University of British Columbia
Student Name and Number:

Math 152 Quiz 3

Section 202 - Winter 2013
Instructor: Jose Gonzalez
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INSTRUCTIONS (Read carefully): This quiz has **two** independent questions. In order to get credit for your answers you must **show all your work**.

(1) Find all the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Solution To find the eigenvalues of A the first step is to find the polynomial $\det(A - \lambda I)$ in the variable λ:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5.$$

Now, we find the eigenvalues as the zeroes of the polynomial $\det(A - \lambda I) = \lambda^2 - 6\lambda + 5$ (that is, we find the values of λ for which the polynomial takes the value zero). We will find the zeroes by factoring the polynomial (we could have instead used the formula for the zeroes of a quadratic polynomial):

$$\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$$

This expression is zero $(\lambda - 1)(\lambda - 5) = 0$ exactly when λ is either equal to 1 or equal to 5.

Therefore the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$.

(2) Consider the following matrix

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -2 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

The number $\lambda = 1$ is an eigenvalue of this matrix. Find the eigenvectors of the matrix A that correspond to the eigenvalue $\lambda = 1$.

Solution To find the eigenvectors corresponding to an eigenvalue λ one solves the homogeneous linear system $(A - \lambda I)\vec{x} = \vec{0}$. In this case $\lambda = 1$ and:

$$A - \lambda I = \begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 1 & 3 & -4 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the system to solve is

$$\begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve the sytem $(A - \lambda I)\vec{x} = \vec{0}$ using Gaussian Elimination:

$$\begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 1 & 3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -4 \\ 2 & -3 & 1 \\ -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -4 \\ 0 & -9 & 9 \\ 0 & 6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \\ 0 & 6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The basic variables are x and y. The variable z is free. Now, we express the basic variables in terms of the free variables. From the last matrix we get the equations

$$x = z \qquad \text{and} \qquad y = z.$$

Therefore, the eigenvectors of the matrix A that correspond to the eigenvalue $\lambda = 1$ are exactly the vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where z can be any real number (or any complex number if you want to work over the complex numbers).