Practice Midterm 2
MATH 9A

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1. The velocity of a body moving along the x-axis is $v = 2t^2 - 7t + 6$. When is the body moving forward?

   We need to determine when $v > 0$. This will tell us when the body is moving forward. Note, $v = 2t^2 - 7t + 6$ factors as $(2t - 3)(t - 2)$. Hence, the critical values of s are $t = \frac{3}{2}$ and $t = 2$. If we test v on $[0, \frac{3}{2})$ we see $v > 0$. Testing v on $(\frac{3}{2}, 2)$ we see that $v < 0$. Testing v on $(2, \infty)$ we see that $v > 0$. Hence, the body is moving forward when $t < \frac{3}{2}$ and when $t > 2$.

2. Suppose for functions f and g, we know $f(1) = 5$, $g(1) = -1.5$, $f'(1) = 11$, $g'(1) = -8$, $f(2) = 3$, $g(2) = -1$, $f'(2) = .3$, $g'(2) = -5$. Find the derivative of

   (a) $f(x + g(x))$ at $x = 2$

   (b) $g(f(x) + 2g(x))$ at $x = 1$.

   (a) If $h(x) = f(x + g(x))$, then $h'(x) = f'(x + g(x))(1 + g'(x))$. Hence,

   $$h'(2) = f'(2 + g(2))(1 + g'(2)) = f'(1)(1 - 5) = 11(-4) = -44.$$

   (b) If $h(x) = g(f(x) + 2g(x))$, then $h'(x) = g'(f(x) + 2g(x))(f'(x) + 2g'(x))$. Hence,

   $$h'(1) = g'(f(1)+2g(1))(f'(1)+2g'(1)) = g'(5+2(-1.5))(11+2(-8)) = -5(-5) = 25.$$

3. Find the tangent line to the parametric curve defined by $x = \tan(2t)$, $y = \sec^2(t)$ when $t = -\frac{\pi}{6}$.

   First we need to find $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. $\frac{dy}{dt} = 2\sec t \sec t \tan t = 2\sec^2 t \tan t$ and $\frac{dx}{dt} = 2\sec^2(2t)$. So $\frac{dy}{dx} = \frac{\sec^2 t \tan t}{\sec^2(2t)}$. The slope of the tangent line at $t = -\frac{\pi}{6}$ is $\frac{dy}{dx}(-\frac{\pi}{6}) = \frac{\frac{4}{3} \cdot -\frac{1}{\sqrt{3}}}{4} = -\frac{1}{3\sqrt{3}}$.

   Therefore, the tangent line is $y - \frac{4}{3} = -\frac{1}{3\sqrt{3}}(x + \sqrt{3})$ or $y = -\frac{1}{3\sqrt{3}}x + 1$.

4. If $x^2 + y^3 = 5$, find $\dfrac{d^2x}{dy^2}$ at the point $(-2, 1)$.

First we need to find $\dfrac{dx}{dy}$. Note, this is taking the derivative of everything with respect to y.

$2x\dfrac{dx}{dy} + 3y^2 = 0$. Hence, $\dfrac{dx}{dy} = -\dfrac{3y^2}{2x}$.

Now we take the derivative of $\dfrac{dx}{dy}$ with respect to y again to get

$$\frac{d^2x}{dy^2} = \frac{-12xy + 6y^2\frac{dx}{dy}}{4x^2} = \frac{-12xy - \frac{9y^4}{x}}{4x^2}.$$

Now plug in the point $(-2, 1)$ to get $\dfrac{d^2x}{dy^2}(-2, 1) = \dfrac{57}{32}$.

5. Two planes are flying at 35,000 feet along straight line courses that intersect at right angles. Plane A is approaching the intersection at a speed of 300 miles per hour and plane B is approaching the intersection at a speed of 400 miles per hour. At what rate is the distance between the planes changing when A is 50 miles from the intersection point and B is 120 miles from the intersection point?

Draw a right triangle ABC, where the right angle is at vertex C. Label side AC as x, BC as y and AB as z.

Note we have the relation $x^2 + y^2 = z^2$.

As the planes fly toward the intersection point C, all of the sides of the triangle are decreasing.

Take the derivative of the expression above with respect to t to get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. We know that $\frac{dx}{dt} = -300$ mph, $\frac{dy}{dt} = -400$ mph, $x = 50$ and $y = 120$. $50, 120, 130$ is a pythagorean triple so $z = 130$.

Plugging in all the known quantities we see that

$$\frac{dz}{dt} = \frac{-15,000 - 48,000}{130} = -485$$

Hence, the distance between the planes is decreasing at a rate of 485 mph.

6. The edge of a cube is measured as 2 inches with an error of 1%. The cube's volume is to be computed from this measurement. Estimate the percentage error in the volume computation.

The volume of a cube is given by $V = s^3$. The percentage error is given by $\frac{dV}{V} \cdot 100$. $dV = 3s^2 ds$ and $\frac{dV}{V} = 3\frac{ds}{s}$. But $\frac{ds}{s} = .01$. Hence, $\frac{dV}{V} \cdot 100 = 3\%$

7. Find the absolute extrema of $f(x) = 1 - 2\sqrt[3]{x^2}$ on the interval $[-8, 1]$.

The extrema can occur at the endpoints or at the critical values. The critical values are found by taking the derivative and setting it to 0 or finding where it is undefined. $f'(x) = -\frac{4}{3\sqrt[3]{x}}$. f' is never 0 but 0 is a critical value as $f'(0)$ is undefined. Now compare $f(-8) = -7$, $f(0) = 1$ and $f(1) = -1$. We see that the maximum is 1 and the minimum is -7.

8. Use Rolle's Theorem to show that $f(x) = \sqrt{x(4-x)}$ has a horizontal tangent line on the interval $[0, 4]$. Then find the x value where f has a horizontal tangent line.

   Note that the domain of f is $[0, 4]$ and f is continuous on the whole domain and differentiable on $(0, 4)$, as $f'(x) = \dfrac{2-x}{\sqrt{x(4-x)}}$, which is not defined at 0 or 4. $f(0) = f(4) = 0$. As we have all of the hypotheses of Rolle's Theorem, we can conclude that $f'(c) = 0$ for some $c \in (0, 4)$. Observing the derivative above, we see that $c = 2$.

9. Find the intervals where $f(x) = x^3 - 2x^2 + 4$ is increasing.

   $f'(x) = 3x^2 - 4x = x(3x - 4)$. Hence the critical values of f are $x = 0$ and $x = \frac{4}{3}$. We break the number line into the following three intervals to test where the derivative is positive or negative.

$(-\infty, 0)$	$(0, \frac{4}{3})$	$(\frac{4}{3}, \infty)$
$f'(-1) = 7$	$f'(1) = -1$	$f'(2) = 4$
$+$	$-$	$+$

   Looking at the table we see that f is increasing on the intervals $(-\infty, 0]$ and $[\frac{4}{3}, \infty)$.

10. Find the x values where $f(x) = (x+1)^3(x-2)^2$ has critical points. Using the first derivative test, identify the critical values as either having a local maximum, a local minimum or neither.

$$f'(x) = 2(x+1)^3(x-2)+3(x+1)^2(x-2)^2 = (x+1)^2(x-2)[2(x+1)+3(x-2)] = (x+1)^2(x-2)(5x-4).$$

   We see the critical values of f are $x = -1, x = \frac{4}{5}, x = 2$. We break the number line into the following four intervals to test where the derivative is positive or negative.

$(-\infty, -1)$	$(-1, \frac{4}{5})$	$(\frac{4}{5}, 2)$	$(2, \infty)$
$f'(-2) = 56$	$f'(0) = 8$	$f'(1) = -4$	$f'(3) = 176$
$+$	$+$	$-$	$+$

   As there is no sign change at $x = -1$, f has neither a local maximum nor a local minimum at $x = -1$. As the sign changes from positive to negative at $x = \frac{4}{5}$, f has a local maximum at $x = \frac{4}{5}$. As the sign changes from negative to positive at $x = 2$, f has a local minimum at $x = 2$.

11. Find the intervals where $f(x) = 6x^2 - x^4$ is concave down.

   Concavity is determined by the second derivative. $f'(x) = 12x - 4x^3$ and $f''(x) = 12 - 12x^2$. The x values where the concavity may change are $x = -1, 1$. We break the number line into the following three intervals to test where the second derivative is positive or negative.

$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f''(-2) = -36$	$f''(0) = 12$	$f''(2) = -36$
$-$	$+$	$-$

   Hence, f is concave down on the intervals $(-\infty, -1]$ and $[1, \infty)$.