Definition 5.7 *A braided monoidal category* $(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ *is* balanced *if it is equipped with a natural automorphism* $\theta : 1_{\mathcal{V}} \Rightarrow 1_{\mathcal{V}}$ *called the* balancing *or* twist map *satisfying*

$$\theta_I = Id_I$$

and

$$\theta_{A\otimes B} = \sigma_{B,A}(\sigma_{A,B}(\theta_A \otimes \theta_B)).$$

Notice that the second condition may be rewritten as

$$[\sigma_{B,A}]^{-1}(\theta_{A\otimes B}) = \sigma_{A,B}(\theta_A \otimes \theta_B).$$

In Chapter 12 we will give another characterization of braided monoidal categories in terms of a "multiplication" on a monoidal category.

Another concept familiar from the case of categories of vector-spaces is the notion of a dual object.

Definition 5.8 *A* right *(resp.* left*)* dual to an object X *in a monoidal category* $\mathcal{V} = (\mathcal{V}, \otimes, I, \alpha, \lambda, \rho)$ *is an object* X^* *(resp.* ${}^*X$*) equipped with maps* $\epsilon : X \otimes X^* \to I$ *and* $\eta : I \to X^* \otimes X$ *(resp.* $e : {}^*X \otimes X \to I$ *and* $h : I \to X \otimes {}^*X$*) such that the composites*

$$X \xrightarrow{\rho^{-1}} X \otimes I \xrightarrow{X\otimes\eta} X \otimes (X^* \otimes X) \xrightarrow{\alpha^{-1}} (X \otimes X^*) \otimes X \xrightarrow{\epsilon\otimes X} I \times X \xrightarrow{\lambda} X$$

and

$$X^* \xrightarrow{\lambda^{-1}} I\otimes X^* \xrightarrow{\eta\otimes X^*} (X^*\otimes X)\otimes X^* \xrightarrow{\alpha} X^*\otimes(X\otimes X^*) \xrightarrow{X^*\otimes\eta} X^*\otimes I \xrightarrow{\rho} X^*$$

(resp.

$$X \xrightarrow{\lambda^{-1}} I \otimes X \xrightarrow{h\otimes X} (X \otimes {}^*X) \otimes X \xrightarrow{\alpha} X \otimes ({}^*X \otimes X) \xrightarrow{X\otimes e} X \otimes I \xrightarrow{\rho} X$$

and

$${}^*X \xrightarrow{\rho^{-1}} {}^*X\otimes I \xrightarrow{{}^*X\otimes h} X\otimes(X\otimes {}^*X) \xrightarrow{\alpha^{-1}} ({}^*X\otimes X)\otimes {}^*X \xrightarrow{e\otimes {}^*X} I\otimes {}^*X \xrightarrow{\lambda} {}^*X)$$

are identity maps.

Notice that in the case of a symmetric monoidal category

$$(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda, \sigma),$$

a right dual to any object is canonically a left dual by taking $e = \sigma\epsilon$ and $h = \eta\sigma$.

This type of duality is an abstraction from the sort of duality which exists in categories of finite dimensional vector-spaces. It is not hard to show that the canonical isomorphism from the second dual of a vector-space to the space generalizes to give canonical isomorphisms $k : {}^*(X^*) \to X$ and $\kappa : ({}^*X)^* \to X$. In general, however, there may not even be any maps from X^{**} or ${}^{**}X$ to X (cf. [22]). In cases where every object admits a right (resp. left) dual, it is easy to show that a choice of right (resp. left) dual for every object extends to a contravariant functor, whose application to maps will be denoted f^* (resp. ${}^*f$), and that the canonical maps noted above become natural isomorphisms between the compositions of these functors and the identity functor. Likewise, it is easy to show that $(A \otimes B)^*$ is canonically isomorphic to $B^* \otimes A^*$, and similarly for left duals.

In the case of a braided monoidal category every right dual is also a left dual, but in general the left dual structure is non-canonical (cf. [22]). In symmetric monoidal categories, we return to the familiar: right duals are canonically left duals. In non-symmetric braided monoidal categories it is possible to provide a canonical left dual structure on all right duals only in the presence of additional structure on the category: the category must be balanced and the balancing be related to the duality structure in a natural way.

Definition 5.9 *A braided monoidal category* $\mathcal{C}$ *is* ribbon (or tortile) *if all objects admit right duals, and it is equipped with a balancing* $\theta : 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$*, which moreover satisfies*

$$\theta_{A^*} = \theta_A^*.$$

Definition 5.10 *A symmetric monoidal category* $\mathcal{C}$ *is* rigid *if all objects admit (right) duals.*

$$\begin{tikzcd}
X^{**} \arrow[r, "\phi_{X^*}"] \arrow[d, "(\phi_X^{-1})^*"'] & {}^*(X^*) \arrow[d, "k"] \\
({}^*X)^* \arrow[r, "\kappa"'] & X
\end{tikzcd}$$

Figure 5.2: Two-Sided Dual Condition

Definition 5.11 *A monoidal category is* sovereign *if it is equipped with a choice for each object X of a right dual X^* and a left dual ${}^*X$, and a natural isomorphism $\phi_X : X^* \to {}^*X$ satisfying the condition of Figure 5.2.*

We then have the following theorem, which is due to Deligne [15] (cf. also [61], where a more detailed proof may be found):

Theorem 5.12 *Every ribbon category is a sovereign category when equipped with the left-duals obtained by letting ${}^*X = X^*$ with structure given by $e = \sigma^{-1}\epsilon$ and $h = \eta\sigma$, and conversely.*

sketch of proof: The proof is reduced to a sequence of lemmas. Throughout, we use Mac Lane's coherence theorem to justify the suppression of all instances of monoidal structure maps.

Lemma 5.13 *The identity maps on the right duals are components of a natural isomorphism from the right dual functor to the left dual functor (with the given structure maps).*

sketch of proof of Lemma 5.13: This amounts to saying that the functors ${}^*(-)$ and $(-)^*$ are equal. This is immediate by construction for objects, but must be checked for maps. The reader familiar with the diagrams that can be used to represent maps in braided monoidal categories can easily recover the proof given in [61]. Briefly, one first shows that

$${}^*f = [{}^*X \otimes h_X][e_X \otimes X^*]f^*$$

by using the naturality of σ (twice) and the right duality structure of $(-)^*$ (once). One then uses the left duality structure to obtain the desired result. □

Lemma 5.14 *Any natural isomorphism $\phi : X^* \to {}^*X$ is induced by a natural automorphism of the identity functor $\theta : X \to X$, and conversely any natural automorphism of the identity functor induces a natural isomorphism from $(-)^*$ to ${}^*(-)$.*

proof of Lemma 5.14 This is immediate from the previous lemma and the dinaturality properties of ϵ and η. Given ϕ, θ is given by

$$\theta_X = [\eta_X \otimes X][\phi_X \otimes X \otimes X]\sigma_{X,X}[e \otimes X],$$

while given θ, ϕ is given by

$$\phi_X = [h \otimes X^*]\sigma^{-1}_{X,X^*}[\theta_X \otimes X^* \otimes {}^*X][\epsilon_X \otimes {}^*X].$$

□

Lemma 5.15 *A natural isomorphism $\phi : X^* \to {}^*X$ provides a sovereign category structure for the right and left dual structures given in the statement of the theorem if and only if the corresponding natural automorphism $\theta : Id_{\mathcal{C}} \Rightarrow Id_{\mathcal{C}}$ satisfies the balancing axioms of Definition 5.7.*

sketch of proof of Lemma 5.15

The proof that the balancing condition implies sovereignty is done by calculating the two composites in the diagram obtained from that of Figure 5.2 by inverting both vertical maps. By using the naturality

conditions on the braiding and the dinaturality of the structure maps for the right duals, it follows that $\kappa_X k_X^{-1}$ equals

$$[h_{{}^*X} \otimes ({}^*X)^*][{}^*X \otimes \sigma_{(X^*)^*,({}^*X)^*}][\epsilon_X \otimes (X^*)^*].$$

(Recall that for any object Y, ${}^*Y = Y^*$.) Observing that $\theta_I = Id_I$, it follows from the naturality of θ that

$$\kappa_X k_X^{-1} = \\ [h_{{}^*X} \otimes ({}^*X)^*][{}^*X \otimes \sigma_{(X^*)^*,({}^*X)^*}][\theta_{{}^*X \otimes ({}^*X)^*} \otimes {}^*(X^*)][\epsilon_X \otimes {}^*(X^*)].$$

Similarly, recalling the definition of ϕ in terms of θ and the definition of $(-)^*$ on maps, one can use the triangle condition and dinaturality of the unit and counit of the structure maps for $(-)^*$ and the naturality and invertibility of the braiding to show that

$$\phi_X^* \phi_{X^*} = \\ [h_{{}^*X} \otimes ({}^*X)^*][{}^*X \otimes \sigma_{(X^*)^*,({}^*X)^*}][\theta_{{}^*X} \otimes \theta_{({}^*X)^*} \otimes {}^*(X^*)] \\ [\sigma^2 \otimes {}^*(X^*)][\epsilon_X \otimes {}^*(X^*)].$$

It thus follows that if θ satisfies the balancing axiom

$$\theta_{A \otimes B} = [\theta_A \otimes \theta_B]\sigma_{A,B}\sigma_{B,A},$$

then ϕ defined in the theorem gives a sovereign structure on the category for the given right duals and left duals obtained by "twisting" with the braiding.

The key to the reverse implication is to consider in detail the condition that ϕ be a *monoidal* natural transformation. Let

$$b_{X,Y} : (X \otimes Y)^* \to Y^* \otimes X^*$$

be the canonical isomorphism which makes $(-)^*$ into a monoidal functor. In this case, the condition that b be the structure maps for the monoidal functor is equivalent to the condition

$$\eta_{X \otimes Y}[b \otimes X \otimes Y] = \eta_Y[Y^* \otimes \eta_X \otimes X^*]$$

and a similar condition relating ϵ and b^{-1}.

Composing both sides of this equation with σ and applying the naturality of σ to both sides shows that

$$h_{X \otimes Y}[X \otimes Y \otimes B] = h_X[X \otimes h_Y \otimes {}^*X][X \otimes Y \otimes \sigma^2],$$

and a similar calculation for the condition on ϵ and ϵ shows that the structure map for ${}^*(-)$ as a monoidal functor is $b\sigma^{-2}$. The condition that ϕ be a monoidal natural transformation becomes

$$b[\phi \otimes \phi] = \phi b \sigma^{-2}$$

or equivalently,

$$b_{X,Y}[\phi_X \otimes \phi_Y]\sigma^2 b_{X,Y}^{-1} = \phi_{X \otimes Y}.$$

Now recalling the definition of θ in terms of ϕ, and calculating $\theta_{X \otimes Y}$ by substituting the left-hand side of the last equation for $\phi_{X \otimes Y}$, applying the defining property of b and using the naturality and invertibility of the braiding, we obtain the balancing condition for θ as defined in terms of ϕ.

Thus we establish the lemma and the theorem. □ □

In the case of categories of modules over a bialgebra A, the structures discussed in this chapter correspond to the additional structures discussed at the end of the previous chapter. We state without proof:

Theorem 5.16 *If A is a bialgebra over K, then the following implications hold for the category A-**mod** with the induced monoidal structure of Example 3.7:*

*1. If A is a Hopf algebra , then A-**mod** has right (and left) duals.*