Practice Midterm 1
MATH 9C

Prof. Janet Vassilev

April 25, 2006

1. Determine if the following improper integral converges or diverges:

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$$

Note $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to 1}\int_b^e \frac{1}{x\sqrt{\ln x}}\,dx.$

Using substitution, with $u = \ln x$, $du = \frac{1}{x}dx$, $u(b) = \ln b$, $u(e) = 1$.

We see that $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to 1}\int_{\ln b}^1 \frac{1}{\sqrt{u}}\,du = \lim_{b\to 1} 2 - \sqrt{\ln b} = 2$. Hence, $\displaystyle\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$ converges.

2. Consider the sequence $a_n = \dfrac{1+(-1)^n}{n}$.

   (a) Is a_n bounded below? If so, find a lower bound.

   As $0 \le \dfrac{1+(-1)^n}{n} \le \dfrac{2}{n}$, a_n is bounded below by 0.

   (b) Is a_n non increasing? Explain your answer.

   a_n is not non increasing as $a_{2n-1} = 0$ for $n \ge 1$, but $a_{2n} = \frac{2}{n}$ so a_n increases from every odd to the next even.

   (c) Find $\lim_{n\to\infty} a_n$.

   Using the sandwich theorem, as $0 \le \dfrac{1+(-1)^n}{n} \le \dfrac{2}{n}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} a_n = 0$.

3. List the first three partial sums of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$.

$S_1 = \frac{1}{6}$, $S_2 = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$, $S_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{10}$.

4. Suppose a_n and b_n are sequences and $\sum_{n=1}^{\infty} a_n + b_n$ converges. Is it true that $\sum_{n=1}^{\infty} a_n$ converges? Why or why not?

   Let $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n+1}$. $a_n + b_n = \frac{1}{n} - \frac{1}{n+1}$ and $\sum_{n=1}^{\infty} a_n + b_n$ is a converging telescoping series, but $\sum_{n=1}^{\infty} a_n$ is a diverging p series. So the answer is false.

5. Find the sum of the series $\sum_{n=0}^{\infty} \frac{2(-1)^n}{5^n}$

   $$\sum_{n=0}^{\infty} \frac{2(-1)^n}{5^n} = \frac{2}{1+\frac{1}{5}} = \frac{5}{3}$$

6. Suppose a_n is a sequence which is negative, increasing and $a_n = f(n)$ for $n \geq 1$ for some continuous function $f(x)$ on $[1, \infty)$. How might one determine if $\sum_{n=1}^{\infty} a_n$ is convergent or not?

   Taking the absolute value of a_n, $|a_n| = -a_n > 0$. Since $a_n \leq a_{n+1}$, $|a_n| = -a_n \geq -a_{n+1} = |a_{n+1}|$, so $|a_n|$ is a decreasing positive function that agrees with the continuous function $|f(x)|$ on $[1, \infty)$. Hence, $|a_n|$ satisfies the hypotheses of the integral test. Now if $\sum_{n=1}^{\infty} |a_n|$ is convergent then so is $\sum_{n=1}^{\infty} a_n$ is convergent and if $\sum_{n=1}^{\infty} |a_n|$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent since $\sum_{n=1}^{\infty} a_n = -\sum_{n=1}^{\infty} |a_n|$.

7. Determine if the following series converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2-n+1}$

   Compare with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is a diverging p-series.

   $\lim_{n\to\infty} \frac{\frac{2n+1}{n^2-n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(2n+1)}{n^2-n+1} = 2 \neq 0$. So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{2n+1}{n^2-n+1}$ diverges.

   (b) $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

   Compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a converging p-series.

   $\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$. So by the limit comparison test, $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges.

   (c) $\sum_{n=1}^{\infty} \frac{5}{n^n}$

   Using the root test, $\lim_{n\to\infty} \sqrt[n]{\frac{5}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{5}}{n} = 0 < 1$. We conclude that $\sum_{n=1}^{\infty} \frac{5}{n^n}$ converges by the root test.

(d) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Here we will use the ratio test. We need to compute $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2n+2)!}}{\frac{n!}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)!(2n)!}{n!(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)}{(2n+2)(2n+1)} = 0 < 1$. Hence, $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges by the ratio test.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

We will use the alternating series test. $a_n = \frac{1}{\sqrt{n}} > 0$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Therefore, by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

(f) $\sum_{n=1}^{\infty} n\sin(\frac{1}{n})$

$\lim_{n\to\infty} n\sin(\frac{1}{n}) = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$ as $\frac{1}{n} \to 0$ as $n \to \infty$. So by the n-th term test, $\sum_{n=1}^{\infty} n\sin(\frac{1}{n})$ diverges.

8. Determine if the following series are conditionally convergent, absolutely convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

Note that $\left|\frac{(-3)^n}{n!}\right| = \frac{3^n}{n!}$. Using the ratio test, $\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$. Hence, $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ is absolutely convergent by the ratio test.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

$|\frac{(-1)^n n}{n^2+1}| = \frac{n}{n^2+1}$. Compare with $\frac{1}{2n}$. $\frac{n}{n^2+1} > \frac{1}{2n}$. $\sum_{n=1}^{\infty} \frac{1}{2n}$ is $\frac{1}{2}$ times the divergent p series $\sum_{n=1}^{\infty} \frac{1}{n}$. Now by the direct comparison test, we obtain $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is a diverging series. But $\frac{n}{n^2+1} > 0$ is a decreasing sequence with $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges by the alternating series test. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent.

(c) $\sum_{n=1}^{\infty} \frac{(2n)^{2n}}{(2n^2+1)^n}$

Note $\frac{(2n)^{2n}}{(2n^2+1)^n} = \left|\frac{(2n)^{2n}}{(2n^2+1)^n}\right|$. We will use the root test. We see that

$\lim_{n\to\infty} \sqrt[n]{\frac{(2n)^{2n}}{(2n^2+1)^n}} = \lim_{n\to\infty} \frac{4n^2}{(2n^2+1)} = 2$. Thus $\sum_{n=1}^{\infty} \frac{(2n)^{2n}}{(2n^2+1)^n}$ diverges.