Answers

1. $\displaystyle\int_0^4 \frac{dx}{(x-1)^2}$ discontinuous at $x=1$ inside of $(0,4)$

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^2} = \lim_{b\to 1^-} -\frac{1}{x-1}\Big/_0^b = \lim_{b\to 1^-} -\left(\frac{1}{b-1} - (-1)\right)$$

$$= \lim_{b\to 1^-} -\left(\frac{1}{b-1}+1\right) = +\infty \Rightarrow \int_0^4 \frac{dx}{(x-1)^2} \text{ is } \boxed{\text{divergent}}$$

[note: it is unnecessary to check $\lim_{b\to 1^+}$, because Both limits must be finite to have convergence.]

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2. $$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b x^{-p}\,dx$$

$$= \lim_{b\to+\infty} \frac{x^{-p+1}}{-p+1}\Big/_1^b$$

→ Then, Assume $p>1 \Rightarrow \displaystyle\lim_{b\to+\infty}\frac{1}{-p+1}\left(b^{-p+1}-1\right)$

$$= \frac{1}{-p+1}(0-1) = \frac{1}{p-1} < \infty \Rightarrow \text{convergent.}$$

→ by integral test, we know $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges

→ Then, Assume $p<1 \Rightarrow \displaystyle\lim_{b\to+\infty}\frac{1}{1-p}\left(\underbrace{b^{1-p}}_{\to\infty}-1\right) = +\infty$

because $1-p>0$

3]

$$\frac{n!}{2^n} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{2\cdot 2\cdot 2\cdot \ldots \cdot 2} = \left(\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot \ldots \cdot\frac{n}{2}\right) > \frac{n}{2}$$

for n large enough $\Rightarrow$ $\frac{n!}{2^n}$ is <u>not</u> <u>bounded</u> $\Rightarrow$ divergent.

4]

$$0 \le \sin^2(n) \le 1 \Rightarrow 0 \le \frac{\sin^2(n)}{2^n} \le \frac{1}{2^n}$$

claim: $\frac{1}{2^n} \le \frac{1}{n^2}$

proof: $\frac{1}{2^n} \overset{?}{\le} \frac{1}{n^2} \Rightarrow n^2 \overset{?}{\le} 2^n$

This is true for $n \ge 4 \Rightarrow \frac{1}{2^n} \le \frac{1}{n^2}$

$$\Rightarrow \frac{\sin^2(n)}{2^n} \le \frac{1}{2^n} \le \frac{1}{n^2} \Rightarrow \text{convergent}$$

by Squeeze Theorem.

5]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

a geo-series with $r = \frac{-1}{2}$ and $a = 1$

since $|r| = \frac{1}{2} < 1 \Rightarrow$ convergent

with sum $= \frac{a}{1-r} = \frac{2}{3}$

6] $$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\frac{5}{2^n} + \sum_{n=0}^{\infty}\frac{1}{3^n}$$

$$= 5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$$

Two geo series

$\Rightarrow$ $\boxed{\text{convergent}}$ with sum: $\frac{5}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = \boxed{\frac{23}{2}}$

7] $$\sum_{n=0}^{\infty}(-1)^n\frac{n}{e^n}$$

First, Test for Absolute convergence: $\sum_{n=0}^{\infty}\left|(-1)^n\frac{n}{e^n}\right| = \sum_{n=0}^{\infty}\frac{n}{e^n}$.

$\sum_{n=0}^{\infty}\frac{n}{e^n}$ converges by integral test. (i.e. $\int_0^{\infty}\frac{n}{e^n}dn < \infty$)

$\Rightarrow$ absolute convergence

8] use alternating series test.

$f(x) = \frac{\ln(x)}{x-\ln(x)} \Rightarrow f'(x) = \frac{(\frac{1}{x})(x-\ln(x)) - (\ln(x))(1-\frac{1}{x})}{(x-\ln(x))^2} = \frac{1-(\frac{\ln(x)}{x})-\ln(x)+(\frac{\ln(x)}{x})}{(x-\ln(x))^2}$

$= \frac{1-\ln(x)}{(x-\ln(x))^2} = \frac{1-\ln(x)}{(x-\ln(x))^2} < 0 \Rightarrow u_n \geq u_{n+1} > 0$ with n sufficiently large

and $\lim_{n\to\infty}\frac{\ln(n)}{n-\ln(n)} \overset{L'H}{=} \lim_{n\to\infty}\frac{1/n}{1-1/n} = 0 \Rightarrow$ convergence

BUT: $n \leq n \Rightarrow n-\ln(n) \leq n \Rightarrow \frac{1}{n-\ln(n)} \geq \frac{1}{n}$

$\Rightarrow \frac{\ln(n)}{n-\ln(n)} \geq \frac{1}{n} \Rightarrow$ divergent. $\therefore$ $\boxed{\text{conditionally conv.}}$

9.

$$\sum_{n=2}^{\infty} \left| (-1)^n \left(\frac{\ln(n)}{\ln(n^2)} \right)^n \right| = \sum_{n=2}^{\infty} \left(\frac{\ln(n)}{\ln(n^2)} \right)^n$$

$$= \sum_{n=2}^{\infty} \left(\frac{\ln(n)}{2\ln(n)} \right)^n = \sum_{n=2}^{\infty} \left(\frac{1}{2} \right)^n$$

$\Rightarrow$ conv. geo series.

in fact, $\boxed{\text{absolutely conv.}}$

10.

Take limit.

$$\lim_{n\to\infty} \left| (-1)^n \frac{(2n)!}{2^n n! n} \right| = \lim_{n\to\infty} \frac{(2n)!}{2^n n! n} = \lim_{n\to\infty} \frac{(n+1)(n+2)\cdot\ldots\cdot(2n)}{2^n n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+2)\cdot\ldots\cdot(n+(n-1))}{2^{n-1}} > \lim_{n\to\infty} \left(\frac{n+1}{2} \right)^{n-1} = \infty \neq 0$$

$\Rightarrow$ divergent.