbol{P}^n$ whose Plücker coordinates define the given point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $\boldsymbol{P}^N$. Choose $(d+1)$ points $P_i' = (p_i'(0), \cdots, p_i'(n))$ with $i = 0, \cdots, d$ which span L. Then the $(d+1) \times (d+1)$-matrix $[p_i'(k_\gamma)]$ is invertible because its determinant is by hypothesis a nonzero multiple of $p(k_0 \cdots k_d) = 1$. Altering the P_i' by the inverse matrix, we may assume $[p_i'(k_\gamma)]$ is the identity matrix. Then for any sequence $j_0 \cdots j_d$ the determinant $\det[p_i'(j_\beta)]$ is obviously equal to $p(j_0 \cdots j_d)$. Now fix λ and j with $0 \leqq \lambda \leqq d$ and $0 \leqq j \leqq n$, and put $j_\beta = k_\beta$ for $\beta \neq \lambda$ and $j_\lambda = j$. Then $[p_i'(j_\beta)]$ clearly coincides with the identity matrix outside the λ-th column. So we have

$$p_\lambda'(j) = \det[p_i'(j_\beta)] = p(j_0 \cdots j_d) = p_\lambda(j),$$

where the last equation is the definition of $p_\lambda(j)$ made above. Thus we have $P_\lambda' = P_\lambda$ for each λ and so $L' = L$.

We have now reached our goal and proved the following theorem:

THEOREM 1. *There is a natural bijective correspondence between the d-planes in* $\boldsymbol{P}^n$ *and the points of* $\boldsymbol{P}^N$ *with* $N = \binom{n+1}{d+1} - 1$, *whose coordinates satisfy the quadratic relations* (QR).

In the course of the proof, we also established the following result:

PROPOSITION 2. *There is a natural bijective correspondence between the set of points of P^N whose coordinates $p(j_0 \cdots j_d)$ satisfy the quadratic relations* (QR) *and the requirement $p(k_0 \cdots k_d) \neq 0$ and the affine $(d+1)(n-d)$-space of $(d+1)(n+1)$ matrices $[p_i(j)]$ with $i = 0, \cdots, d$ and $j = 0, \cdots, n$ such that the $(d+1) \times (d+1)$-submatrix $[p_i(k_\gamma)]$ with $i, \gamma = 0, \cdots, d$ is the identity. Moreover, such a matrix $[p_i(j)]$ corresponds to the point of P^N with coordinates $p(j_0 \cdots j_d) = \det[p_i(i_\beta)]$ and such a point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of P^N corresponds to the $(d+1) \times (n+1)$-matrix with entries*

$$p_i(j) = p(k_0 \cdots k_{i-1} j k_{i+1} \cdots k_d) / p(k_0 \cdots k_d).$$

By virtue of this proposition, the set of points of P^N whose coordinates satisfy the quadratic relations (QR) is covered by $(N+1)$ copies of affine $(d+1)(n-d)$-space, so it is a submanifold of P^N of dimension $(d+1)(n-d)$. It is called the **Grassmann manifold** (of d-planes in n-spaces) and denoted by $G_{d,n}$. In these terms, Theorem (1) says that the d-planes in P^n are represented by the points of the $(d+1)(n-d)$-dimensional Grassmann manifold $G_{d,n}$.

For example, the lines in P^3 are represented by the points of the 4-dimensiona Grassmann manifold $G_{1,3}$, which can be described as the points of P^5 whose coordinates $p(j_0 j_1)$ satisfy the single quadratic relation,

$$p(01)p(23) - p(02)p(13) + p(03)p(12) = 0.$$

3. Schubert conditions. We are now going to work out a necessary and sufficient determinantal condition for a d-plane in P^n to intersect a given sequence of linear spaces in P^n in a prescribed way.

Let $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_d$ be a strictly increasing sequence (or flag) of $(d+1)$ linear spaces in P^n. A d-plane L in P^n is said to satisfy the **Schubert condition** defined by this sequence if $\dim(A_i \cap L) \geqq i$ for all i. The set of all such d-planes L corresponds to a subset of $G_{d,n}$, which is denoted by $\Omega(A_0 \cdots A_d)$.

For example, fix a line A_0 in P^3 and take A_1 to be P^3 itself. Then the subset $\Omega(A_0 A_1)$ of $G_{1,3}$ represents the set of lines L in P^3 satisfying $\dim(L \cap A_0) \geqq 0$ and $\dim(L \cap A_1) \geqq 1$. Since the second condition is automatically satisfied, $\Omega(A_0 A_1)$ represents the set of lines L intersecting A_0.

PROPOSITION 3. *Let $0 \leqq a_0 < \cdots < a_d \leqq n$ be a sequence of integers and for $i = 0, \cdots, d$ let A_i be the a_i-dimensional linear space in P^n whose points are of the form $(p(0), \cdots, p(a_i), 0, \cdots, 0)$. Then $\Omega(A_0 \cdots A_d)$ consists exactly of those points $(\cdots, p(j_0 \cdots j_d), \cdots)$ in $G_{d,n}$ satisfying $p(j_0 \cdots j_d) = 0$ whenever $j_i > a_i$ holds for some i.*

Proof. Consider a d-plane L in P^n which satisfies the Schubert condition $\dim(A_i \cap L) \geqq i$ for $i = 0, \cdots, d$. By induction on i, we may clearly pick a point

$P_i = (p_i(0), \cdots, p_i(n))$ in $A_i \cap L$ such that $P_0, \cdots, P_i$ are linearly independent. Then $P_0, \cdots, P_d$ form a basis of L. So, in the construction of section two, L is represented by the point of $G_{d,n}$ with coordinates $p(j_0 \cdots j_d) = \det[p_i(j_\beta)]$. Suppose we have $j_\lambda > a_\lambda$ for a certain λ. Since P_i lies in A_i, we have $p_i(j) = 0$ for $j = (a_i + 1), \cdots, n$, and hence the matrix $[p_i(j_\beta)]$ takes the form,

$$[p_i(j_\beta)] = \begin{array}{c} \\ \left[\begin{array}{cc} * & \overbrace{\boxed{0}}^{(d-\lambda+1)} \\ * & * \end{array}\right] \end{array} \begin{array}{l} \\ \left.\vphantom{\begin{array}{c}0\\0\end{array}}\right\} \lambda \\ \\ \end{array}$$

It is now easy to see that $p(j_0 \cdots j_d) = 0$ either by (Laplace) expansion of the determinant along the last $(d - \lambda + 1)$ columns or by induction on $(d - \lambda + 1)$, the cases $(d - \lambda + 1) = 1$ and $(d - \lambda + 1) = 2$ being clear.

Conversely consider a point $(\cdots, p(j_0 \cdots j_d), \cdots)$ on $G_{d,n}$ satisfying $p(j_0 \cdots j_d) = 0$ whenever $j_i > a_i$ holds for some i. Choose a nonzero coordinate $p(k_0 \cdots k_d)$ which maximizes the sum $\sum_{\gamma=0}^{d} k_\gamma$. Replacing each $p(j_0 \cdots j_d)$ by $p(j_0 \cdots j_d)/p(k_0 \cdots k_d)$, we may assume $p(k_0 \cdots k_d) = 1$. Now, in section two, we saw that the point $(\cdots, p(j_0 \cdots j_d), \cdots)$ represents the d-plane L spanned by the points $P_i = (p_i(0), \cdots, p_i(n))$ with $p_i(j) = p(\cdots k_{i-1} j k_{i+1} \cdots)$ for $j = 0, \cdots, n$ and for $i = 0, \cdots, d$.

Fix $j > a_i$, we shall show that $p_i(j)$ is zero. Since $p(k_0 \cdots k_d)$ is not zero, we have $k_i \leqq a_i$ and so $k_i < j$. Consequently, the sum $\sum_{\gamma=0}^{d} k_\gamma$ is strictly less than the sum $(j + \sum_{\gamma \neq i} k_\gamma)$. Hence $p_i(j) = p(\cdots k_{i-1} j k_{i+1} \cdots)$ is zero by the maximality of $\sum_{\gamma=0}^{d} k_\gamma$.

Therefore P_i lies in A_i. Hence the $(i + 1)$ linearly independent points $P_0, \cdots, P_i$ lie in $(A_i \cap L)$. So L satisfies the Schubert condition $\dim(A_i \cap L) \geqq i$ for $i = 0, \cdots, d$. Thus $(\cdots, p(j_0 \cdots j_d), \cdots)$ lies in $\Omega(A_0 \cdots A_d)$.

PROPOSITION 4. *Let $A_0 \subsetneqq \cdots \subsetneqq A_d$ and $B_0 \subsetneqq \cdots \subsetneqq B_d$ be two strictly increasing sequences of linear spaces in $\boldsymbol{P}^n$ and assume $\dim(A_i) = \dim(B_i)$ for $i = 0, \cdots, d$. Then there is an invertible linear transformation of $\boldsymbol{P}^N$ into itself which carries $G_{d,n}$ into itself and $\Omega(B_0 \cdots B_d)$ into $\Omega(A_0 \cdots A_d)$.*

Proof. Since we have $\dim(A_i) = \dim(B_i)$ for each i, there obviously is an invertible $(n + 1) \times (n + 1)$-matrix $[a_{ij}]$ such that the linear transformation T of $\boldsymbol{P}^n$ into itself defined by the formula

$$T(p(0), \cdots, p(n)) = \left(\sum_{i=0}^{n} p(i) a_{i0}, \cdots, \sum_{i=0}^{n} p(i) a_{in} \right)$$

carries B_i onto A_i for each i. Clearly, T carries a d-plane L in $\boldsymbol{P}^n$ into another one $T(L)$, and if L satisfies the Schubert condition $\dim(B_i \cap L) \geqq i$ for all i, then $T(L)$

satisfies the Schubert condition $\dim(A_i \cap T(L)) \geqq i$ for all i because we have $T(B_i) = A_i$.

Choose $(d+1)$ points $P_i = (p_i(0), \cdots, p_i(n))$ with $i = 0, \cdots, d$ which span L. Then the $(d+1)$ points $T(P_i)$ span $T(L)$. Now, $T(P_i)$ is of the form $(q_i(0), \cdots, q_i(n))$ with

$$q_i(j) = \sum_{\alpha=0}^{n} p(\alpha)a_{\alpha j} \text{ for } j = 0, \cdots, n,$$

and a straightforward computation shows that the Plücker coordinates $q(j_0 \cdots j_d) = \det[q_i(j_\beta)]$ of $T(L)$ are certain fixed linear combinations of the Plücker coordinates $p(j_0 \cdots j_d) = \det[p_i(j_\beta)]$ of L.

In other words, there is a linear transformation $\Lambda[a_{ij}]$ of $\boldsymbol{P}^N$ into itself which carries $G_{d,n}$ into itself and $\Omega(B_0 \cdots B_d)$ into $\Omega(A_0 \cdots A_d)$. Since $[a_{ij}]$ is nonsingular, it is evident that $\Lambda[a_{ij}]$ is invertible and $\Lambda([a_{ij}]^{-1})$ is its inverse.

COROLLARY 5. *Let $B_0 \subsetneq \cdots \subsetneq B_d$ be a strictly increasing sequence of linear spaces in $\boldsymbol{P}^n$. Then $\Omega(B_0 \cdots B_d)$ consists of those points in $G_{d,n}$ whose coordinates $q(j_0 \cdots j_d)$ satisfy certain linear equations; in other words, $\Omega(B_0 \cdots B_d)$ is the intersection of $G_{d,n}$ and a certain linear space in $\boldsymbol{P}^N$. Moreover, the linear space is a hyperplane if and only if we have $\dim(B_0) = (n-d-1)$ and $\dim(B_i) = (n-d+i)$ for $i = 1, \cdots, d$.*

Proof. For $i = 0, \cdots, d$ put $a_i = \dim(B_i)$ and let A_i be the a_i-dimensional linear space in $\boldsymbol{P}^n$ whose points are of the form $(p(0), \cdots, p(a_i), 0, \cdots, 0)$. By Proposition 4, there is a linear transformation S of $\boldsymbol{P}^N$ into itself such that a point P of $G_{d,n}$ lies in $\Omega(B_0 \cdots B_d)$ if and only if $S(P)$ lies in $\Omega(A_0 \cdots A_d)$. By virtue of Proposition 3, $S(P)$ lies in $\Omega(A_0 \cdots A_d)$ if and only if each of its coordinates $q(j_0 \cdots j_d)$ is zero whenever $j_i > a_i$ holds for some i. Since each $q(j_0 \cdots j_d)$ is a certain linear combination of the coordinates $p(j_0 \cdots j_d)$ of P, we conclude that P lies in $\Omega(B_0 \cdots B_d)$ if and only if the $p(j_0 \cdots j_d)$ satisfy certain linear equations. Moreover, the number of linearly independent equations is obviously the number of sequences $j_0 \cdots j_d$, such that $j_i > a_i$ holds for some i, and it is evident that there is only one such sequence if and only if we have $a_0 = (n-d-1)$ and $a_i = (n-d+i)$ for $i = 1, \cdots, d$. Thus, the Corollary is proved.

We are now in a good position to determine the number of lines L in $\boldsymbol{P}^3$ which (simultaneously) intersect four given lines L_1, L_2, L_3, L_4. In section two, we saw that the lines L are represented by the set $G_{1,3}$ of points $(p(01), p(02), p(03), p(12), p(13), p(23))$ of $\boldsymbol{P}^5$ which satisfy the single quadratic relation

$$p(01)p(23) - p(02)p(13) + p(03)p(12) = 0.$$

At the beginning of this section, we noted that the lines L intersecting a given line A are represented by the points of the subset $\Omega(A\boldsymbol{P}^3)$ of G_{13}; hence, the lines L intersecting the four given lines L_1, L_2, L_3, L_4 are represented by the points of the intersection

$$Q = \bigcap_{i=1}^{4} \Omega(L_i P^3).$$

Now, by Corollary 5, for each i we have $\Omega(L_i P^3) = G_{1,3} \cap H_i$ for a suitable hyperplane H_i of P^5. Put $M = \bigcap_{i=1}^{4} H_i$; then we have $Q = G_{1,3} \cap M$. If the H_i are linearly independent, then M is a line. Then, by using the quadratic relation defining $G_{1,3}$ to express Q as the zeros of a certain quadratic polynomial in a parameter of M, it is easy to see that Q consists of two points, which may coincide. (They coincide exactly when M is tangent to $G_{1,3}$.) If the H_i are linearly dependent, then M is a linear space of dimension two or more and it is easy to see that Q must be infinite. Thus, the number of lines L which intersect L_1, L_2, L_3, and L_4 is either infinity or two or one (counted twice).

It is not hard to choose the lines L_1, L_2, L_3, L_4 in such a way that Q consists of only one point. Consequently, the "principle of conservation of number" will not be valid unless multiplicities are taken into account. For example, take L_1, L_2, and L_3 to be three skew lines. Fix a point P_1 on L_1. Let π_2 be the plane of P_1 and L_2 and let π_3 be the plane of P_1 and L_3. Since L_2 and L_3 do not intersect, the planes π_2 and π_3 are distinct. Take L_4 to be the line of intersection of these two planes. Then L_4 passes through P_1 and it intersects L_2 in a point P_2 and L_3 in a point P_3. The points P_1, P_2, and P_3 are distinct because the lines L_1, L_2 and L_3 are skew, so any two of the points determine L_4. Now let L be any line intersecting L_1, L_2, L_3 and L_4. If L passes through P_2 and P_3, then L coincides with L_4 because P_2 and P_3 determine L_4. Suppose L does not pass through P_2. Since L intersects L_2 and L_4, it must then lie in the plane of L_2 and L_4, which is π_2. So L passes through the point of intersection of π_2 and L_1, which is P_1. Similarly L must also pass through P_3. Then L coincides with L_4 because P_1 and P_3 determine L_4. Thus L_4 is the only line intersecting L_1, L_2, L_3, and L_4.

In the above example we saw that for any three skew lines L_1, L_2, L_3 in P^3 there is a unique line which passes through a given point P_1 of L_1 and intersects L_2 and L_3. Hence, if we had chosen L_4 to be L_1 itself, then there would be an infinite number of lines intersecting L_1, L_2, L_3, and L_4, one for each point of L_1. Of course, the number of lines intersecting four given lines is also infinite if the four all pass through the same point or if they all lie in the same plane.

Since an infinite number of solutions do appear in some special cases of an enumerative problem, the "principle of conservation of number" must be stated in the following way: If the number of solutions is finite in a given special case, then the number of solutions is the same in the general case as well, multiplicities, of course, being taken into account. In some problems, as in determining the lines in 3-space which intersect three given lines, the number of solutions is infinite. In these problems, the "principle of conservation of number" does not strictly apply. However, as Schubert himself realized, something is conserved under specialization. In the next section, we shall see that what is conserved is a cohomology class.

4. The Schubert calculus. In this section we explain the symbolic formalism, known as Schubert calculus, for solving enumerative problems. The foundational material here is far deeper than before and the main proofs are far more difficult, so we shall not go into them. However, we shall indicate the various ways to approach them and give references in the next section.

We shall base our development upon algebraic topology. In section two, we saw that $G_{d,n}$ is a complex manifold of dimension $(d+1)(n-d)$. From algebraic topology, we know that the cohomology group with the integers as coefficients $H^i(G_{d,n}; \boldsymbol{Z})$ is zero when i is not in the interval $[0, 2(d+1)(n-d)]$ and that the direct sum

$$H^*(G_{d,n}; \boldsymbol{Z}) = \bigoplus_i H^i(G_{d,n}; \boldsymbol{Z})$$

is a graded ring under cup-product. Moreover, $G_{d,n}$ is oriented, so there is a natural isomorphism of the $2(d+1)(n-d)$-th cohomology group with $\boldsymbol{Z}$; the image in $\boldsymbol{Z}$ of an element u is called the **degree** of u and denoted by $\deg(u)$.

A harder result is that we can assign a natural cohomology class (that is, an element of $H^*(G_{d,n}; \boldsymbol{Z})$) to each subset of $G_{d,n}$ defined by a system of polynomial equations. Such a subset is called a **subvariety** of $G_{d,n}$. If two subvarieties are members of the same continuous system of subvarieties, then both are assigned the same cohomology class. (Intuitively, the two are homotopic.)

The subsets $\Omega(A_0 \cdots A_d)$ are subvarieties of $G_{d,n}$ by Corollary 5; they are called **Schubert varieties** and their cohomology classes are called **Schubert cycles**. We are now going to prove that the cohomology class of $\Omega(A_0 \cdots A_d)$ depends only on the integers $a_i = \dim(A_i)$ for $i = 0, \cdots, d$. Indeed, consider the continuous system of subvarieties $(\Lambda M)\Omega(A_0 \cdots A_d)$ parametrized by the nonsingular $(n+1) \times (n+1)$-matrices M, where ΛM denotes the linear transformation of $\boldsymbol{P}^N$ into itself induced by the matrix M, (see the proof of Proposition 4). This system clearly includes $\Omega(A_0 \cdots A_d)$ and by Proposition 4 it includes every subvariety $\Omega(B_0 \cdots B_d)$ with $\dim(B_i) = a_i$ for $i = 0, \cdots, d$. Since all the subvarieties in a continuous system are assigned the same cohomology class, the cohomology class of $\Omega(A_0 \cdots A_d)$ depends only on the a_i. We are now justified in denoting this Schubert cycle by $\Omega(a_0 \cdots a_d)$.

Perhaps the most important result in the theory of cohomology classes is this: When several subvarieties intersect properly in a finite set of points, then the number of points, counted with multiplicity, is equal to the degree of the product of the corresponding cohomology classes. Roughly put, the theorem holds because passing to cohomology classes turns intersection into cup-product. For example, suppose each subvariety represents the d-planes in $\boldsymbol{P}^n$ which satisfy certain geometric conditions. Then the number of d-planes which simultaneously satisfy all the conditions, multiplicities being taken into account, can be determined by formally computing with the corresponding cohomology classes. Since the cohomology classes all remain the same when the subvarieties vary in a continuous system, this number will remain

constant when the geometric conditions are varied (or specialized) in a continuous way. This conclusion is an interpretation of Schubert's "principle of conservation of number."

We now state the first main theorem of Schubert calculus. It asserts that the Schubert cycles completely determine the cohomology of $G_{d,n}$.

THEOREM (The basis theorem). *Considered additively $H^*(G_{d,n}; Z)$ is a free abelian group and the Schubert cycles $\Omega(a_0 \cdots a_d)$ form a basis.*

By construction, the cohomology class of a subvariety X of $G_{d,n}$ lies in $H^{2p}(G_{d,n}; Z)$ when X is irreducible of dimension $[(d+1)(n-d)-p]$. Irreducibility means that X is not the union of two smaller subvarieties in a nontrivial way. The dimension of X is then r if an open subset of X is canonically a manifold of dimension r.

We now prove that $\Omega(A_0 \cdots A_d)$ is irreducible of dimension $\sum_{i=0}^{d}(a_i - i)$ with $a_i = \dim(A_i)$. First, suppose A_i consists of the points $(p(0), \cdots, p(n))$ with $p(j) = 0$ when $j > a_i$ and consider the space S of all $(d+1) \times (n+1)$-matrices $[p_i(j)]$ with $p_i(j) = 0$ when $j > a_i$ for $i = 0, \cdots, d$. Let S_0 be the open subset of S of matrices whose maximal minors $p(j_0 \cdots j_d) = \det[p_\alpha(j_\beta)]$ are not all zero. In the course of proving Proposition 3 we saw that sending a matrix $[p_i(j)]$ to the point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of P^N defines a map π of S_0 onto $\Omega(A_0 \cdots A_d)$. Since S is an affine space, it follows by an elementary argument that S_0 is irreducible and consequently that $\Omega(A_0 \cdots A_d)$ is irreducible. Now, let S_1 be the subset of S of matrices $[p_i(j)]$ whose submatrix $[p_i(a_j)]$ is the $(d+1) \times (d+1)$ identity. Then S_1 lies in S_0 and as we saw when proving Proposition 3, $\pi(S_1)$ is the open subset of $\Omega(A_0 \cdots A_d)$ of points $(\cdots, p(j_0 \cdots j_d), \cdots)$ with $p(a_0 \cdots a_d) \neq 0$. However, Proposition 2 implies that π induces an analytic isomorphism of S_1 with $\pi(S_1)$. Since S_1 is obviously an affine space of dimension $\sum_{i=0}^{d}(a_i - i)$, the dimension of $\Omega(A_0 \cdots A_d)$ is therefore this number.

We may now rephrase the basis theorem in the following way:

THEOREM (The basis theorem). *Each even dimensional integral cohomology group $H^{2p}(G_{d,n}; Z)$ is a free abelian group and the Schubert cycles $\Omega(a_0 \cdots a_d)$ with $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$ form a basis. Each odd dimensional group is zero.*

For example, consider the Grassmann manifold $G_{0,n}$ of points in P^n. The Plücker coordinates of a point are obviously its ordinary coordinates; hence, we have $G_{0,d} = P^n$ and $\Omega(A_0) = A_0$. Now, the basis theorem says that $H^{2p}(P^n; Z)$ for $0 \leqq p \leqq n$ is a free cyclic group generated by the class $\Omega(n-p)$ of an $(n-p)$-dimensional linear space. The other groups are zero.

For a second example, consider the Grassmann manifold $G_{1,3}$ of lines in P^3. Here, the basis theorem says that there are exactly five nonzero cohomology groups:

the middle one $H^4(G_{1,3}; \boldsymbol{Z})$ is free abelian on two generators $\Omega(0.3)$ and $\Omega(1.2)$ and the others $H^{2p}(G_{1,3}; \boldsymbol{Z})$ for $p = 0, 1, 3, 4$ are free cyclic on generators respectively $\Omega(2.3)$, $\Omega(1.3)$, $\Omega(0.2)$, $\Omega(0.1)$. Moreover, it is evident that $\Omega(0.1)$ is the class of a point, that $\Omega(2.3)$ is the class of $G_{1,3}$ and that in view of Corollary 5, $\Omega(1.3)$ is the class of a hyperplane section.

The following proposition complements the basis theorem with some very useful information.

PROPOSITION. *The basis* $\{\cdots, \Omega(a_0 \cdots a_d), \cdots\}$ *of the group* $H^{2p}(G_{d,n}; \boldsymbol{Z})$ *and the basis* $\{\cdots, \Omega(n-a_d, \cdots, n-a_0), \cdots\}$ *of the group* $H^{2[(d+1)(n-d)-p]}(G_{d,n}; \boldsymbol{Z})$ *are dual under the pairing* $v, w \mapsto \deg(v \cdot w)$ *of Poincaré duality.*

In other words, the proposition says that an arbitrary element v of $H^{2p}(G_{d,n}; \boldsymbol{Z})$ can be written uniquely in the form

$$v = \sum \delta(n-a_d, \cdots, n-a_0)\Omega(a_0 \cdots a_d),$$

where the integers $\delta(n-a_d, \cdots, n-a_0)$ can be found by using the formula

$$\delta(n-a_d, \cdots, n-a_0) = \deg(v \cdot \Omega(n-a_d, \cdots, n-a_0)).$$

In particular, if v is the cohomology class of an irreducible subvariety X of $G_{d,n}$, then each integer $\delta(n-a_d, \cdots, n-a_0)$ is nonnegative because it is the number of points with multiplicity in the intersection of X and $\Omega(B_0 \cdots B_d)$ for suitably chosen linear spaces B_i. Schubert called these integers the degrees (*Gradzahlen*) of X.

Let Y be an irreducible subvariety of $G_{d,n}$ of dimension p and let the integers $\varepsilon(a_0 \cdots a_d)$ be its degrees. If the intersection $X \cap Y$ is a finite set of points, then the number $i(X \cap Y)$ of points counted with multiplicity is, as we know, the degree of the product of

$$\sum \delta(n-a_d, \cdots, n-a_0)\Omega(a_0 \cdots a_d) \text{ and } \sum \varepsilon(a_0 \cdots a_d)\Omega(n-a_d, \cdots, n-a_0).$$

Therefore, by the proposition we have

$$i(X \cap Y) = \sum \delta(n-a_d, \cdots, n-a_0)\varepsilon(a_0 \cdots a_d).$$

This formula constitutes a generalization of Bézout's theorem. Bézout's theorem deals with the case $G_{0,n} = \boldsymbol{P}^n$. We saw above that the cohomology class v of an $(n-p)$-dimensional irreducible subvariety X of $\boldsymbol{P}^n$ is of the form $v = \delta(p)\Omega(n-p)$ and by the proposition $\delta(p)$ is the number of points with multiplicity in the intersection of X and a suitably chosen p-dimensional linear space. Thus $\delta(p)$ is the degree of X in the usual sense. Let Y be a p-dimensional irreducible subvariety of $\boldsymbol{P}^n$ and let $\varepsilon(n-p)$ be its degree. Suppose X and Y intersect in a finite set of points. Then the formula above becomes $i(X \cap Y) = \delta(p)\varepsilon(n-p)$; in other words, the number of points counted with multiplicity in $X \cap Y$ is the product of the degree of X and the degree of Y. This result is known as Bézout's theorem.

The basis theorem implies that the product of any two Schubert cycles can be uniquely expressed as a linear combination of other Schubert cycles with integers as coefficients. The second and third main theorems allow us to compute such expressions explicitly. The second expresses an arbitrary Schubert cycle as a determinant in the following $(n-d+1)$ special Schubert cycles:

$$\sigma(h) = \Omega(h, n-d+1, \cdots, n) \text{ for } h = 0, \cdots, (n-d).$$

THEOREM (The determinantal formula). *For all sequences of integers* $0 \leqq a_0 < \cdots < a_d \leqq n$ *the following formula holds in the cohomology ring* $H^*(G_{d,n}; \boldsymbol{Z})$:

$$\Omega(a_0 \cdots a_d) = \begin{vmatrix} \sigma(a_0) & \cdots & \sigma(a_0 - d) \\ \vdots & & \vdots \\ \sigma(a_d) & \cdots & \sigma(a_d - d) \end{vmatrix}$$

where we agree to put $\sigma(h) = 0$ *for* $h \notin [0, (n-d)]$.

This theorem, together with the basis theorem, implies that the special Schubert cycles generate the cohomology ring as a $\boldsymbol{Z}$-algebra. Moreover, it reduces the problem of determining the product of two arbitrary Schubert cycles to the case where one (or for that matter, each) is a special Schubert cycle. This case is handled by the third main theorem, which follows.

THEOREM (Pieri's formula). *For all sequences of integers* $0 \leqq a_0 < \cdots < a_d \leqq n$ *and for* $h = 0, \cdots, (n-d)$, *the following formula holds in the cohomology ring* $H^*(G_{d,n}; \boldsymbol{Z})$:

$$\Omega(a_0 \cdots a_d) \cdot \sigma(h) = \Sigma \Omega(b_0 \cdots b_d),$$

where the sum ranges over all sequences of integers $b_0 < \cdots < b_d$ *satisfying* $0 \leqq b_0 \leqq a_0 < b_1 \leqq a_1 < \cdots < b_d \leqq a_d$ *and* $\Sigma_{i=0}^d b_i = \Sigma_{i=0}^d a_i - (n - d - h)$.

Let us use these results to determine the number of lines L in $\boldsymbol{P}^3$ which (simultaneously) intersect four given lines L_1, L_2, L_3, L_4. In section three, we saw that such lines L are represented by the points of the intersection

$$Q = \bigcap_{i=1}^{4} \Omega(L_i, \boldsymbol{P}^3).$$

So, we want to compute the degree of $\Omega(1.3)^4$. By definition we have $\Omega(1.3) = \sigma(1)$ and Pieri's formula gives $\Omega(1.3) \cdot \sigma(1) = \Sigma \Omega(b_0 \cdot b_1)$ with $0 \leqq b_0 \leqq 1 < b_1 \leqq 3$ and $b_0 + b_1 = 3$. Hence we obtain $\Omega(1.3)^2 = \Omega(0.3) + \Omega(1.2)$. Now, the proposition yields $\Omega(0.3)^2 = 0$, $\Omega(1.2)^2 = 0$ and $\deg(\Omega(0.3) \cdot \Omega(1.2)) = 1$. Hence we find $\deg(\Omega(1.3)^4) = 2$. Alternately, a second application of Pieri's formula yields $\Omega(1.3)^3 = 2\Omega(0.2)$ and a third yields $\Omega(1.3)^4 = 2\Omega(0.1)$. Since $\Omega(0.1)$ is the class of a single point, its degree

is one. Thus, we again find $\deg(\Omega(1.3)^4) = 2$. Therefore, if Q is a finite set of points, then the number of points with multiplicity in Q is two. Thus the number of lines is either infinity or two or one (counted twice).

In the preceding example we obtained the formula $\Omega(1.3)^3 = 2\Omega(0.2)$. Since the various subvarieties in a continuous system are all assigned the same cohomology class, this formula suggests that the set of lines which simultaneously intersect three skew lines can be continuously deformed into the union of two sets of lines which lie in a plane and pass through a fixed point. In fact, we shall now see that this is the case.

Specialize the three lines L_1, L_2, L_3 so that L_1 and L_2 intersect in a point P and so that L_3 intersects the plane F of L_1 and L_2 in a point Q not equal to P. Then a line intersecting L_1 and L_2 must either lie in F or pass through P, and conversely a line lying in F or passing through P intersects L_1 and L_2. So a line intersecting L_1, L_2 and L_3 must either lie in F and pass through Q or pass through P and lie in the plane F' of P and L_3, and conversely a line lying in F and passing through Q or lying in F' and passing through P intersects L_1, L_2 and L_3. In other words, we have

$$\bigcap_{i=1}^{3} \Omega(L_i \cdot P^3) = \Omega(Q \cdot F) + \Omega(P \cdot F').$$

When the subvarieties of $G_{d,n}$ defined by more general geometric conditions are considered, the power of the calculus becomes staggering. Schubert's book contains many examples and we now give two.

Let us compute the number of lines L in P^3 which simultaneously intersect four given curves C_1, C_2, C_3, C_4. Let $c_i \in H^4(P^3; Z)$ be the cohomology class of C_i and ℓ the class of a line. We have $c_i = \delta_i \ell$, where δ_i is the degree of C_i, (see the discussion of Bezout's theorem after the proposition). So it is not surprising (and is justified below) that the lines L which intersect a given C_i are represented by the points of a subvariety X_i of $G_{1,3}$ and that the cohomology class x_i of X_i is of the form $x_i = \delta_i \Omega(1.3)$. Hence we have

$$x_1 x_2 x_3 x_4 = 2\delta_1\delta_2\delta_3\delta_4\Omega(0.1)$$

in view of the computations in the example above. So when the number of lines intersecting C_1, C_2, C_3, C_4 is finite and multiple solutions are taken into account, the number of lines is $2\delta_1\delta_2\delta_3\delta_4$. This result is indicated geometrically by specializing each C_i so that it becomes a union of δ_i lines, then the number of lines (simultaneously) intersecting C_1, C_2, C_3, C_4 is obviously $\delta_1\delta_2\delta_3\delta_4$ times the number of lines intersecting four lines and the latter number, we know, is 2.

To analyze each X_i rigorously, we need to consider the subset Z of the product $P^3 \times G_{1,3}$ consisting of the pairs (P, Q) such that the point P of P^3 lies on the line represented by Q. With a certain amount of elementary computations like those in sections one and two, one can show that Z is a complex manifold of dimension 5 which can be described by a system of (bihomogenous quadratic) polynomial

equations. Let $p: P^3 \times G_{1,3} \to P^3$ and $q: P^3 \times G_{1,3} \to G_{1,3}$ be the projections. Then we clearly have $X_i = q(Z \cap p^{-1}C_i)$ set-theoretically and it is easy to show that $x_i = q_*(z \cdot p^*c_i)$, where z is the cohomology class of Z, where p^* is the natural operation on cohomology induced by p and where q_* is the Poincaré dual of q^*. Similarly we have $\Omega(1.3) = q_*(z \cdot p^*\ell)$. Consequently the relation $c_i = \delta_i \ell$ implies the relation $x_i = \delta_i q_*(z \cdot p^*\ell) = \delta_i \Omega(1.3)$ as asserted.

Finally, we sketch a proof that two quadrics in P^4 have, in general, sixteen lines in common. A quadric Q in P^n is defined as the set of zeros of a single homogeneous polynomial F of degree two and the $m = \binom{n+2}{2}$ coefficients of F may be used to represent Q by a point q of P^{m-1}. First, we observe that the lines L in P^4 which lie on a general quadric are represented by the points ℓ of a 3-dimensional irreducible subvariety of $G_{1,4}$. Indeed, let W be the subset of $P^{14} \times G_{1,4}$ consisting of the pairs (q, ℓ) where q represents a quadric Q in P^4 and ℓ represents a line L lying in Q. Let $p: W \to P^{14}$ and $r: W \to G_{1,4}$ be the projections. A fiber of r represents the quadrics Q which contain a given line L. Let F_1, F_2, F_3 be independent homogeneous linear equations defining L. Then the polynomial F defining Q is obviously of the form

$$F = G_1F_1 + G_2F_2 + G_3F_3,$$

where G_i is a suitable homogeneous linear equation. Hence all such polynomials F form a vector space of dimension $(5 + 4 + 3) = 12$, so the fiber of r is P^{11}. Therefore W is an irreducible subvariety of dimension $[11 + \dim(G_{1,4})] = 17$. A general fiber of p, which represents the lines lying on a general quadric Q, is therefore irreducible of dimension $(17 - 14) = 3$.

Let Q be a general quadric in P^4, let X be the 3-dimensional irreducible subvariety of $G_{1,4}$ representing the lines lying in Q, and let x be the cohomology class of X. By the basis theorem, we have $x = \lambda\Omega(0.4) + \mu\Omega(1.3)$ and by the proposition, we have $\lambda = \deg(x \cdot \Omega(0.4))$ and $\mu = \deg(x \cdot \Omega(1.3))$. Now, no line lying in Q can pass through a point P of P^4 not in Q. Hence $X \cap \Omega(P, P^4)$ is empty, so we have $\lambda = 0$. On the other hand, a general 3-dimensional linear space A_1 intersects Q in a quadric Q_1 in this copy of P^3 and exactly four lines lying in Q_1 meet a general line A_0 lying in A_1 because A_0 intersects Q_1 in two distinct points and therefore meets a line of each ruling at each point. Hence $X \cap \Omega(A_0 \cdot A_1)$ consists of four points, so we have $\mu = 4$. Let Q' be another general quadric in P^4. Then the number of lines common to Q and Q', multiplicities being taken into account, is therefore equal to $\deg(x^2) = 4^2 \deg(\Omega(1.3)^2) = 16$.

5. Some comments, references and open questions. — Nearly everything discussed so far remains valid in characteristic p. The cohomology theory used in section four has been completely algebrized, and the material of sections two and three generalizes virtually without change over any ground field. In what follows, we shall work over an arbitrary ground field k and discuss restrictions on k as needed.

The work of Hodge and Pedoe [8] is by far the most complete reference. Their

treatment is purely algebraic and largely independent of the blanket assumption of characteristic zero.

Concerning section two.—Proceeding as in the first part of the proof of Theorem 1, however, using (Laplace) expansion of the determinants along several columns, one proves that the Plücker coordinates of a d-plane in P^n satisfy more quadratic relations, namely

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\, p(i_0 \cdots i_{\lambda-1}\sigma i_\lambda \cdots \sigma i_d)p(\sigma j_0 \cdots \sigma j_\lambda j_{\lambda+1} \cdots j_d) = 0,$$

where the sum ranges over all permutations σ of $(i_\lambda \cdots i_d j_0 \cdots j_\lambda)$ such that $\sigma i_\lambda < \cdots < \sigma i_d$ and $\sigma j_0 < \cdots < \sigma j_\lambda$. The quadratic relations (QR) occur when we take $\lambda = d$.

For each sequence of integers $i_0 \cdots i_d$ satisfying $0 \leqq i_0 < i_1 < \cdots < i_d \leqq n$ take an indeterminate $X(i_0 \cdots i_d)$ and then, by using the formulas (A), define $X(i_0 \cdots i_d)$ for any sequence of integers $i_0 \cdots i_d$ satisfying $0 \leqq i_j \leqq n$ for $j = 0, \cdots, d$. In these terms, we can now say that $G_{d,n}$ is contained in the set of zeros of all the homogeneous quadratic polynomials of the form

$$\text{(QP)} \qquad \sum_{\sigma} \operatorname{sgn}(\sigma) X(i_0 \cdots i_{\lambda-1}\sigma i_\lambda \cdots \sigma i_d) X(\sigma j_0 \cdots \sigma j_\lambda j_{\lambda+1} \cdots j_d),$$

where the sum ranges over the same permutations as above. Now, Theorem 1 says that $G_{d,n}$ can be expressed as the set of zeros of the particular such polynomials with $\lambda = d$. Consequently, $G_{d,n}$ can be expressed as the set of zeros of all the polynomials of the form (QP) as well. It can be shown formally (by a proof like that of (9) on page 379 of Vol. II of [8]) that each polynomial of the form (QP) is a linear combination with rational numbers as coefficients of the particular ones with $\lambda = d$ and it is an open question whether integers may be used as coefficients.

Let I be the ideal in the polynomial ring $R = k[\cdots, X(i_0 \cdots i_d), \cdots]$ generated by the polynomials of the form (QP) and let J be the subideal generated by the particular ones with $\lambda = d$. It can be shown that I is a prime ideal.

It then follows from the fact that I and J have the same zeros, that I is the radical of J. An interesting open question is whether I is always equal to J. They are equal in characteristic zero and would always be equal if the integers could be used as coefficients above.

The ring R/I is called the **homogeneous coordinate ring** of $G_{d,n}$ and plays an important role in the study of its geometry. The ring is naturally graded and the m-th graded piece consists of the residue classes of the homogeneous polynomials of degree m. Hodge and Littlewood (see [8], vol. II, chap. XIV, §9) have proved an explicit formula, known as the **postulation formula**, which expresses the dimension of m-th graded piece, for every m, as the value of a certain polynomial.

Igusa [9] (Theorem 1, p. 310) proved that R/I is a normal domain and derived several important results in invariant theory from this fact. The ring R/I is in fact a

unique factorization domain (see Samuel [**21**], Proposition 8.5, p. 38); this fact easily yields Severi's result that every $[(d+1)(n-d)-1]$-dimensional irreducible subvariety of $G_{d,n}$ is the intersection of $G_{d,n}$ and the set of zeros of a single homogeneous polynomial.

More recently, it has been proved (see Hochster [**6**] and Laksov [**16**]) that R/I is a Cohen-Macaulay ring. It follows by general principles that there is an exact sequence

$$O \to F_r \to F_{r-1} \to \cdots \to F_1 \to R \to R/I \to 0,$$

where the F_i are free R-modules and r is equal to $[N-(d+1)(n-d)]$. It is an interesting open problem to give an explicit natural such sequence, or in other words to find the syzygies of the ideal I of R.

Concerning section three.—For each Schubert subvariety $\Omega(A_0 \cdots A_d)$ of $G_{d,n}$, let $I(A_0 \cdots A_d)$ be the ideal of R generated by the quadratic polynomials of the form (QP) and the linear polynomials corresponding to the linear equations of Corollary 5. An important method for proving a result about the ring R/I is to prove more generally a corresponding result for each ring $R/I(A_0 \cdots A_d)$ by induction on the dimension of $\Omega(A_0 \cdots A_d)$. For example, this method is used to establish the postulation formula and the Cohen-Macaulay nature of R/I.

Another reason for interest in the rings $R/I(A_0 \cdots A_d)$ is that locally each $\Omega(A_0 \cdots A_d)$ can be described as the zeros of certain minors in the affine space of $(d+1) \times (n-d)$-matrices. For example, suppose that A_i consists of the points in P^n of the form $(p(0), \cdots, p(a_i), 0, \cdots, 0)$ and that for some $s \leqq d$ we have $a_i = (d-s+i)$ for $i = 0, \cdots, s$. Then Proposition 3 asserts that a point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $G_{d,n}$ lies in $\Omega(A_0 \cdots A_d)$ if and only if $p(j_0 \cdots j_d)$ is zero whenever we have $a_i < j_i$ for some i At the end of section two, we noted that the points $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $G_{d,n}$ with $p(0 \cdots d) \neq 0$ are in natural bijective correspondence with the space of $(d+1) \times (n+1)$ matrices $[p_\alpha(j)]$ such that the $(d+1) \times (d+1)$ submatrix consisting of the first $(d+1)$ columns is the identity. Now, suppose that $p(j_0 \cdots j_d)$ is zero whenever we have $a_i < j_i$ for some i. Fixing $i \geqq s$ and considering all sequences

$$0 \leqq j_0 < \cdots < j_{i-1} \leqq d \leqq a_i < j_i < \cdots < j_d \leqq n$$

we easily conclude that all $(d-i+1) \times (d-i+1)$-minors of the $(d+1) \times (n-a_i)$-submatrix of $[p_\alpha(j)]$ consisting of the last $(n-a_i)$ columns are zero. Conversely, suppose that all such minors are zero whenever we have $i \geqq s$. Consider a determinant $p(j_0 \cdots j_d) = \det[p_\alpha(j_\beta)]$ with $a_i < j_i$ for some i. Since $i < s$ clearly implies $a_s < j_s$, we may assume $i \geqq s$. Then (Laplace) expansion of the determinant along the last $(d-i+1)$ columns shows that it is zero. Thus the points $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $\Omega(A_0 \cdots A_d)$ with $p(0 \cdots d) \neq 0$ can be described as the zeros of all the $(d-i+1) \times (d-i+1)$-minors from the last $(n-a_i)$ columns for all $i \geqq s$ in the affine space of $(d+1) \times (n-d)$-matrices.

The zeros of determinantal equations are called **determinantal varieties** and have been studied for a long time (see Room [19]). Many of their properties can be easily deduced from corresponding properties of Schubert varieties. For example, let I' be the ideal of $k[X_{ij}]$ generated by the corresponding determinantal polynomials. The ring $k[X_{ij}]/I'$, known as the **coordinate ring** of the determinantal variety, is Cohen-Macaulay because a corresponding ring $R/I(A_0 \cdots A_d)$ is. Particular cases of this result were proved by Macaulay [17]; however, the general result was first established by Hochster and Eagon [7] without reference to the Grassmann manifold.

The syzygies of the ideal I' would be known if the syzygies of the corresponding ideal $I(A_0 \cdots A_d)$ were known, but in both cases it is an open problem to find the syzygies. In special cases they have been determined by Macaulay and Eagon—Northcott [3]. Recently, Kempf [10] has found a powerful way of determining syzygies, which gives an elegant treatment of some of the known cases and leads to the solution of new cases; (recently this was proved by Svanes [24]).

The Schubert varieties are, in general, singular. (Over the complex numbers, a singularity is a point where a subvariety is not a complex submanifold.) In fact, a point of $\Omega = \Omega(A_0 \cdots A_d)$ is singular if and only if the corresponding d-plane L in P^n satisfies $\dim(A_i \cap L) \geqq i$ for all i, as usual, and also $\dim(A_j \cap L) \geqq j + 1$ for some j. Hence, the singular locus of Ω is a union of other Schubert varieties, and so the stratification $\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_m = \phi$, where Ω_i is the singular locus of Ω_{i-1}, is exceedingly well-behaved. Moreover, as we noted above, Ω is locally Cohen-Macaulay and so, since its singular locus is sufficiently small (of codimension at least two), Ω is also normal. Thus, the singularities are very nice. However, it remains to be proved that (trivial exceptions aside) these singularities are rigid—that any infinitesmal family varying an open piece of Ω must be analytically isomorphic to the trivial or product family. The rigidity is known in a very special case and it has applications to the theory of smoothing singularities (see Kleiman-Landolfi [14]).

Concerning section four.—Let us work over the complex numbers for a while. Most of the results of cohomology theory we used have become standard algebraic topology, but the assignment of a cohomology class to an algebraic subvariety of an algebraic manifold has not become standard. While early triangulations of such subvarieties have more recently been found unsatisfactory, today it is relatively easy to define the cohomology class either by using integration or relative (or local) cohomology and the difficulty lies in establishing the desired properties. A recent account of the theory is found in the article [2] of Borel and Haefliger.

The basis theorem was first proved by Ehresmann (see [4] §10, pp. 416–418). He observed that the Schubert varieties furnish a cellular decomposition of the Grassmann manifold because each Schubert variety contains an open subset which is an affine space (as we noted on the way to reformulating the basis theorem) and because the complement of this open set in the Schubert variety is the union of certain smaller Schubert varieties. The basis theorem then follows from some general results

about cell complexes which were included for this purpose and which have become standard. Ehresmann (see [4] §11, pp. 418–422) also proved the proposition complementing the basis theorem by a simple direct computation involving suitably chosen Schubert varieties to represent the Schubert cycles in question. He did not mention either the determinantal formula or Pieri's formula.

Another approach to Schubert calculus is by way of algebraic groups. When proving Proposition 4 in section three, we saw that the group $GL(n+1)$ of invertible $(n+1) \times (n+1)$-matrices acts on the Grassmann manifold $G_{d,n}$. It is easy to see that the action is transitive and that the d-plane in P^n whose points are of the form $(p(0), \cdots, p(d), 0, \cdots, 0)$ is left fixed by the matrices of the form

$$\begin{array}{c} \overbrace{\quad}^{d+1} \\ d+1 \left\{ \left[\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right] \right. \end{array}$$

These matrices form a (parabolic) subgroup of $GL(n+1)$ and $G_{d,n}$ can obviously be considered as the quotient of $GL(n+1)$ by this subgroup. This observation suggests looking more generally at any quotient of a semi-simple algebraic group by a parabolic subgroup. The decomposition into Schubert cells can be correspondingly generalized by means of the Bruhat decomposition (see Borel [1], Theorem, page 347), and Kostant [15] has discovered a close connection between the (generalized) Schubert calculus and representation theory. In the case of the Grassmann manifold, the explicit formulas of (ordinary) Schubert calculus result from classical formulas of representation theory. In the general case, the situation is not fully understood.

Over an algebraically closed field of any characteristic, there are several purely algebraic theories which can take the place of classical cohomology. By far the most difficult to develop are the so called "Weil cohomologies" such as ℓ-adic cohomology. Over the complex numbers these theories are equivalent to classical cohomology and in any characteristic they have properties like the Künneth formula, Poincaré duality and classes for subvarieties. There are several less sophisticated theories (see Samuel [20]) which formally resemble the part of cohomology generated by the classes of subvarieties, but which may be weaker, that is, contain more information. The most popular of these is the weakest and is known as the **Chow ring** (see [23] and [25]). These theories constitute the topological and algebraic intersection theories (mentioned in section one), and we shall refer to any one of them as a **generalized cohomology theory**. At any rate, they are all equivalent for the Grassmann manifolds and the other varieties with cellular decompositions.

In Hodge-Pedoe [8], a generalized cohomology theory is developed in characteristic zero and the basis theorem for the Grassmann manifold $G_{d,n}$ is proved by induction on n. Then, the proposition complementing the basis theorem is proved by the same direct computation Ehresmann used. Next, Pieri's formula is deduced from the basis theorem and the proposition by another direct computation of the

same type. Finally, the determinantal formula is deduced formally from Pieri's formula. In fact, with a generalized cohomology theory and the basis theorem given, the remaining three results can always be derived without difficulty in this way in any characteristic.

The Grassmann manifold $G_{d,n}$ can obviously be thought of as representing the $(d+1)$-dimensional (vector) subspaces of an $(n+1)$-dimensional vector space. From this point of view, it is natural to consider the trivial vector bundle of rank $(n+1)$ on $G_{d,n}$ and its canonical subbundle E whose fiber over a point of $G_{d,n}$ is the $(d+1)$-dimensional (vector) subspace of the $(n+1)$-dimensional vector space represented by the point. This subbundle E is universal in the sense that for any variety X and for any subbundle of rank $(d+1)$ of the trivial bundle of rank $(n+1)$ on X, there is a unique map of X into $G_{d,n}$ such that the subbundle E on $G_{d,n}$ induces the given subbundle on X.

A general theory of Chern classes with values in any generalized cohomology theory has been worked out (see Grothendieck [**5**]), and the special Schubert cycle $\sigma(h)$ is exactly the $(n-d-h)$-th Chern class of the quotient of the trivial bundle of rank $(n+1)$ on $G_{d,n}$ by the universal subbundle (see Kleiman [**12**], p. 297). The results of Schubert calculus now yield a description of the generalized cohomology of $G_{d,n}$ as the ring generated by these Chern classes. Grothendieck (see [**23**], Théorème 1, p. 4–19) has given a formal derivation of this description, without any mention of Schubert varieties or cycles.

The determinantal formula is related to a very useful formula of Porteous in differential geometry and it appears in the study of the singularities of a map (see [**18**]). The determinantal formula is also the key to proving the existence of certain special divisors on curves (see Kempf [**11**] and Kleiman-Laksov [**13**]), and in his article [**11**], Kempf gives a nice direct proof of the formula.

Another source of interest in Schubert varieties is the problem of smoothing cycles. The problem is to show that the class of any subvariety Z of a nonsingular algebraic variety V is the difference of two classes each the class of a nonsingular subvariety. When $\dim(Z) < (\dim(V)+2)/2$ holds, then some multiple of the class of Z is such a difference and the proof involves a careful study of the geometry of certain Schubert varieties (see Kleiman [**12**]). However, it is suspected that the general problem has a negative solution and in fact that the Schubert cycle $\sigma(1)$ on the Grassmann manifold $G_{2,5}$ is not the difference of two cycles each the class of a nonsingular subvariety, nor is any multiple of $\sigma(1)$.

The examples from enumerative geometry we considered, while simple, illustrate fairly well the use of Schubert calculus. Classically relatively complicated geometric situations were studied. They often involved tangency conditions such as requiring a line to be an n-fold tangent to a given curve or requiring a line to intersect a given surface and lie in the tangent plane of the surface at the point of intersection. In principle, the method is always the same: describe the problem in terms of subvarieties of a Grassmann manifold; find the degrees of each subvariety; and use the

formulas of Schubert calculus to compute the product of the classes of the subvarieties. Moreover, each degree is the number of points of intersection of a subvariety with a certain Schubert variety, or in other words, it is the number of solutions to a certain simpler enumerative problem. In practice, finding the degrees can be difficult and may, as in the case of tangency conditions, involve more sophisticated algebraic geometry.

Although we have given the "principle of conservation of number" a rigorous mathematical interpretation, it is usually difficult to use it because it is difficult to know what the correct multiplicities are. For example, consider the lines in P^3 intersecting lines L_1, L_2, L_3, L_4; how can we tell by direct geometric means that if L_1, L_2 and L_3 are skew and L_4 intersects each of them, then the one solution (found at the end of section three) should be counted with multiplicity two, or, for that matter, how can we tell that if L_1 intersects L_2 and L_3 intersects L_4, then the two solutions (found in section one) should each be counted with multiplicity one? In the general case of an enumerative problem, it is possible to prove, in characteristic zero and often in characteristic p, that the solutions all appear with multiplicity one. Thus, for example, we may assert that the number of distinct lines in P^3 meeting four curves C_1, C_2, C_3, C_4 of degree $\delta_1, \delta_2, \delta_3, \delta_4$ is, in general, $2\delta_1\delta_2\delta_3\delta_4$ and that two quadrics in P^4 have, in general, sixteen lines in common. In analyzing the latter example, we used geometric means to see that there are four lines which simultaneously lie on a general quadric, lie on a general 3-plane and intersect a general line in this 3-plane. Here we are able to say that each solution appears with multiplicity one because the quadric, the 3-plane and the line in the 3-plane satisfy no special conditions.

In more abstract terms, we can assert in characteristic zero (see [8], p. 338) that for any two irreducible subvarieties X and Y of $G_{d,n}$, the components all appear with multiplicity one in the intersection of X and the image of Y under the linear transformation of $G_{d,n}$ into itself induced by any sufficiently general invertible $(d+1) \times (n+1)$-matrix. It would be interesting to know what happens in characteristic p.

Supported in part of NSF GP 28936.

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PRIME FACTORS OF CONSECUTIVE INTEGERS

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For each positive integer k, there exists a corresponding positive integer m such that in any sequence of m consecutive integers greater than k there is at least one having a prime factor greater than k. A simple demonstration of this fact comes from a modification of Euclid's proof that there are infinitely many primes. Let P be the product of all primes not larger than k, and let $a_1 < a_2 < \cdots < a_{\phi(P)}$ be the positive integers not greater than P which are prime relative to P. For any a_i and any integer r, the number $rP + a_i$ is prime relative to P so, if greater than 1, has