

# Practice Midterm 1

## MATH 9A

### Solutions

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October 17, 2005

1. Compute  $\lim_{x \rightarrow 0} \frac{\sqrt{9-x} - 3}{3x}$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{9-x} - 3}{3x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{9-x} - 3)(\sqrt{9-x} + 3)}{3x(\sqrt{9-x} + 3)} \\ &= \lim_{x \rightarrow 0} \frac{9 - x - 9}{3x(\sqrt{9-x} + 3)} \\ &= \lim_{x \rightarrow 0} \frac{-1}{3(\sqrt{9-x} + 3)} \\ &= -\frac{1}{18}\end{aligned}$$

2. Using the definition of the limit, show that  $\lim_{x \rightarrow 4} 6 - 2x = -2$ .

Recall,  $\lim_{x \rightarrow 4} 6 - 2x = -2$  if for every  $\epsilon$  there exists a  $\delta$  such that if  $0 < |x - 4| < \delta$ , then  $|6 - 2x - (-2)| < \epsilon$ .

We observe that  $|6 - 2x - (-2)| < \epsilon$  is equivalent to  $|8 - 2x| < \epsilon$ . Which is similarly equivalent to  $2|4 - x| < \epsilon$ . As  $|4 - x| = |x - 4|$ , we see a final equivalence with  $|x - 4| < \frac{\epsilon}{2}$ .

Hence, if we choose,  $\delta = \frac{\epsilon}{2}$ , then we see that if we choose  $x$  such that  $0 < |x - 4| < \delta$ , then  $|6 - 2x - (-2)| < \epsilon$ . Hence,  $\lim_{x \rightarrow 4} 6 - 2x = -2$ .

3. Suppose  $f$  is an odd function and we know  $\lim_{x \rightarrow -2^-} f(x) = 3$ . Can we say anything about  $\lim_{x \rightarrow 2^-} f(x)$  or  $\lim_{x \rightarrow 2^+} f(x)$ ? Explain your answer.

As  $f$  is an odd function and we know  $f(-x) = -f(x)$ . Hence, if  $\lim_{x \rightarrow -2^-} f(x) = 3$ , we see that  $\lim_{x \rightarrow 2^+} f(x) = -3$  since  $|x| > 2$  if  $x < -2$ . We cannot say anything about  $\lim_{x \rightarrow 2^-} f(x)$  as we know not, what  $\lim_{x \rightarrow -2^+} f(x)$  is or if it exists at all.

4. Find the horizontal and vertical asymptote(s) of the function  $f(x) = \frac{2x^2 - 5x - 3}{x^2 - 4x + 3}$ .

To find the horizontal asymptote(s) we need to determine  $\lim_{x \rightarrow \infty} \frac{2x^2 - 5x - 3}{x^2 - 4x + 3}$  and  $\lim_{x \rightarrow -\infty} \frac{2x^2 - 5x - 3}{x^2 - 4x + 3}$ . Note however, since the highest power of  $x$  in both the numera-

tor and denominator is 2, we need only compare the coefficients of  $x^2$  in both the numerator and the denominator. Hence,  $\lim_{x \rightarrow \infty} \frac{2x^2 - 5x - 3}{x^2 - 4x + 3} = \lim_{x \rightarrow -\infty} \frac{2x^2 - 5x - 3}{x^2 - 4x + 3} = 2$ . Hence,  $f(x)$  has horizontal asymptote  $y = 2$ .

To find the vertical asymptote(s), we need to find the values,  $x$  for which  $f(x)$  is not defined. Then we determine whether the left or right hand limit near these values approaches some form of infinity.  $f(x)$  is undefined at  $x = 1$  and  $x = 3$ . At  $x \neq 3$   $f(x) = \frac{2x + 1}{x - 1}$ . So

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{2x + 1}{x - 1} = +\infty$$

since the top approaches 3 and the bottom is a very small positive number. Hence  $x = 1$  is a vertical asymptote.

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{2x + 1}{x - 1} = \frac{7}{2}.$$

So  $x = 1$  is the only vertical asymptote.

5. Find  $\lim_{x \rightarrow 0} \frac{\sin^2(3x)}{x^2}$ .

$$\lim_{x \rightarrow 0} \frac{\sin^2(3x)}{x^2} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{3 \sin(3x)}{3x} \cdot \frac{3 \sin(3x)}{3x} = 9$$

as  $\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1$ .

6. Using the Intermediate Value Theorem, show that  $f(x) = x^3 - 4x + 11$  has a root.

Note that  $f(0) = 11 > 0$  and  $f(-3) = -5 < 0$ . Now, by the Intermediate Value Theorem, there exists a  $c \in (-3, 0)$  such that  $f(c) = 0$ . Hence,  $f(x)$  has a root.

7. Find the slope of the tangent line to  $f(x) = x - 3x^2$  at  $(1, -2)$  using the definition

$$m_{tan} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}.$$

$$\begin{aligned} m_{tan} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h) - 3(1+h)^2 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1+h-3-6h-3h^2+2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-5h-3h^2}{h} \\ &= \lim_{h \rightarrow 0} -5-3h = -5 \end{aligned}$$

8. Find the equation of the tangent line to  $f(x) = \frac{2\sqrt{x}}{\sqrt{x}-1}$  at the point  $(4, 4)$ .

$$f'(x) = \frac{(\sqrt{x}-1)\frac{1}{\sqrt{x}} - 2\sqrt{x}\frac{1}{2\sqrt{x}}}{(\sqrt{x}-1)^2} = -\frac{1}{\sqrt{x}(\sqrt{x}-1)^2}$$

The slope of the tangent line to  $f(x)$  at  $x = 4$  is  $f'(4) = -\frac{1}{2}$ .

Hence, the equation of the tangent line is  $y = -\frac{1}{2}(x-4) + 4$  or  $y = -\frac{1}{2}x + 6$ .

9. The position of a point moving along the  $x$  axis is  $x(t) = 3 - 3\cos t$  units. Find the speed of the point at time  $t = \frac{\pi}{6}$

The speed of the point at time  $t = \frac{\pi}{6}$  is given by  $x'(\frac{\pi}{6})$ .  $x'(t) = 3\sin t$ . Hence, the speed of the point is  $\frac{3}{2}$  units per second.

10. Refer to the graph below to answer the following:

You should refer to the graph that I sketched on the board during class Thursday.

- (a) Where is  $f(x)$  continuous?

$f(x)$  is continuous on the intervals  $[-1, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4]$ .

- (b) Where is  $f(x)$  differentiable?

$f(x)$  is differentiable on the intervals  $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4]$ .

- (c) Identify the  $x$  values of any jump discontinuities.

$f(x)$  has jump discontinuities at  $x = 1$  and  $x = 2$ .

- (d) Identify the  $x$  values of any removable discontinuities.

$f(x)$  has a removable discontinuity at  $x = 3$ .

- (e) Find  $\lim_{x \rightarrow 2^+} f(x)$ .

$$\lim_{x \rightarrow 2^+} f(x) = 1.$$

11. Find the first two derivatives of  $f(x) = x \tan x$ .

$$f'(x) = x \sec^2 x + \tan x.$$

$$f''(x) = x(\sec x \sec x \tan x + \sec x \sec x \tan x) + \sec^2 x + \sec^2 x = 2 \sec^2 x(x \tan x + 1)$$