Practice

Math 101. Midterm 1.

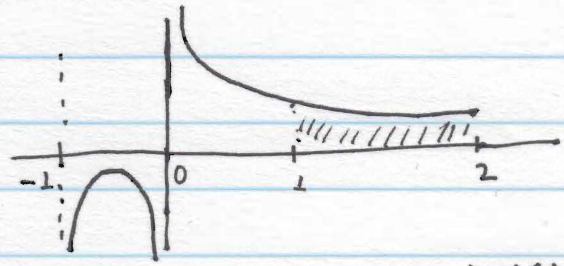
Solutions to problems in the Feb 20/2014 review section.

① What is the antiderivative $\int \frac{1}{x+x^2}\,dx$?

$$\int \frac{1}{x+x^2}\,dx \overset{*}{=} \int \left[\frac{1}{x} - \frac{1}{x+1}\right] dx = \ln|x| - \ln|x+1| + C.$$

$$* \quad \frac{1}{x+x^2} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{(A+B)x+A}{x+x^2}.$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \Rightarrow A=1,\ B=-1$$



② What if we integrate it over $[1,2]$?

$$\int_1^2 \frac{1}{x+x^2}\,dx \overset{FTC}{=} \left(\ln|x| - \ln|x+1|\right)\Big|_1^2 = (\ln 2 - \ln 3) - (\ln 1 - \ln 2) = \ln\left(\frac{4}{3}\right).$$

③ What if we integrate it over $[-1,1]$ instead? Improper!!

$$\underbrace{\int_{-1}^{1} \frac{1}{x+x^2}\,dx}_{} = \underbrace{\lim_{a\to -1^+} \int_a^{-1/2} \frac{1}{x+x^2}\,dx}_{Ⓐ} + \underbrace{\lim_{b\to 0^-} \int_{-1/2}^{b} \frac{1}{x+x^2}\,dx}_{Ⓑ} + \underbrace{\lim_{c\to 0^+} \int_c^1 \frac{1}{x+x^2}\,dx}_{Ⓒ}.$$

Would need that Ⓐ, Ⓑ, Ⓒ exist (converge).

But not! look for Ⓒ:

$$\lim_{c\to 0^+} \int_c^1 \frac{1}{x+x^2}\,dx = \lim_{c\to 0^+} \left(\ln|x| - \ln|x+1|\right)\Big|_c^1 = \lim_{c\to 0^+} \left(\ln 1 - \ln 2 - \ln c + \ln(c+1)\right)$$

diverges!!

④ What if we integrate it over $[1,\infty)$ instead?

$$\int_1^\infty \frac{1}{x+x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x+x^2}\,dx = \lim_{t\to\infty} \left(\ln|x| - \ln|x+1|\right)\Big|_1^t = \lim_{t\to\infty} \left(\ln t - \ln(t+1) - \ln 1 + \ln 2\right)$$

$$= \lim_{t\to\infty} \ln\left(\frac{2t}{t+1}\right) = \ln 2.$$

⑤ What is the average value of $(\tan x)^{-2}$ on the interval $[\pi/6, \pi/4]$.

$$\text{Avg} = \frac{1}{\pi/4 - \pi/6}\int_{\pi/6}^{\pi/4} \frac{1}{\tan^2 x}\,dx = \frac{1}{\frac{\pi}{12}}\cdot\left[-\frac{1}{\tan x} - x\right]\Bigg|_{\pi/6}^{\pi/4} = \frac{12}{\pi}\left[-1+\sqrt{3}-\frac{\pi}{12}\right].$$

Antiderivative:

$$\int \frac{1}{\tan^2 x}\,dx = \int\frac{\sec^2 x - \tan^2 x}{\tan^2 x}\,dx = \underbrace{\int \frac{\sec^2 x}{\tan^2 x}\,dx}_{\substack{u=\tan x\\ du=\sec^2 x}} - \int dx = \int \frac{1}{u^2}\,du - x$$

$$= -\frac{1}{u} - x + C$$

$$= -\frac{1}{\tan x} - x + C.$$

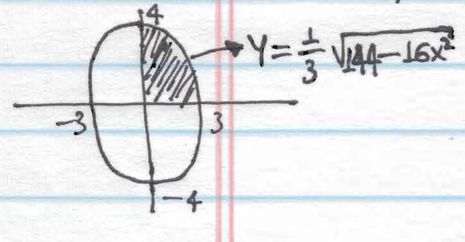
⑥

$$\frac{d}{d\theta}\int_{4+\sin\theta}^{3\cos\theta} x^2\cdot e^{-x}\,dx = \frac{d}{d\theta}\int_{4+\sin\theta}^{0} x^2\cdot e^{-x}\,dx + \frac{d}{d\theta}\int_{0}^{3\cos\theta} x^2\cdot e^{-x}\,dx$$

$$= -\frac{d}{d\theta}\int_{0}^{4+\sin\theta} x^2\cdot e^{-x}\,dx + \frac{d}{d\theta}\int_{0}^{3\cos\theta} x^2\cdot e^{-x}\,dx$$

$$= (4+\sin\theta)^2\cdot e^{-(4+\sin\theta)}\cdot\cos\theta + (3\cos\theta)^2\cdot e^{-(3\cos\theta)}\cdot 3\cdot(-\sin\theta).$$

⑦ Find area of the ellipse



$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

$$\Rightarrow y = \frac{\pm 1}{3}\sqrt{144-16x^2}$$

$$y = \pm\frac{4}{3}\sqrt{9-x^2}$$

$$A = 4\cdot\int_0^3 \frac{1}{3}\cdot\sqrt{144-16x^2}\,dx$$

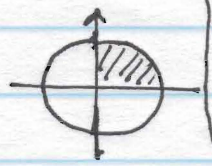
$$= 4\cdot\int_0^3 \frac{1}{3}\cdot 4\cdot\sqrt{9-x^2}\,dx$$

$$= \frac{16}{3}\int_0^3\sqrt{9-x^2}\,dx.$$

$$= \frac{16}{3}\cdot\frac{1}{4}\cdot\pi\cdot 3^2 = 12\pi.$$

$$A = \frac{16\cdot 9}{3\cdot 2}\left[\arcsin\left(\frac{x}{3}\right)\right]_0^3 = \frac{16\cdot 9}{3\cdot 2}\cdot\arcsin(1) = \frac{16\cdot 9}{3\cdot 2}\cdot\frac{\pi}{2} = 12\pi$$

✓



$x = 3\sin\theta$
$dx = 3\cos\theta$

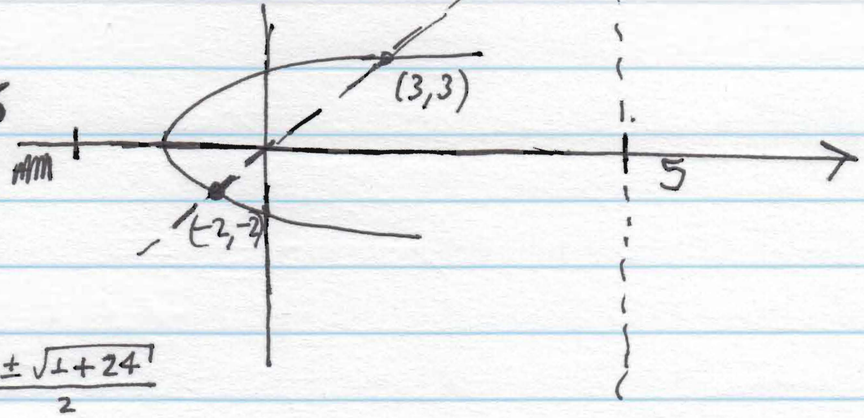
Alternatively: To practice trig substitution: $\int\sqrt{9-x^2}\,dx = \ldots$

$$\int\sqrt{3^2-x^2}\,dx = \int\sqrt{3^2-3^2\sin^2\theta}\cdot 3\cdot\cos\theta = \int 3^2\cdot\cos^2\theta\,d\theta = 3^2\cdot\int\cos^2\theta\,d\theta = 3^2\cdot\int\frac{\cos 2\theta+1}{2}\,d\theta$$

$$= 3^2\cdot\frac{\sin 2\theta}{4} + \frac{3^2}{2}\cdot\theta + C = \frac{3\cdot\sin\theta\cdot\sqrt{3^2-3^2\sin^2\theta}}{2} + \frac{3^2}{2}\theta + C = \frac{x\cdot\sqrt{9-x^2}}{2} + \frac{9}{2}\cdot\arcsin\left(\frac{x}{3}\right) + C.$$

(8) Area of region bounded between curves:

$$\begin{cases} y^2 = x+6 \\ y = x \end{cases}$$



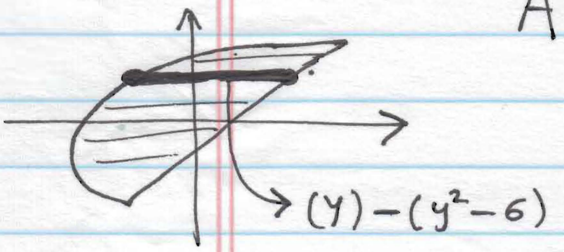
$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0 \qquad x = \frac{1 \pm \sqrt{1+24}}{2}$

$\boxed{x=3}$ $\boxed{y=3}$ $\qquad$ $\boxed{x=-2}$ $\boxed{y=-2}$

$$A = \int_{-2}^{3} \left[(y) - (y^2-6) \right] dy$$

$$= \left(\frac{y^2}{2} - \frac{y^3}{3} + 6y \right) \Big|_{-2}^{3} = \frac{125}{6}.$$



(9) What about volume if make it rotate about line $x=5$??

$$V = \int_{-2}^{3} \left[\pi R_y^2 - \pi r_y^2 \right] dy$$

$$V = \int_{-2}^{3} \left[\pi \cdot (5-(y^2-6))^2 - \pi \cdot (5-y)^2 \right] dy.$$

$$= \pi \cdot \left(\frac{y^5}{5} - \frac{23}{3} y^3 + 5y^2 + 96y \right) \Big|_{-2}^{3} = \frac{875}{3} \pi.$$

(10) Find Riemann Sum for simple integral on $[0,1]$.

$$\int_0^1 e^{3x+4} dx = \lim_{n \to \infty} \sum_{i=0}^{n} e^{3 \cdot \left(\frac{i}{n}\right)+4} \cdot \frac{1}{n}.$$

$$= e^4 \cdot \int_0^1 e^{3x} = e^4 \cdot \frac{1}{3} \cdot e^{3x} \Big|_0^1 = \frac{e^4}{3}(e^3-1). = \frac{e^7}{3} - \frac{e^4}{3}.$$