University of British Columbia
Student Name and Section:

Math 101 Quiz 4

Sect. 202-203 - Winter 2014
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Instructions: This quiz aims to help you learn some material important for our class. Use it for your own benefit.

Hints: You will get the same credit whether you use this version with hints or the alternative version with no hints.

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(1) (*5 points*) Determine all possible the values of the constant $p > 0$ for which the following series converges: $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+\cos(n)}{(n^3+\sin(n))^p}$

Strategy: The given series has positive terms and then it should behave just like $\sum \frac{n^2}{n^{3p}} = \sum \frac{1}{n^{3p-2}}$. So, we expect it to converge for $3p-2>1$ and to diverge for $3p-2 \leq 1$. Then we consider separately the three cases $p>1$, $p=1$ and $p<1$.

Solution: We consider three cases:

Case $p > 1$ The function $f(x) = \frac{3x^2+\cos(x)}{(x^3+\sin(x))^p}$ is positive and continuous for $x \geq 1$. Its derivative is

$$f'(x) = \frac{(6x-\sin(x))(x^3+\sin(x))^p - p(x^3+\sin(x))^{p-1}(3x^2+\cos(x))^2}{(x^3+\sin(x))^{2p}} = \frac{(6x-\sin(x))(x^3+\sin(x)) - p(3x^2+\cos(x))^2}{(x^3+\sin(x))^{p+1}}$$
$$= \frac{(6-9p)x^4 + (\text{Expression eventually dominated by } 2x^3)}{(x^3+\sin(x))^{p+1}}$$

Since $6-9p<0$, then $f'(x)$ is eventually negative; so, $f(x)$ is eventually decreasing. Then, you can use integral test.

Show that the series converges for all $p > 1$ using integral test:

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{3x^2+\cos(x)}{(x^3+\sin(x))^p}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{3x^2+\cos(x)}{(x^3+\sin(x))^p}\,dx = \lim_{t\to\infty} \left.\frac{1}{(-p+1)\cdot(x^3+\sin(x))^{p-1}}\right|_1^t$$
$$= \lim_{t\to\infty}\left[\frac{1}{(-p+1)\cdot(t^3+\sin(t))^{p-1}} - \frac{1}{(-p+1)\cdot(1^3+\sin(1))^{p-1}}\right] = \frac{1}{(p-1)\cdot(1+\sin(1))^{p-1}}.$$

Since the improper integral converges then the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+\cos(n)}{(n^3+\sin(n))^p}$ converges as well.

Case $p = 1$ Show that the series diverges by limit comparison test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

By limit comparison test we have:

$$\lim_{n\to\infty}\frac{\frac{3n^2+\cos(n)}{n^3+\sin(n)}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{3n^3+n\cos(n)}{n^3+\sin(n)} = \lim_{n\to\infty}\frac{3+\frac{\cos(n)}{n^2}}{1+\frac{\sin(n)}{n^3}} = \frac{3+0}{1+0} = 3.$$

Then both series $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+\cos(n)}{n^3+\sin(n)}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ converge or both diverge. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series) then $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+\cos(n)}{n^3+\sin(n)}$ diverges as well.

Case $p < 1$ Show that the series diverges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+\cos(n)}{(n^3+\sin(n))^1}$ (which you just solved).

Given $p < 1$, for each $n \geq 1$ we have the following inequality:

$$(n^3 + \sin(n))^1 \geq (n^3 + \sin(n))^p$$

Therefore,

$$0 \leq \frac{3n^2 + \cos(n)}{(n^3 + \sin(n))^1} \leq \frac{3n^2 + \cos(n)}{(n^3 + \sin(n))^p}$$

Since the series $\sum_{n=1}^{\infty} \frac{3n^2 + \cos(n)}{n^3 + \sin(n)}$ diverges (that was precisely Case $p = 1$), then by comparison test the series $\sum_{n=1}^{\infty} \frac{3n^2 + \cos(n)}{(n^3 + \sin(n))^p}$ diverges as well.

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(2) (*5 points*) Find the radius of convergence R and the interval of convergence of the series: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n!)^2 \cdot (x-40)^n}{(2n)! \cdot 3^{2n+1}}$

Strategy: Use ratio test to find the radius of convergence. You'll get something that looks like $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = A|Bx+C| < 1$, which you transform into $|x -$ "center"$| <$ "radius". To find the interval of convergence you need replace x with each of the two end points "center + radius" and "center - radius", and determine if each of those two series converges or diverges.

Solution: First we use ratio test to find R:

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{|x-40|^{n+1}}{|x-40|^n} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{3^{2n+1}}{3^{2n+3}} \\ &= \lim_{n\to\infty} (n+1)^2 \cdot |x-40| \cdot \frac{1}{(2n+2)(2n+1)} \cdot \frac{1}{3^2} \\ &= \frac{|x-40|}{36} \end{aligned}$$

Now we set $\frac{|x-40|}{36} < 1$, which is equivalent to $|x-40| < 36$. Then the radius of convergence is $R = 36$.

Now, verify that the sequence $\left\{ \frac{(n!)^2 \cdot 4^n}{(2n)! \cdot 3} \right\}$ is increasing (so, in particular it does not have limit zero):

Let $B_n = \frac{(n!)^2 \cdot 4^n}{(2n)! \cdot 3}$. Let us verify that $B_{n+1} \geq B_n$ for all $n \geq 1$.

$$\begin{aligned} B_{n+1} \geq B_n &\iff \frac{((n+1)!)^2 \cdot 4^{n+1}}{(2n+2)! \cdot 3} \geq \frac{(n!)^2 \cdot 4^n}{(2n)! \cdot 3} \\ &\iff \frac{(n+1)^2 \cdot 4}{(2n+2)(2n+1)} \geq 1 \iff 4n^2 + 8n + 4 \geq 4n^2 + 6n + 2 \\ &\iff 2n + 2 \geq 0. \end{aligned}$$

As the last inequality is true for all $n \geq 1$, then so are the previous ones, and we conclude that the sequence $\left\{ \frac{(n!)^2 \cdot 4^n}{(2n)! \cdot 3} \right\}$ is increasing.

Use the previous assertion to test the two end points and get the interval of convergence:

The end points of the interval of convergence are $a + R = 40 + 36 = 76$ and $a - R = 40 - 36 = 4$.

When $x = 76$ we get $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n!)^2 \cdot (76-40)^n}{(2n)! \cdot 3^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n!)^2 \cdot 4^n}{(2n)! \cdot 3}$ and this series diverges (as we just saw that the sequence of terms cannot have limit zero).

When $x = 4$ we get $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n!)^2 \cdot (4-40)^n}{(2n)! \cdot 3^{2n+1}} = \sum_{n=0}^{\infty} \frac{(n!)^2 \cdot 4^n}{(2n)! \cdot 3}$ and this series diverges (as we just saw that the sequence of terms cannot have limit zero).

Therefore the interval of convergence is $(4, 76)$, or in other words $4 < x < 76$.

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(3) (*5 points*) Find the function $y = f(x)$ that satisfies the differential equation $f'(x) = \frac{2x}{3y^2}$ and the initial condition $f(0) = 2$.

Strategy: This is a separable first order differential equation. Write $\frac{dy}{dx} = \frac{2x}{3y^2}$; separate the expressions containing the variables x and y; integrate both sides; solve for y; and finally use the initial condition to find the value of the arbitrary constant that appeared when integrating.

Solution:

We separate variables and integrate:

$$\frac{dy}{dx} = \frac{2x}{3y^2}$$

$$3y^2 dy = 2x dx$$

$$\int 3y^2 \, dy = \int 2x \, dx$$

$$y^3 = x^2 + C$$

Since $f(0) = 2$, then $2^3 = 0^2 + C$, so $C = 8$. Then,

$$y^3 = x^2 + 8$$

and the final answer is:

$$\boxed{f(x) \;=\; y = \sqrt[3]{x^2 + 8}}$$