Finite Analysis and the Discrete Fourier Transform

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The Setup

Let G be a finite abelian group

$$G \simeq \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}.$$

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Definition (The function space $L^2(G)$)

$$L^2(G) = \{f : G \to \mathbb{C}\}$$

$$\langle f, g \rangle_{L^2(G)} = \sum_{x \in G} f(x)\overline{g}(x)$$

$$\|f\|^2_{L^2(G)} = \langle f, f \rangle_{L^2(G)}$$

Proposition

$L^2(G)$ is a $|G|$-dimensional $\mathbb{C}$-vector space with an orthonormal basis consisting of the delta functions

$$\delta_a(x) = \begin{cases} 1 & \textit{if } x = a \\ 0 & \textit{if } x \neq a. \end{cases}$$

Proof.

Obvious. □

Definition

For $f, g \in L^2(G)$

$$\tau_s f(x) = f(s+x)$$
$$f * g(x) = \sum_{a \in G} f(a)g(x-a).$$

Convolving with a delta function amounts to translation:

$$f * \delta_a(x) = \sum_{y \in G} f(y)\delta_a(x-y) = f(x-a) = \tau_{-a} f(x)$$

The Dual Group

A **character** of G is a group homomorphism from G into the 1-torus (*i.e.* unit circle), $\mathbb{T} \subset \mathbb{C}$, under multiplication.

Definition

The set of all characters of G under pointwise multiplication is called **the dual group** $\widehat{G}$

$$\widehat{G} = \{\chi \in \mathrm{Hom}_{\mathsf{Grp}}(G, \mathbb{T})\}$$
$$\chi_1\chi_2(x) = \chi_1(x)\chi_2(x)$$
$$e_{\widehat{G}} = 1(x) \equiv 1$$
$$\chi^{-1}(x) = \overline{\chi}(x) = \chi(-x)$$

Proposition (Orthogonality of characters)

If $\chi, \psi \in \widehat{G}$ *then*

$$\langle \chi, \psi \rangle = \begin{cases} |G| & \textit{if } \chi = \psi \\ 0 & \textit{otherwise} \end{cases}$$

Proof.

First observe that $\langle \chi, \psi \rangle = \langle \chi\psi^{-1}, 1 \rangle$, then consider $\langle \chi, 1 \rangle$. □

Definition (L^2 on the dual group)

The space of functions $L^2(\widehat{G})$ is defined by

$$L^2(\widehat{G}) = \{F : \widehat{G} \to \mathbb{C}\}$$

$$\langle F, G \rangle_{L^2(\widehat{G})} = \sum_{\chi \in \widehat{G}} F(\chi)\overline{G}(\chi)$$

$$\|F\|^2_{L^2(\widehat{G})} = \langle F, F \rangle_{L^2(\widehat{G})}$$

$G \hookrightarrow \widehat{G}$

For $a = (a_1, \cdots, a_r), x = (x_1, \cdots, x_r) \in G$ there is a character

$$e_a(x) = \prod_{j=1}^{r} e_{a_j}(x_j)$$

given by componentwise complex exponentiation

$$\mathbb{Z}_{m_j} \ni x_j \xrightarrow{e_{a_j}} \exp\left(\frac{2\pi i a_j}{m_j} x_j\right) \in \mathbb{T}$$

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For all $\xi, x \in G$

$$\begin{aligned} e_\xi(x) &= e_x(\xi) \\ \overline{e_\xi}(x) &= e_{-\xi}(x) = e_\xi(-x) \end{aligned}$$

The Discrete Fourier Transform

Definition

The **discrete Fourier transform** (DFT) is the map $L^2(G) \to L^2(\widehat{G})$ defined by

$$\widehat{f}(\chi) = \langle f, \chi \rangle_{L^2(G)} = \sum_{x \in G} f(x)\overline{\chi}(x).$$

The **inverse DFT** is defined by

$$f^{\vee}(\chi) = \sum_{x \in G} f(x)\chi(x).$$

Observations

- For all $a \in G$,

$$\widehat{\delta_a}(\chi) = \sum_{x \in G} \delta_a(x)\overline{\chi}(x) = \overline{\chi}(a)$$

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- $\overline{\widehat{f}} = \check{\overline{f}}$ and $\widehat{\overline{f}} = \overline{\check{f}}$, because

$$\overline{\widehat{f}}(\chi) = \overline{\sum_{x \in G} f(x)\overline{\chi}(x)} = \sum_{x} \overline{f}(x)\chi(x) = \check{\overline{f}}(\chi)$$

and

$$\widehat{\overline{f}}(\chi) = \sum_{x \in G} \overline{f}(x)\overline{\chi}(x) = \overline{\sum_{x} f(x)\chi(x)} = \overline{\check{f}}(\chi)$$

Lemma

For all $f, g \in L^2(G)$,

$$\sum_{x \in G} \widehat{f}(x) g(x) = \sum_{x \in G} f(x) \widehat{g}(x).$$

Proof.

Switch the order of summation, re-arrange, then use the property that $e_{-x}(\xi) = e_{-\xi}(x)$. □

Basic DFT Properties

Proposition

Let $f, g \in L^2(G)$ *and* $x \in G$.

1. *The DFT* $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ *is bijective and linear*

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3. *The DFT is invertible:*

$$\frac{1}{|G|}(\widehat{f})^{\vee} = f = \frac{1}{|G|}(f^{\vee})\widehat{\ }$$

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4. *Plancherel/Parseval:*

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5. *Translation transforms to scalar multiplication:* $(\tau_s f)\widehat{\ }(\chi) = \chi(s)\widehat{f}(\chi)$

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- Let $f \in C^2(\mathbb{R})$ and suppose we have sampled its values at n points $x_0, \cdots, x_{n-1}$ of equal spacing $\delta = \frac{1}{n}$. For simplicity denote $f(x_j) = f(j)$.

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- Assume in addition that f satisfies periodic boundary conditions; thereby we may regard f as being defined on the "finite circle" $\mathbb{Z}_n$, interpreting all of f's arguments mod n.

Definition (Finite Difference Operators)

The "first derivative" is

$$D_1 f(j) = \frac{f(j+1) - f(j)}{\delta},$$

the "second derivative" is

$$D_2 f(j) = D_1\big(D_1 f(j)\big) = \big[n^2(A - 2I)\big] f(j+1).$$

Where I is the identity operator, $If(j) = f(j)$ and A is the adjacency operator

$$Af(j) = f(j-1) + f(j+1)$$

Definition

The **combinatorial Laplacian** Δ_c is defined by

$$\Delta_c = A - 2I$$

so that

$$D_2 f(j) = n^2 \Delta_c f(j+1).$$

Note $\lim_{n\to\infty} n^2\Delta_c = \Delta$, which is to say that $n^2\Delta_c$ approximates the second derivative on the circle.

- On $\mathbb{R}/\mathbb{Z}$, Δ has eigenfunctions

$$f_a(x) = \exp(2\pi i a x)$$

for $a \in \mathbb{Z}, x \in \mathbb{R}$ with eigenvalues

$$\lambda_a = -4\pi^2 a^2$$

- On $\mathbb{Z}_n$ we should expect $n^2\Delta_c$ to have eigenfunctions

$$f_{a,n}(j) = \exp\left(\frac{2\pi i a}{n} j\right)$$

for $a = 0, \cdots, n-1$ and $j \in \mathbb{Z}_n$ with eigenvalues

$$\lambda_{a,n} \approx -4\pi^2 a^2$$

Theorem (Contiuous case)

If $f \in C^1[a, b]$ *then*

$$\int |f(x)|^2 \, dx \leq (2\pi)^{-2} \int |f'(x)|^2 \, dx.$$

Wirtinger's Inequality

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Theorem (Discrete case)

If $z \in \mathbb{C}^k$ *then*

$$\sum_{j=0}^{k-1} |z_j|^2 \leq (2\sin(\pi/k))^{-2} \sum_{j=0}^{k-1} |z_{j+1} - z_j|^2$$

The Uncertainty Principle

Theorem (Continuous Versions)

1. *Let* $f \in \mathcal{S}(\mathbb{R})$. *For all* $b, \beta \in \mathbb{R}$

$$\frac{\|f\|_2^4}{16\pi^2} \leq \|(x-b)f\|_2^2 \; \|(\xi-\beta)\widehat{f}\|_2^2$$

2. *If* $f, \widehat{f} \in C_c^\infty(\mathbb{R}^n)$ *then* $f \equiv 0$

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2. *If* $f, \widehat{f} \in C_c^\infty(\mathbb{R}^n)$ *then* $f \equiv 0$

Theorem (Discrete Version)

If $f : G \to \mathbb{C}$ *is not identically zero then*
$$\left|\operatorname{supp} f\right| \left|\operatorname{supp} \widehat{f}\right| \geq |G|.$$

1. The Gaussian $f(x) = e^{-\pi x^2}$ is a well-known example for which equality holds in the continuous version.

   It is also porportional to its Fourier transform and so is a kind of fixed point.

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3. The Legendre symbol from number theory is a fixed point of the DFT that does **not** saturate the uncertainty principle!

References

Terras [3] for the DFT. Folland [1], Katznelson [2] for the continuous fourier transform.

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