# FINITE ANALYSIS AND THE DISCRETE FOURIER TRANSFORM

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## 1. The Discrete Fourier Transform on Finite Abelian Groups

### 1.1. The setup.

Let $G$ be a finite abelian group,

$$G \simeq \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}$$

say $|G| = d$, and define the space of functions to be transformed

**Definition.**

$$\begin{aligned} L^2(G) &= \{f : G \to \mathbb{C}\} \\ \langle f, g \rangle_{L^2(G)} &= \sum_{x \in G} f(x)\overline{g}(x) \\ \|f\|^2_{L^2(G)} &= \langle f, f \rangle_{L^2(G)} \\ &= \sum_{x \in G} f(x)\overline{f}(x) \end{aligned}$$

**Proposition 1.** *$L^2(G)$ is a $d$-dimensional $\mathbb{C}$-vector space with an orthonormal basis consisting of the delta functions*

$$\delta_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

*Proof.* **Linear independence.** Suppose $\alpha_a \in \mathbb{C}$ $(a \in G)$ satisfy $\sum_{a \in G} \alpha_a \delta_a = 0$. For each $x \in G$ we have $x = a$ for some $a$ and therefore $\alpha_a \delta_a(x) = 0$ if and only if $\alpha_a = 0$. As each term $\alpha_a \delta_a(x) = 0$ for all $x \in G$ we see that $\alpha_a = 0$ necessarily for all $a$.

**Spanning.** Given $f \in L^2(G)$ it is clear that $f(x) = \sum_{a \in G} f(a)\delta_a(x)$ for all $x \in G$.

**Orthogonality.** If $a \neq b$ then

$$\langle \delta_a, \delta_b \rangle = \sum_{x \in G} \delta_a(x)\overline{\delta}_b(x) = \sum \delta_a(x)\delta_b(x) = 0 + 1 \cdot \delta_b(a) + \delta_a(b) \cdot 1 + 0 = 0.$$

*Date*:

**Normality.** Well,

$$\|\delta_a\|_2^2 = \langle \delta_a, \delta_a \rangle = \sum_{x \in G} \delta_a(x)\overline{\delta}_a(x) = \sum_{x \in G} \left|\delta_a\right|^2(x) = 1 \qquad \square$$

Consequently, there is a vector space isomorphism $L^2(G) \to \mathbb{C}^d$

$$f \mapsto \big(f(a_1), \cdots, f(a_d)\big)$$

which will be used in proving the discrete case of the Wirtinger inequality.

To complete the setup for the first result we define also the operations of **translation** and **convolution** on $L^2(G)$: given $s \in G$ and $f, g \in L^2(G)$

$$\tau_s f(x) = f(s + x)$$
$$f * g(x) = \sum_{a \in G} f(a)g(x - a)$$

1.2. **The dual group.** A **character of** $G$ is a group homomorphism from $G$ into the torus, under multiplication.

**Definition.** The set of all characters of $G$ under pointwise multiplication is called **the dual group** $\widehat{G}$ of $G$,

$$\widehat{G} = \big\{\chi \in \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{T})\big\} \qquad \chi_1\chi_2(x) = \chi_1(x)\chi_2(x).$$

In this group the "trivial character" $\chi(x) \equiv 1$ is the identity and the inverse of $\chi \in \widehat{G}$ is $\chi^{-1}(x) = \overline{\chi}(x) = \chi(-x)$ since

$$\chi(x)\overline{\chi}(x) = \left|\chi(x)\right|^2 = 1 \quad \forall x \in G$$

for all $\chi \in \widehat{G}$.

**Proposition 2.** *Orthogonality of characters.*
*If* $\chi, \psi \in \widehat{G}$ *then*

$$\langle \chi, \psi \rangle = \begin{cases} |G| & \text{if } \chi = \psi \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Observe that it suffices to consider $\psi = 1$ since $\langle \chi, \psi \rangle = \langle \chi\overline{\psi}, 1 \rangle$. Denote

$$S := \langle \chi, 1 \rangle = \sum_{x \in G} \chi(x).$$

It is clear that $S = |G|$ if $\chi = 1$, so assume there exists $a \in G$ such that $\chi(a) \neq 1$. Multiply $S$ by $\chi(a)$

$$S\chi(a) = \left(\sum_{x \in G} \chi(x)\right)\chi(a) = \sum_{x \in G} \chi(x + a)$$

and make the change of variable $y = x + a$, observing that $\sum_x = \sum_y$. Then

$$S\chi(a) = \sum_{y\in G} \chi(y) = S$$

and since $\chi(a) \neq 1$ it follows that $S = 0$. □

The space of functions $L^2(\widehat{G})$ is defined by

**Definition.**

$$L^2(\widehat{G}) = \{F : \widehat{G} \to \mathbb{C}\}$$
$$\langle F, G\rangle_{L^2(\widehat{G})} = \sum_{\chi\in\widehat{G}} F(\chi)\overline{G}(\chi)$$
$$\|F\|^2_{L^2(\widehat{G})} = \langle F, F\rangle_{L^2(\widehat{G})}$$

Given $a \in G$, write $a = (a_1, \cdots, a_r)$ and for a given $x = (x_1, \cdots, x_r) \in G$ define a character by

$$e_a(x) = \prod_{j=1}^{r} e_{a_j}(x_j)$$

where $e_{a_j}(x) = \exp\left(\frac{2\pi i a_j}{m_j} x_j\right)$. Noting that $e_{a+b} = e_a e_b$, observe that $G \hookrightarrow \widehat{G}$ by $a \mapsto e_a$. The following properties are obvious and will be used in the sequel, so we record them here.

**Lemma 3.** *For all $x, \xi \in G$ we have*

1. $e_\xi(x) = e_x(\xi)$
2. $\overline{e_\xi}(x) = e_{-\xi}(x) = e_\xi(x)$

*Proof.* Immediate from the definition. □

## 1.3. **Basic properties of the DFT.**

**Definition.** The **discrete Fourier transform** (DFT) is the map $L^2(G) \to L^2(\widehat{G})$ defined by

$$\widehat{f}(\chi) = \sum_{x\in G} f(x)\overline{\chi}(x).$$

The **inverse DFT** is defined by

$$f^\vee(\chi) = \sum_{x\in G} f(x)\chi(x).$$

*Remark.*

- For all $a \in G$, $\widehat{\delta}_a(\chi) = \sum_{x\in G} \delta_a(x)\overline{\chi}(x) = \overline{\chi}(a)$

- $f * \delta_a = \tau_{-a} f$, because
$$f * \delta_a(x) = \sum_{y \in G} f(y)\delta_a(x-y) = f(x-a)$$
- $\widehat{\overline{f}} = \overline{\check{f}}$ and $\overline{\widehat{f}} = \check{\overline{f}}$, because
$$\widehat{\overline{f}}(\chi) = \overline{\sum_{x \in G} f(x)\overline{\chi}(x)} = \sum_x \overline{f}(x)\chi(x) = \check{\overline{f}}(\chi)$$
and
$$\widehat{\overline{f}}(\chi) = \sum_{x \in G} \overline{f}(x)\overline{\chi}(x) = \overline{\sum_x f(x)\chi(x)} = \overline{\check{f}}(\chi)$$

**Lemma 4.** *For all $f, g \in L^2(G)$ we have*
$$\sum_{x \in G} f(x)\widehat{g}(x) = \sum_{x \in G} \widehat{f}(x) g(x)$$

*Proof.*
$$\begin{aligned}
\sum_{x \in G} \widehat{f}(x) g(x) &= \sum_{x \in G} \left( \sum_{\xi \in G} f(\xi) e_{-x}(\xi) \right) g(x) \\
&= \sum_{x \in G} \sum_{\xi \in G} f(\xi) g(x) e_{-x}(\xi) \\
&= \sum_{\xi \in G} \sum_{x \in G} f(\xi) g(x) e_{-x}(\xi) \\
&= \sum_{\xi \in G} f(\xi) \sum_{x \in G} g(x) e_{-\xi}(x) \\
&= \sum_{\xi \in G} f(\xi) \widehat{g}(\xi)
\end{aligned}$$
□

**Proposition 5.** *Basic DFT Properties. Let $f, g \in L^2(G)$ and $x \in G$.*

1. *The DFT is an isomorphism* $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ *is bijective and linear*
2. *Convolution* $(f * g)\widehat{\ } = \widehat{f}\widehat{g}$
3. *Inversion* $f(x) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi(x)$, *that is*
$$\frac{1}{|G|}(\widehat{f})^{\vee} = f = \frac{1}{|G|}(f^{\vee})\widehat{\ }$$
4. *Plancherel / Parseval* $\frac{1}{|G|}\langle f, h \rangle_{L^2(G)} = \langle \widehat{f}, \widehat{h} \rangle_{L^2(\widehat{G})}$
5. *Translation* $(\tau_s f)\widehat{\ }(\chi) = \chi(s)\widehat{f}(\chi)$

*Remark.*

- Showing that $L^2(G) \simeq L^2(\widehat{G})$ allows one to deduce that $G \simeq \widehat{G}$, since in this case it is true that $|G| = |\widehat{G}|$ and there is an injective map $G \hookrightarrow \widehat{G}$. So once (1) is proven we are free to express every $\chi \in \widehat{G}$ as $\chi = e_\xi$ for some $\xi \in G$, and we say that $G$ is "reflexive" or "self-dual".
- The proof of (1) is facilitated by having an explicit inverse, so (3) will be proved first. As the latter is linear in $f$, it will suffice to prove that inversion holds for $\delta_a$. This last remark also applies to (4), although we can employ the isomporphism $G \simeq \widehat{G}$ to prove the formula.

*Proof of 2.* By definition

$$\begin{aligned}
(f * g)\widehat{\ }(\chi) &= \sum_{x \in G} f * g(x)\overline{\chi}(x) \\
&= \sum_{x \in G} \left( \sum_{a \in G} f(a) g(x-a) \right) \overline{\chi}(x) \\
&= \sum_{a \in G} \left( \sum_{x \in G} f(a) g(x-a) \right) \overline{\chi}(x) \\
&= \sum_{a \in G} f(a) \left( \sum_{x \in G} g(x-a) \right) \overline{\chi}(x).
\end{aligned}$$

Pause to observe that

$$\overline{\chi}(x) = \overline{\chi}(x-a+a) = \overline{\chi}(x-a)\,\overline{\chi}(a)$$

and continue the above equality

$$(f * g)\widehat{\ }(\chi) = \sum_{a \in G} f(a)\overline{\chi}(a) \sum_{x \in G} g(x-a)\overline{\chi}(x-a).$$

Make the change of variable $b = x - a$ and note that $b$ runs through all of $G$ as $x$ does, so

$$(f * g)\widehat{\ }(\chi) = \sum_{a \in G} f(a)\overline{\chi}(a) \sum_{b \in G} g(b)\overline{\chi}(b) = \widehat{f}(\chi) \cdot \widehat{g}(\chi)$$

□

*Proof of 3.* By definition

$$\begin{aligned}(\widehat{f})^\vee(y) &= \sum_{\chi\in\widehat{G}} \widehat{f}(\chi)\chi(y) \\ &= \sum_{\chi\in\widehat{G}} \left(\sum_{x\in G} f(x)\overline{\chi}(x)\right)\chi(y) \\ &= \sum_{\chi\in\widehat{G}}\sum_{x\in G} f(x)\chi(y-x).\end{aligned}$$

Switch the order of summation

$$\begin{aligned}(\widehat{f})^\vee(y) &= \sum_{x}\sum_{\chi} f(x)\chi(y-x) \\ &= \sum_{x} f(x)\sum_{\chi}\chi(y-x)\end{aligned}$$

and note that by regarding $(y-x)\in G \hookrightarrow \widehat{\widehat{G}}$ whereby $\chi \mapsto \chi(y-x)$ we have, by orthogonality of characters,

$$\sum_{\chi}\chi(y-x) = \begin{cases} |\widehat{G}| & \text{if } x = y \\ 0 & \text{otherwise.}\end{cases}$$

Therefore $(\widehat{f})^\vee(y) = |\widehat{G}|f(y)$, which establishes inversion.

Since $L^2(G)$ is a $|G|$-dimensional $\mathbb{C}$omplex vector space, we see that so too is $L^2(G)$, whence $|G| = |\widehat{G}|$. Replacing $|\widehat{G}|$ by $|G|$ in the inversion formula gives the desired result; similar considerations show that $f = \frac{1}{|G|}(f^\vee)\widehat{\ }$. ☐

*Proof of 1.* Linearity is obvious; (3) shows that $\mathcal{F}$ is a bijection. ☐

*Proof of 4.* Set

$$g = \overline{\widehat{\overline{h}}} = (\overline{h})^\vee$$

and note that by inversion,

$$\widehat{g} = \frac{1}{|G|}\overline{h}.$$

By the lemma,

$$\langle\widehat{f},\widehat{h}\rangle = \sum_{x\in G}\widehat{f}(x)\overline{\widehat{h}(x)} = \sum_{x\in G}\widehat{f}(x)g(x) = \sum_{x\in G} f(x)\widehat{g}(x) = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{h}(x) = \frac{1}{|G|}\langle f,h\rangle$$

as was to be shown. ☐

*Proof of 5.* By definition

$$\big(\tau_s f\big)\widehat{\ }(\chi) = \sum_{x\in G} \tau_s f(x)\overline{\chi}(x) = \sum_{x\in G} f(x+s)\overline{\chi}(x)$$

make the change of variable $y = x+s$ and note that $\sum_x = \sum_y$ whence

$$\big(\tau_s f\big)\widehat{\ }(\chi) = \sum_{y\in G} f(y)\overline{\chi}(y-s) = \sum_{y\in G} f(y)\overline{\chi}(y)\chi(s) = \chi(s)\hat{f}(\chi) \qquad \square$$

1.4. **The case $G = \mathbb{Z}_n$.** If we specialize to the case $G = \mathbb{Z}_n$ and use the basis of $L^2(\mathbb{Z}_n)$ given by the delta functions then the DFT has the following matrix representation

$$\mathcal{F}_n = \left[\omega^{-(j-1)(k-1)}\right]_{1\le j,k\le n}$$
$$\omega = \exp\left(\frac{2\pi i}{n}\right)$$

For example,

$$\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

The brave reader would attempt to prove that the DFT can be inverted by computing $\det \mathcal{F}_n$.

## 2. Analogues between Continuous and Finite problems

2.1. **The Combinatorial Laplacian and its spectrum.** The Laplacian $\Delta = \frac{d^2}{dx^2}$ on $\mathbb{R}/\mathbb{Z}$ can be approximated by an operator known as the "combinatorial Laplacian" on a certain type of graph. Let $f : \mathbb{R} \to \mathbb{C}$ be $C^2$ and suppose we have sampled its values at $n$ points $x_0, \cdots, x_{n-1}$ of equal spacing $\delta = \frac{1}{n}$. For simplicity denote $f(x_j) = f(j)$. Assume in addition that $f$ satisfies periodic boundary conditions; thereby we may regard $f$ as being defined on the "finite circle" $\mathbb{Z}_n$, interpreting all of $f$'s arguments mod $n$.

**Definition.** The **finite difference operator** $D_1$ on $L^2(\mathbb{Z}_n)$ is defined by

$$D_1 f(j) = \frac{f(j+1) - f(j)}{\delta}.$$

Observe that

$$\lim_{\delta\to 0^+} D_1 f(j) = f'(x_j).$$

Similarly, the "second derivative" of $f$ is defined as

$$\begin{aligned} D_2(f) &= D_1\big(D_1(f)\big) \\ &= \frac{f(j+2) - f(j+1) - f(j-1) + f(j)}{\delta^2} \\ &= \frac{f(j+2) + f(j) - 2f(j+1)}{\delta^2} \\ &= n^2\big[(A - 2I)f(j+1)\big] \end{aligned}$$

where $I$ is the identity operator and $A$ is the so-called **adjacency operator** on $\mathbb{Z}_n$. It works like this:

$$Af(j) = f(j-1) + f(j+1)$$

which represents the sum of $f$ evaluated at all points adjacent to $j$ on the Cayley graph $G(\mathbb{Z}_n, \{\pm 1 \pmod n\})$

**Definition.** The **combinatorial Laplacian** $\Delta_c$ is defined by $\Delta_c = A - 2I$ so that $D_2 f(j) = n^2 \Delta_c f(j+1)$.

As before, it is clear that $\lim_{n\to\infty} \Delta_c = \Delta$, which is to say that $n^2\Delta_c$ is a difference operator approximating the second derivative operator on the circle.

On the circle $\mathbb{R}/\mathbb{Z}$, $\Delta$ has eigenfunctions $\exp(2\pi i a x)$ for $a \in \mathbb{Z}, x \in \mathbb{R}$ with eigenvalues $-4\pi^2a^2$. So on $\mathbb{Z}_n$ we should expect $n^2\Delta_c$ to have eigenfunctions $\exp\left(\frac{2\pi i a}{n} j\right)$ for $a = 0, \cdots, n-1$ and $j \in \mathbb{Z}_n$ with eigenvalues $\approx -4\pi^2 a^2$. This is the case:

$$\begin{aligned} n^2\Delta_c \exp\left(\frac{2\pi i a}{n} j\right) &= n^2\left[(A - 2I)\exp\left(\frac{2\pi i a}{n} j\right)\right] \\ &= n^2\left[A \exp\left(\frac{2\pi i a}{n} j\right) - 2\exp\left(\frac{2\pi i a}{n} j\right)\right] \\ &= n^2\left[\exp\left(\frac{2\pi i a}{n}(j+1)\right) + \exp\left(\frac{2\pi i a}{n}(j-1)\right) - 2\exp\left(\frac{2\pi i a}{n} j\right)\right] \\ &= n^2\left[\exp\left(\frac{2\pi i a}{n}\right) + \exp\left(\frac{-2\pi i a}{n}\right) - 2\right]\exp\left(\frac{2\pi i a}{n} j\right) \end{aligned}$$

re-writing the eigenvalue shows that we can recover the Laplacian's spectrum by letting $n \to \infty$:

$$\begin{aligned}
\lambda &= n^2\left[\exp\left(\frac{2\pi i a}{n}\right) + \exp\left(\frac{-2\pi i a}{n}\right) - 2\right] \\
&= n^2\left[2\cos\left(\frac{2\pi a}{n}\right) - 2\right] \\
&= 2n^2\left[\cos\left(\frac{2\pi a}{n}\right) - 1\right] \\
&= -2n^2 \sum_{k=1}^{\infty}(-1)^k \frac{\left(\frac{2\pi a}{n}\right)^{2k}}{(2k)!} \\
&= -2n^2\left[\frac{4\pi^2 a^2}{2n^2} + \sum_{k=2}^{\infty}(-1)^k \frac{\left(\frac{2\pi a}{n}\right)^{2k}}{(2k)!}\right] \\
&= -4\pi^2 a^2 + \text{terms dependent on } \frac{a}{n}
\end{aligned}$$

## 2.2. **Wirtinger's Inequality.**

### 2.2.1. *Continuous statement and proof.*

**Theorem 6.** *If $f \in C^1[a,b]$ then*

$$\int \left|f(x)\right|^2 \, dx \le \left(2\pi\right)^{-2} \int \left|f'(x)\right|^2 \, dx.$$

*Proof.* By a change of variable it suffices to assume $a = 0, b = \frac{1}{2}$. Extend $f$ to $[-\frac{1}{2}, \frac{1}{2}]$ by defining $f(-x) = -f(x)$; observe that this gives $\int_{[-\frac{1}{2},\frac{1}{2}]} f(x)\, dx = 0$. Now extend $f$ to be periodic on $\mathbb{R}$ and verify that $f \in C^1(\mathbb{T})$.

The hypotheses of Fourier inversion are satisfied so we can write

$$f(x) = \sum_{-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} \qquad \text{and} \qquad f'(x) = \sum_{-\infty}^{\infty} (f')\widehat{\ }(n) e^{-2\pi i n x}$$

where $\widehat{f}$ denotes the usual Fourier coefficnents. Using the ping-pong table from the beginning of the class (well, the analogous version for periodic functions) we can write $(f')\widehat{\ }(n) = -2\pi i n \widehat{f}(n)$, thus obtaining

$$(2\pi)^{-2}\left|(f')\widehat{\ }(n)\right|^2 = n^2\left|\widehat{f}(n)\right|^2 \ge \left|\widehat{f}(n)\right|^2.$$

The inequality remains valid for all partial sums. Since all series converge absolutely, taking the limit of the partial sums gives

$$\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 \leq (2\pi)^{-2} \sum_{-\infty}^{\infty} \left|(f')\widehat{\ }(n)\right|^2$$

and by the Parseval identity $\big(\|f\|_{L^2} = \|\widehat{f}\|_{l^2}\big)$ it follows that

$$\|f\|_2^2 \leq (2\pi)^{-2}\|f'\|_2^2$$

as desired. □

2.2.2. *Finite statement and proof.*

**Theorem 7.** *If* $z \in \mathbb{C}^k$ *then*[1]

$$\sum_{j=0}^{k-1} |z_j|^2 \leq \big(2\sin(\pi/k)\big)^{-2} \sum_{j=0}^{k-1} |z_{j+1} - z_j|^2$$

*Proof.* Write the finite derivative operator as

$$D_1 z_j = z_{j+1} - z_j = [\delta_{-1} - \delta_0]\,(j) * z_j$$

and transform

$$(D_1 z_j)\widehat{\ } = (z_{j+1} - z_j)\widehat{\ } = ([\delta_{-1} - \delta_0]\,(j) * z_j)\widehat{\ }$$

giving

$$(D_1 z_j)\widehat{\ } = [\delta_{-1} - \delta_0]\widehat{\ } \cdot \widehat{z}_j = \left[\exp\left(\frac{-2\pi i j}{k}\right) - 1\right] \widehat{z}_j.$$

It follows that

$$\sum_{j=0}^{k-1} \left| (D_1 z_j)\widehat{\ }\right|^2 = \sum_{j=0}^{k-1} \left|\exp\left(\frac{-2\pi i j}{k}\right) - 1\right|^2 |\widehat{z}_j|^2 \geq 4\sin^2(\pi/k) \sum_{j=0}^{k-1} \left|\widehat{z}_j\right|^2 = 4\sin^2(\pi/k) \sum_{j=0}^{k-1} \left|z_j\right|^2$$

where the last equality follows from the Plancherel identity. To justify this bound, observe that

$$\begin{aligned}
\left|\exp\left(\frac{-2\pi i j}{k}\right) - 1\right|^2 &= \big(\cos(2\pi j/k) - 1\big)^2 + \big(\sin(2\pi j/k)\big)^2 \\
&= \cdots \text{ (algebra) } \cdots \\
&= 2\big(1 - \cos(2\pi j/k)\big) = 4\sin^2(\pi j/k)
\end{aligned}$$

and we can minimize this non-trivially by taking $j = 1$, thus

$$\left|\exp\left(\frac{-2\pi i j}{k}\right) - 1\right|^2 \geq 4\sin^2(\pi/k)$$

[1]In this section we consider the DFT as a linear transformation $\mathbb{C}^k \to \mathbb{C}^k$, utilizing the vector space isomorphism mentioned at the end of §1.1

for all $j$. Therefore

$$\|z\|_{l^2} \leq \big(2\sin(\pi/k)\big)^{-2} \, \|D_1 z\|_{l^2}$$

as was to be shown. □

2.2.3. *Remarks on the phrasing of the results.* In some references the function $f$ is required to be real-valued. In this case we need the fact that the integral of $f$ over one period is zero in order to make its real-valued Fourier series do what we want. The complex-valued case presented here makes for a more elegant presentation. The complementary discrete result in this case would then include the hypothesis that $\sum z_j = 0$.

## 2.3. **The Uncertainty Principle.**

### 2.3.1. *Continuous statement and proof.*

**Theorem 8.** *If $f, \widehat{f} \in C_c^\infty(\mathbb{R}^n)$ then $f \equiv 0$.*

*Proof.* Let $f, \widehat{f} \in C_c^\infty(\mathbb{R}^n)$; assume $\operatorname{supp}(f) \subset K$ and $\operatorname{supp}(f) \subset J$, compact subsets of $\mathbb{R}^n$. We show that the vanishing of $\widehat{f}$ on a nonempty open subset necessitates $f \equiv 0$.

Choose $\xi_0 \notin \operatorname{supp}\big(\widehat{f}\big)$. Since $\widehat{f} \in C_c^\infty(\mathbb{R}^n)$ we know $\partial^\alpha \widehat{f} \equiv 0$ for all $\alpha$ on an open set $U \ni \xi_0$. Changing coordinates allows us to take $\xi_0 = 0$: Substitute $e^{-2\pi i \xi_0 \cdot x} f$ for $f$, which leads to the substitution of

$$\big(e^{-2\pi i \xi_0 \cdot x} f\big)\widehat{\ } = \tau_{\xi_0} \widehat{f}(\xi) = \widehat{f}(\xi - \xi_0)$$

for $\widehat{f}$. Now $\widehat{f}(0) = 0$, and we may assume $\partial^\alpha \widehat{f}$ vanishes (for all $\alpha$) on an open set $V \ni 0$. Note that in this case $V$ is merely $U - \xi_0 = \{x - \xi_0 : x \in U\}$. Write

$$\widehat{f}(\xi) = \int_K f(x) e^{-2\pi i \xi \cdot x} \, dx$$

and observe that since $e^{-2\pi i \xi \cdot x}$ is infinitely differentiable at 0 we can use its MacLaurin series

$$\begin{aligned} e^{-2\pi i \xi \cdot x} &= \sum_{j=0}^{\infty} \frac{1}{j!} [e^{-2\pi i \xi \cdot x}]^{(j)}(0) (-2\pi i \xi \cdot x)^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} (-2\pi i \xi \cdot x)^j \end{aligned}$$

to write

$$\widehat{f}(\xi) = \int_K f(x) \Big( \sum_{j=0}^{\infty} \frac{1}{j!} (-2\pi i \xi \cdot x)^j \Big) \, dx = \int_K \Big( \sum_{j=0}^{\infty} \frac{1}{j!} (-2\pi i \xi \cdot x)^j \Big) f(x) \, dx$$

The Maclaurin series of a function converges uniformly to the function within its disk of convergence. In the case of the exponential function, the radius is infinite. Hence

$$\lim_{m\to\infty}\Big\|\sum_{j=0}^{m}\frac{1}{j!}(-2\pi i\xi\cdot x)^j f(x) - e^{-2\pi i\xi\cdot x}f(x)\Big\|_u = 0.$$

The compact support of $f$ guarantees that $\sum_{j=0}^{m}\frac{1}{j!}(-2\pi i\xi\cdot x)^j f(x)$ and $e^{-2\pi i\xi\cdot x}f(x)$ are integrable (for all $m$). Integrability and uniform convergence allow us to invoke the standard theorem of interchanging the integral and summation ([3], Corollary p.152). Thus

$$\widehat{f}(\xi) = \sum_{j=0}^{\infty}\int_K \frac{1}{j!}(-2\pi i\xi\cdot x)^j f(x).$$

The following special case of the $n$-dimensional multinomial theorem ([1], §8.1 Exercise #2a)

$$(-2\pi i\xi\cdot x)^j = \sum_{|\alpha|=j}\frac{j!}{\alpha!}(-2\pi i\xi\cdot x)^\alpha$$

is used to break this integral down further:

$$\begin{aligned}
\widehat{f}(\xi) &= \sum_{j=0}^{\infty}\int_K \frac{1}{j!}(-2\pi i\xi\cdot x)^j f(x)\,dx \\
&= \sum_{j=0}^{\infty}\int_K \frac{1}{j!}\Big(\sum_{|\alpha|=j}\frac{j!}{\alpha!}(-2\pi i\xi\cdot x)^\alpha\Big) f(x)\,dx \\
&= \sum_{j=0}^{\infty}\int_K \Big(\sum_{|\alpha|=j}\frac{1}{\alpha!}(-2\pi i\xi\cdot x)^\alpha\Big) f(x)\,dx \\
&= \sum_{j=0}^{\infty}\int_K \Big(\sum_{|\alpha|=j}\frac{1}{\alpha!}(-2\pi i x)^\alpha\xi^\alpha\Big) f(x)\,dx \\
&= \sum_{j=0}^{\infty}\int_K \Big(\sum_{|\alpha|=j}\frac{1}{\alpha!}\xi^\alpha(-2\pi i x)^\alpha\Big) f(x)\,dx \\
&= \sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}\xi^\alpha\int_K (-2\pi i x)^\alpha f(x)\,dx \\
&= \sum_{\alpha}\frac{1}{\alpha!}\xi^\alpha\int_K (-2\pi i x)^\alpha f(x)\,dx.
\end{aligned}$$

But ([1], Theorem 8.22(d)) asserts that

$$\begin{aligned}\partial^\alpha \widehat{f}(\xi) &= \big((-2\pi i x)^\alpha f\big)\widehat{} \\ &= \int_K (-2\pi i x)^\alpha f(x) e^{-2\pi i \xi\cdot x}\, dx\end{aligned}$$

in particular

$$\partial^\alpha \widehat{f}(0) = \int_K (-2\pi i x)^\alpha f(x)\, dx.$$

We now know

$$\begin{aligned}\widehat{f}(\xi) &= \sum_\alpha \frac{1}{\alpha!}\xi^\alpha \int_K (-2\pi i x)^\alpha f(x)\, dx \\ &= \sum_\alpha \frac{1}{\alpha!}\xi^\alpha \partial^\alpha \widehat{f}(0)\end{aligned}$$

and yet $\partial^\alpha \widehat{f}(0) = 0$ for all $\alpha$. Thus for all $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \sum_\alpha \frac{1}{\alpha!}\xi^\alpha \partial^\alpha \widehat{f}(0) = \sum_\alpha \frac{1}{\alpha!}\xi^\alpha 0 = 0.$$

Noting that $C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ (since $\int_{\mathbb{R}^n} |f|\, dx \leq \|f\|_u m(K) < \infty$) we appeal to the corollary to the Fourier inversion theorem ([1], Corollary 8.27) and conclude that $f \equiv 0$ almost everywhere. As $f$ is a continuous function we are free to erase "almost everywhere". Therefore $f \equiv 0$ . □

**Lemma 9.** *The multinomial theorem (*[1], *§8.1 Exercise #2a)*
*If* $x = (x_1 + \cdots + x_n) \in \mathbb{R}^n$ *then*

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$

*Proof.* We use the binomial theorem and induction on $n$. For $n = 1$ both sides equal $x_1^k$ since there is only one term $j_1$ in the sum. For the induction step, suppose the multinomial theorem holds for $n$. Then

$$\begin{aligned}(x_1 + \cdots + x_n + x_{n+1})^k &= (x_1 + \cdots + (x_n + x_{n+1}))^k \\ &= \sum_{j_1+\cdots+j_{n-1}+K=k} \binom{k}{j_1, \ldots, j_n - 1, K} x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} \binom{K}{j_n, j_{n+1}} x_n^{j_n} x_{n+1}^{j_{n+1}} \\ &= \sum_{j_1+\ldots j_{n+1}=k} \binom{k}{j_1, \ldots, j_{n+1}} x_1^{j_1} \cdots x_{n+1}^{j_{n+1}}\end{aligned}$$

which completes the induction. The last step follows becuase

$$\binom{k}{j_1, \ldots, j_{n-1}, K}\binom{K}{j_n, j_{n+1}} = \binom{k}{j_1, \ldots, j_{n+1}},$$

which can be seen by writing

$$\frac{k!}{j_1! \cdots j_{n-1}! K!} \frac{K!}{j_n! j_{n+1}!} = \frac{k!}{j_1! \cdots j_{n+1}!} \qquad \square$$

2.3.2. *Finite statement and proof.* The **support** of a function $f \in L^2(G)$ is defined as

$$\operatorname{supp} f = \{x \in G : f(x) \neq 0\}.$$

There are two statistics in which we are interested

$$\|f\|_2^2 = \sum_{x \in G} |f(x)|^2$$
$$\|f\|_\infty = \max_{x \in G} |f(x)|$$

note that $\|f\|_2^2 \leq \|f\|_\infty^2 \left|\operatorname{supp} f\right|$. We'll also need the following crucial estimate

**Lemma 10. (Cauchy-Schwarz Inequality)** *If $f, g \in L^2(G)$ then $|\langle f, g\rangle|^2 \leq \|f\|_2^2 \|g\|_2^2$*

**Theorem 11.** *If $f : G \to \mathbb{C}$ is not identically zero then*

$$\left|\operatorname{supp} f\right| \left|\operatorname{supp} \widehat{f}\right| \geq |G|.$$

"Squeezing" $f$ to be defined at only a few points corresponds to "stretching" $\widehat{f}$ to be defined at more points; both cannot be simultaneously sharply localized.

*Proof.* Such a function has a Fourier representation

$$f(x) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi(x)$$

and since $|\chi(x)| \leq 1$ for all $x$ we have

$$\|f\|_\infty \leq \frac{1}{|G|} \sum_{\chi \in \widehat{G}} |\widehat{f}(\chi)|.$$

Square both sides and use the Cauchy-Schwarz inequality

$$\|f\|_\infty^2 \leq \frac{1}{|G|^2} \|\widehat{f}\|_2^2 \sum_{\chi \in \operatorname{supp} \widehat{f}} 1^2.$$

Plancherel's identity gives $\|\widehat{f}\|_2^2 = |G|\|f\|_2^2$, so

$$\|f\|_\infty^2 \leq \frac{1}{|G|}\|f\|_2^2\left|\text{supp}\widehat{f}\right|$$

and recalling that $\|f\|_2^2 \leq \|f\|_\infty^2\left|\text{supp}f\right|$ we have

$$\|f\|_\infty^2 \leq \frac{1}{|G|}\|f\|_\infty^2\left|\text{supp}f\right|\left|\text{supp}\widehat{f}\right|.$$

Of course, since $f \not\equiv 0$ we can re-arrange this as

$$|G| \leq \left|\text{supp}f\right|\left|\text{supp}\widehat{f}\right|$$

as desired. □

2.3.3. *BONUS! Continuous uncertainty principle, v.2.*

**Theorem 12.** *Let $f \in \mathcal{S}(\mathbb{R})$ (for simplicity). For all $b, \beta \in \mathbb{R}$*

$$\frac{\|f\|_2^4}{16\pi^2} \leq \|(x-b)f\|_2^2 \ \|(\xi-\beta)\widehat{f}\|_2^2$$

This means that both $f$ and $\widehat{f}$ cannot be sharply localized around points $b$ and $\beta$.

**Lemma 13.** $\|f\|_2^4 \leq 4 \ \|xf\|_2^2 \ \|f'\|_2^2$ .

*Proof.* Observe that

$$\int xf(x)f'(x)\ dx = \int \frac{1}{2}x\big(f(x)^2\big)'\ dx = -\frac{1}{2}\int f(x)^2\ dx$$

where the second inequality comes from integration by parts.

In the complex case, note that

$$f(x+iy) = u(x) + iv(y) \qquad \overline{f}(x+iy) = u(x) - iv(y)$$

$$f'(x+iy) = u'(x) + iv'(y) \qquad |f(x+iy)|^2 = u^2(x) + v^2(y)$$

Now

$$\begin{aligned}-2\text{Re}\int x(u-iv)(u'+iv')\,dx &= -2\text{Re}\int x\big((uu'+vv') - (vu'+uv')\big)\,dx \\ &= -2\int x(uu'+vv')\,dx\end{aligned}$$

and we can show that $\int u^2 + v^2\, dx = -2\int x(uu' + vv')\, dx$. Here goes:

$$\begin{aligned} -2\int x(uu'+vv')\,dx &= -2\int xuu' + xvv'\,dx \\ &= -2\int xuu'\,dx + \int xvv'\,dx \end{aligned}$$

the latter is equal (by the real case) to $\int u^2\,dx + \int v^2\,dx = \int u^2 + v^2\,dx$.

Finally, we have

$$\begin{aligned} \|f\|_2^2 &= \int |f(x)|^2\,dx \\ &= -2\int \big|xf(x)f'(x)\big|\,dx \\ &\leq -2\int |xf(x)|\,|f'(x)|\,dx \end{aligned}$$

and by Hölder's inequality ([1], Theorem 6.2)

$$\begin{aligned} -2\int |xf(x)|\,|f'(x)|\,dx &= -2\|xf\cdot f'\|_1 \\ &\leq -2\|xf\|_2\ \|f'\|_2. \end{aligned}$$

Hence $\|f\|_2^2 \leq -2\|xf\|_2\ \|f'\|_2$, and therefore

$$\|f\|_2^4 \leq 4\|xf\|_2^2\ \|f'\|_2^2$$

as was to be shown. □

*Proof of the Uncertainty Principle.* We may reduce the proof to the case $b = \beta = 0$: consider $g(x) = e^{-2\pi i\beta x} f(x-b)$, which may also be regarded as $g(x) = e^{-2\pi i\beta x}\tau_{-b} f(x)$. On the function side, this corresponds to a translation of $f$ by $b$ followed by multiplication with a phase factor. On the Fourier transform side, this corresponds to a translation of $\widehat{f}$ by $\beta$:

$$\begin{aligned} \widehat{g}(\xi) &= \big(e^{-2\pi i\beta x}\tau_{-b} f(x)\big)\widehat{\ }(\xi) \\ &= \int_{\mathbb{R}} e^{-2\pi i\beta x}\tau_{-b} f(x) e^{-2\pi i\xi x}\,dx \\ &= \int_{\mathbb{R}} e^{-2\pi i(\xi+\beta)x}\tau_{-b} f(x)\,dx \\ &= \int_{\mathbb{R}} e^{-2\pi i(\xi+\beta)x} f(x)\,dx \\ &= \widehat{f}(\xi+\beta) \\ &= \tau_{-\beta}\widehat{f}(\xi). \end{aligned}$$

Taking norms, we have $\|\widehat{g}\|_2 = \|\tau_{-\beta}\widehat{f}\|_2$; by ([1], p.238) it follows that $\|\widehat{g}\|_2 = \|\widehat{f}\|_2$. For this reason is is permissible to assume $\beta = 0$, otherwise substituting $\widehat{g}$ for $\widehat{f}$ in the equations to follow. From this we obtain immediately that

$$\begin{aligned}\|g\|_2 &= \|e^{-2\pi i\beta x}\tau_{-b}f\|_2\\ &= \|e^{-2\pi i0x}\tau_{-b}f\|_2\\ &= \|\tau_{-b}f\|_2\\ &= \|f\|_2\end{aligned}$$

meaning we may also assume $b = 0$, replacing $f$ by $g$ if necessary.

Assuming $b = \beta = 0$ we have from the lemma

$$\|f\|_2^4 \leq 4\|xf\|_2^2 \ \|f'\|_2^2.$$

As $\mathcal{F}$ is a unitary isomorphism on $L^2$ ([1], Theorem 8.29) we have $\|f'\|_2 = \|(f')\widehat{\ }\|_2$. Thus

$$\begin{aligned}\|f\|_2^4 &\leq 4\|xf\|_2^2 \ \|(f')\widehat{\ }\|_2^2\\ &= 4\|xf\|_2^2 \ \|2\pi i\xi\widehat{f}\|_2^2\\ &= 4\|xf\|_2^2 \ 4\pi^2\|\xi\widehat{f}\|_2^2\\ &= 16\pi^2\|xf\|_2^2 \ \|\xi\widehat{f}\|_2^2\end{aligned}$$

as was to be shown.

□

Note that if $f$ is normalized so that $\|f\|_2 = 1$ then we have a more refined estimate

$$\frac{1}{4\pi} \leq \|xf\|_2 \ \|\xi\widehat{f}\|_2$$

**Example.** A trivial example for which equality holds in the discrete uncertainty principle is $f(x) = \delta_e(x)$, where $e \in G$ is the identity. Observe that

$$\widehat{\delta_e}(\chi) = \sum_{x\in G}\delta_e(x)\overline{\chi}(x) = \overline{\chi}(e) = 1.$$

Obviously, $\operatorname{supp}\delta_e = \{e\}$ and $\operatorname{supp}\widehat{\delta_e} = \widehat{G}$.

**Example.** Somewhat less trivially, let $H < G$ and define $\delta_H(x) = \sum_{h\in H}\delta_h(x)$. Think $\delta$-comb, and note that $|\operatorname{supp}\delta_H| = |H|$. The DFT of this function is

$$\widehat{\delta_H}(\chi) = \sum_{x\in G}\delta_H(x)\overline{\chi}(x) = \sum_{x\in H}\overline{\chi}(x) = \begin{cases}|H| & \text{if } \chi \in H^{\#}\\ 0 & \text{if } \chi \notin H^{\#}\end{cases}$$

where

$$H^{\#} = \{\chi \in \widehat{G} : \chi|_H \equiv 1\} \simeq (G/H)\widehat{\ }$$

In other words, $\widehat{\delta_H} = |H|\delta_{H^{\#}}$; therefore

$$|\operatorname{supp}\delta_H||\operatorname{supp}\widehat{\delta_H}| = |H||G/H| = |G|.$$

## 3. The Legendre Symbol

**Definition.** For an odd prime $p$ and an integer $x$ not divisible by $p$, the **Legendre Symbol**

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if there exists } a \in \mathbb{Z}_p \text{ such that } a^2 \equiv x \pmod{p} \\ -1 & \text{else} \end{cases}$$

gives the answer to the question "is $x$ a square modulo $p$?"

**Proposition 14.** *As a function of $x$, the Legendre symbol is a group homomorphism $\mathbb{Z}_p^* \to \{\pm 1\}$. (That the Legendre symbol is $p$-periodic is included in this statement).*

*Proof.* Omitted. □

**Lemma 15.** *There are $\frac{p-1}{2} + 1$ squares in $\mathbb{Z}_p$.*

*Proof.* Discarding 0, which is obviously a square, the new statement to prove is that there are $\frac{p-1}{2}$ squares in $\mathbb{Z}_p^*$. For $k = 1, \cdots, \frac{p-1}{2}$ pair $k$ with $p - k$ and observe that

$$(p-k)^2 = p^2 - 2kp + k^2 \equiv k^2 \pmod{p}.$$ □

Define a function $h_p : \mathbb{Z}_p \to \{-1, 0, 1\}$ by

$$h_p(x) = \begin{cases} \left(\frac{x}{p}\right) & \text{if } x \in \mathbb{Z}_p^* \\ 0 & \text{else.} \end{cases}$$

**Lemma 16.** $\widehat{h}_p(-x) = h_p(x)\widehat{h}_p(-1)$

*Proof.* By definition, if $p \nmid x$ then

$$\widehat{h}_p(x) = \sum_{a=0}^{p-1} h_p(a) e^{\frac{2\pi i a x}{p}} = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e^{\frac{2\pi i a x}{p}}$$

where the final equality is true because $\left(\frac{0}{p}\right) = 0$. As we are no longer considering $a = 0$, the sum above is just the sum over $\mathbb{Z}_p^*$. Taking advantage of this, write $b = ax$ (so that $a = bx^{-1}$) and observe that $\sum_a = \sum_b$. So

$$\begin{aligned}\widehat{h}_p(-x) &= \sum_{b=1}^{p-1} \left(\frac{bx^{-1}}{p}\right) e^{\frac{2\pi i b}{p}} \\ &= \left(\frac{x^{-1}}{p}\right) \sum_{b=1}^{p-1} \left(\frac{b}{p}\right) e^{\frac{2\pi i b}{p}} \\ &= \left(\frac{x^{-1}}{p}\right) \widehat{h}_p(-1) \\ &= \left(\frac{x}{p}\right) \widehat{h}_p(-1) \\ &= h_p(x)\ \widehat{h}_p(-1).\end{aligned}$$

When $p\big|x$,

$$\widehat{h}_p(-x) = \sum_{a=0}^{p-1} \left(\frac{a}{p}\right) e^{\frac{2\pi i a x}{p}} = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$$

(here we've used $x \equiv 0 \pmod{p}$ to get rid of the exponential) since there are equal amounts of squares and non-squares in $\mathbb{Z}_p^*$. So

$$0 = \left(\frac{x}{p}\right) = h_p(x) = \widehat{h}_p(-x)$$

which completes the proof. □

**Corollary 17.** $\widehat{h}_p \sim h_p$

*Proof.*

$$\begin{aligned}\widehat{h}_p(x) &= h_p(-x)\widehat{h}_p(-1) \\ &= h_p(x)h_p(-1)\widehat{h}_p(-1) \\ &= Ch_p(x)\end{aligned}$$

for $C = h_p(-1)\widehat{h}_p(-1)$. □

So $h_p$ is proportional to discrete Fourier transform, just as the Gaussian $\varepsilon^{-2\pi i x^2}$ is for the continuous Fourier transform on $\mathbb{R}$. But the uncertainty principle is **not** saturated by $h_p$ (as it is for the the Gaussian): Recall the statement of the uncertainty principle for the discrete Fourier transform

$$|G| \leq \big|\text{supp} f\big| \cdot \big|\text{supp}\widehat{f}\big|.$$

Since $\sqrt{p} < \frac{p-1}{2} + 1$ it follows that

$$|\mathbb{Z}_p| = p < \left(\frac{p-1}{2} + 1\right)^2 = \left|\mathrm{supp} h_p\right| \cdot \left|\mathrm{supp} \widehat{h}_p\right|.$$

## Description of References

Terras [4] is a very complete account of the Fourier transform on finite groups (abelian and non-abelian) with many applications. Terras contains many discrete counterparts of well-known results not presented here such as the Poisson summation formula, the heat kernel, the isoparametric inequality, Bessel functions, etc. Folland [1] is the core reference for the continuous Fourier transform, and also contains Witringer's inequality and the Uncertainty Principle as exercises. Krantz [2] also contains some material on the Uncertainty Principle.

## References

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