Cox rings and pseudoeffective cones on projectivized toric vector bundles

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Toric vector bundles

A *toric variety* is a normal variety X that contains a torus $T \cong (k^*)^d$ as an open subset, together with an action of T on X that extends the natural action of T on itself. [$X = X(\Sigma)$, $d = \dim X$, n = # of rays in Σ, $k = \overline{k}$]

A *toric vector bundle* is a vector bundle $\mathcal{F}$ over a TV with an action of T, such that the projection is equivariant and the action is linear on the fibers.

The main tool to study TVBs is the classification given by Klyachko:

THEOREM: Klyachko's category equivalence TVB $f : \mathcal{F} \to X \iff$ Vector space F with a collection of filtrations: (D T-invariant divisor, $i \in \mathbf{Z}$)

$$\cdots F^D(i-1) \supseteq F^D(i) \supseteq F^D(i+1) \supseteq \cdots$$

Satisfying compatibility: $\forall$ fixed point $p \in X$, $\exists$ decomposition $F = \bigoplus_{u \in \mathbf{Z}^d} F_u$, s.t. if $p \in D$, the filtration $\{F^D(i)\}$ can be recovered as $F^D(i) = \sum_{\langle u, v_D \rangle \geq i} F_u$.

Projectivized TVBs are a large class of rational varieties that share some of the pleasant properties of TVs and other Mori dream spaces. Nonetheless, their geometry is very different from that of toric varieties.

The finite generation problem

A normal projective $\mathbf{Q}$-factorial variety X is a *Mori Dream Space* if $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$, and for some (or equivalently any) line bundles $L_1, \ldots, L_r$ that generate $\mathrm{Pic}(X)$, the associated *Cox Ring* is a finitely generated k-algebra:

$$\mathcal{R}(X) := \mathrm{Cox}(X, L_1, \ldots, L_r) := \bigoplus_{(m_1,\ldots,m_r) \in \mathbf{Z}^r} H^0(X, L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}).$$

Hering, Payne and Mustaţă proved that the nef cone and the Mori cone of projectivized TVBs are rational polyhedral. They asked whether their Cox rings were indeed finitely generated. The aim of this poster is to describe the complete answer to this question.

Cox rings and pseudoeffective cones of rank two projectivized TVBs

[Hausen-Süß, Gonzalez]

There are now various arguments showing that projectivized rank two TVBs are Mori dream spaces. In one of them, the conclusion follows from the structure of the global Okounkov body of these varieties.

THEOREM [Gonzalez] Let $\mathcal{F}$ be a rank two TVB over a smooth TV X. Then there is a flag of smooth invariant subvarieties $Y_\bullet$ on $\mathbf{P}(\mathcal{F})$ such that the semigroup

$$\bigcup_{(m_1,\ldots,m_n,m) \in \mathbf{Z}^{n+1}} \nu_{Y_\bullet}\Big(H^0\left(\mathbf{P}(\mathcal{F}), \pi^*\mathcal{O}_X\left(m_1 D_1 + \cdots + m_n D_n\right) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)\right) \smallsetminus \{0\}\Big) \times \{(m_1, \ldots, m_n, m)\} \subseteq \mathbf{Z}^{n+d+2}$$

is finitely generated. In particular, the global Okounkov body $\Delta_{Y_\bullet}(\mathbf{P}(\mathcal{F}))$ and the Okounkov bodies of all divisors on $\mathbf{P}(\mathcal{F})$ are rational polyhedral. In the proof, we also provide linear inequalities defining these Okounkov bodies in terms of the fan Σ of X and the Klyachko filtrations of $\mathcal{F}$.

COROLLARY of the proof The Cox ring of $\mathbf{P}(\mathcal{F})$ is finitely generated and its pseudoeffective cone is rational polyhedral.

Cox rings and pseudoeffective cones of higher rank projectivized TVBs

[Gonzalez-Hering-Payne-Süß]

The following are some tools that we used in this project. The first is a special case of a result of Castravet and Tevelev generalizing work of Mukai on Hilbert's 14th problem.

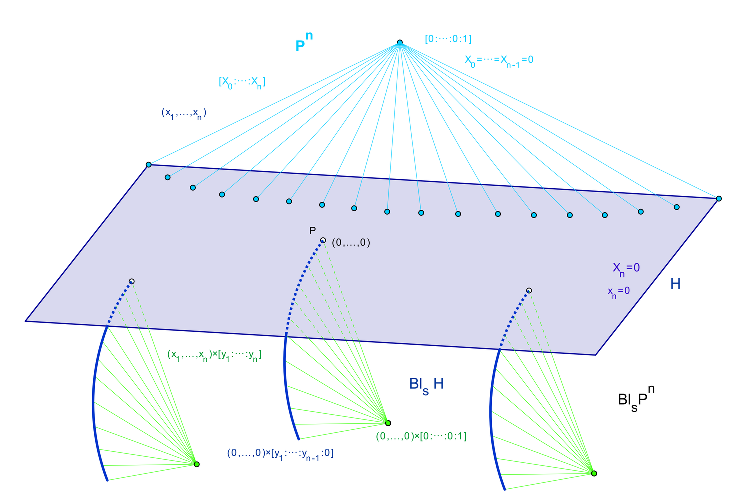
THEOREM [Castravet-Tevelev] For $n > r \geq 2$, the Cox ring of the blow up of $\mathbf{P}^{r-1}$ at n very general points is finitely generated if and only if its semigroup of effective divisors is finitely generated, and this happens if and only if $\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2}$.

Our main tool to study the Cox rings of projectivized TVBs is the following special case of a result of Hausen and Süß.

PROPOSITION Let X be a smooth variety with $\mathrm{Pic}(X)$ finitely generated, endowed with a T-action. Suppose $D_1, \ldots, D_h$ are irreducible divisors in X with positive dimensional generic stabilizers. Suppose, furthermore, that T acts freely on $X \smallsetminus (D_1 \cup \cdots \cup D_h)$ with geometric quotient a smooth variety Y. Then $\mathcal{R}(X)$ is isomorphic to a polynomial ring in s variables over $\mathcal{R}(Y)$.

As an application we obtain the following very useful lemma.

LEMMA Let S be a finite set of points contained in a hyperplane H in $\mathbf{P}^n$. Then the Cox ring of $\mathrm{Bl}_S \mathbf{P}^n$ is isomorphic to a polynomial ring in one variable over the Cox ring of $\mathrm{Bl}_S H$.



A special class of toric vector bundles

We now let X be a smooth projective toric variety, and study the TVBs on X whose Klyachko filtrations have the following form

$$F^{\rho_j}(k) = \begin{cases} F & \text{for } k \leq 0, \\ F_j & \text{for } k = 1, \\ 0 & \text{for } k > 1, \end{cases} \qquad (*)$$

where F_j is either 0 or a codimension one subspace of F, and all of the nonzero F_j are distinct. These filtrations satisfy Klyachko's compatibility condition if and only if, for each maximal cone $\sigma \in \Sigma$, the nonzero hyperplanes F_j such that $\rho_j \prec \sigma$ meet transversely in F.

For some of our results we require the field k to be uncountable in order to take the hyperplanes F_j corresponding to very general configurations of points in the projective space $\mathbf{P}(F)$. The uncountability assumption is not crucial here since one can slightly modify the arguments to get versions of these results over any algebraically closed field k. In fact, one can get suitable configurations of points in $\mathbf{P}(F) = \mathbf{P}^{r-1}(k)$ by linearly embedding one of the configurations of points in $\mathbf{P}^2(k)$ or $\mathbf{P}^3(k)$ (with associated blow up non-MDS) constructed by Totaro in his work on Hilbert's 14th problem, and one can modify the combinatorial assumptions on the base appropriately to get the Klyachko condition to hold, and then one can proceed as in the uncountable case.

Presentations for Cox rings of some projectivized TVBs

[Gonzalez-Hering-Payne-Süß]

Let $\mathcal{F}$ be a toric vector bundle on X given by filtrations of the form $(*)$. After renumbering, say the F_i are distinct hyperplanes for $i \leq s$, and F_j is zero for $j > s$, and let

$$S = \{p_1, \ldots, p_s\}$$

be the set of points corresponding to $F_1, \ldots, F_s$ in the projective space $\mathbf{P}(F)$.

THEOREM The Cox ring $\mathcal{R}(\mathbf{P}(\mathcal{F}))$ is isomorphic to a polynomial ring in $n-s$ variables over $\mathcal{R}(\mathrm{Bl}_S \mathbf{P}(F))$.

On the finite generation of Cox rings of projectivized TVBs

[Gonzalez-Hering-Payne-Süß]

The following result gives a negative answer to the finite generation question of Hering, Mustaţă and Payne.

THEOREM Suppose that k is uncountable and that $n > r \geq d$ and $\frac{1}{r} + \frac{1}{n-r} \leq \frac{1}{2}$. Then there is an irreducible toric vector bundle $\mathcal{F}$ of rank r on $X(\Sigma)$ such that the Cox ring of the projectivization $\mathbf{P}(\mathcal{F})$ is not finitely generated.

Moreover, even projectivized cotangent bundles of toric varieties are not Mori dream spaces in general.

THEOREM[1] Suppose $d \geq 3$ and the characteristic of k is not two or three. Then there exists a smooth projective toric variety $X(\Sigma')$ of dimension d over k such that the Cox ring of the projectivized cotangent bundle on $X(\Sigma')$ is not finitely generated.

Pseudoeffective cones of projectivized TVBs

[Gonzalez-Hering-Payne-Süß]

The pseudoeffective cones of projectivized TVBs are generated by torus invariant divisors, but in general they are not rational polyhedral.

LEMMA Every effective divisor on $\mathbf{P}(\mathcal{F})$ is linearly equivalent to a torus invariant effective divisor.

THEOREM Suppose that k is uncountable, and that $n - d > r \geq d$ and $\frac{1}{r} + \frac{1}{n-d-r} \leq \frac{1}{2}$, and assume there is some cone $\sigma \in \Sigma$ such that every ray of Σ is contained in either σ or $-\sigma$. Then there is an irreducible toric vector bundle $\mathcal{F}$ of rank r on $X(\Sigma)$ such that the pseudoeffective cone of $\mathbf{P}(\mathcal{F})$ is not polyhedral.

1 The variety in the Theorem: The columns of the following matrix span the rays of a smooth projective fan such that the projectivized cotangent bundle of the associated toric threefold is not a Mori dream space.

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 & 0 & -1 & -2 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -2 & 2 & 1 & 1 & 1 & -1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Note that the blow up of $\mathbf{P}^2$ at nine of these points is not a Mori dream space [Totaro]. To get examples in higher dimensions, starting from the example given by a fan Σ in dimension d, we embed $\mathbf{R}^d$ as the last coordinate hyperplane in $\mathbf{R}^{d+1}$, and let Σ' be the fan in $\mathbf{R}^{d+1}$ whose maximal cones are spanned by a maximal cone of Σ together with either $(1, \ldots, 1)$ or $(1, \ldots, 1, -1)$. The projectivized cotangent bundle of the associated toric variety is not a Mori dream space.