Practice Midterm 2 Solutions
MATH 9C

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1. Determine the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x+2)^n$

   We need to use the ratio test (or root test) to determine the radius of convergence. So we look at $\lim_{n\to\infty} \frac{\frac{1}{(n+1)3^{n+1}}|x+2|^{n+1}}{\frac{1}{n3^n}|x+2|^n} = \lim_{n\to\infty} \frac{n|x+2|}{3(n+1)} = \frac{|x+2|}{3}$. By the ratio test the power series converges when $\frac{|x+2|}{3} < 1$. Or when $|x+2| < 3$. Thus $f(x)$ converges on the interval $(-5, 1)$.

   We need to test convergence at the endpoints. Plugging in -5 into the power series we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(-5+2)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p series.

   Plugging in 1 into the power series, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(1+2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test as $\frac{1}{n} > 0$, $\frac{1}{n} > \frac{1}{n+1}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus the interval of convergence for the power series is (-5,1].

2. Determine the interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{2^n}{n!}(3x-1)^n$

   We need to use the ratio test (or root test) to determine the radius of convergence. So we look at $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}|3x-1|^{n+1}}{\frac{2^n}{n!}|3x-1|^n} = \lim_{n\to\infty} \frac{2|3x-1|}{(n+1)} = 0 < 1$. By the ratio test the power series converges for all real numbers x.

3. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence equal to 5, does $f(5)$ necessarily converge?

   If $f(x)$ has radius of convergence 5, we know that f converges on the open interval $(-5, 5)$. We can't determine if $f(5)$ converges or not. We need to know the behavior of $\sum_{n=0}^{\infty} a_n 5^n$. Not knowing a_n, we cannot say anymore.

4. Find the 3rd Taylor polynomial for $f(x) = \frac{1}{x}$ centered at $x = 2$.

   To find the 3rd Taylor polynomial we need to compute 3 derivatives of f and evaluate them at $x = 2$.

$f(x) = \frac{1}{x}$, $f(2) = \frac{1}{2}$
$f'(x) = -\frac{1}{x^2}$, $f'(2) = -\frac{1}{4}$
$f''(x) = \frac{2}{x^3}$, $f''(2) = \frac{1}{4}$
$f'''(x) = -\frac{6}{x^4}$, $f'''(2) = -\frac{3}{8}$.
Thus $P_3(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16}$.

5. Find the 2nd Taylor polynomial for $f(x) = \sin(2x)$ centered at $x = \frac{\pi}{3}$.

   To find the second Taylor polynomial at $x = \frac{\pi}{3}$, we need to compute two derivatives and evaluate them at $x = \frac{\pi}{3}$.

   $f(x) = \sin(2x)$, $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
   $f'(x) = 2\cos(2x)$, $f'(\frac{\pi}{3}) = -1$.
   $f''(x) = -4\sin(2x)$, $f''(\frac{\pi}{3}) = -2\sqrt{3}$.
   Thus $P_2(x) = \frac{\sqrt{3}}{2} - (x - \frac{\pi}{3}) - \sqrt{3}(x - \frac{\pi}{3})^2$.

6. Find the Taylor series for $f(x) = \ln x$ centered at $x = 1$.

   The Taylor series for $f(x) = \ln x$ centered at $x = 1$ is $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$. You can derive this as we derived the Taylor polynomials above, or you can use the Maclaurin series for $\ln(x+1)$ plugging in $(x-1)$ for x as $\ln(x-1+1) = \ln(x)$.

7. Find the Maclaurin series for $f(x) = \dfrac{\sin(2x)}{x}$.

   The Maclaurin series for $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. Thus $\dfrac{\sin(2x)}{x} = \dfrac{1}{x} \cdot \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n}$

8. Find the Maclaurin series for $f(x) = e^{x^2}$.

   The Maclaurin series for $e^x = \sum\limits_{n=0}^{\infty} \frac{1}{n!} x^n$. Thus $e^{x^2} = \sum\limits_{n=0}^{\infty} \frac{1}{n!} x^{2n}$.

9. Show that $y = xe^x$ is a solution to the differential equation $y' = \dfrac{(x+1)y}{x}$.

   $y' = xe^x + e^x = y + \frac{y}{x} = \frac{(x+1)y}{x}$. Thus $y = xe^x$ is a solution to the above differential equation.

10. Find the general solution to $y' = (y^2 + 1)\sec^2(x)$.

    This is a separable differential equation. Separating the variables we obtain $\frac{dy}{y^2+1} = \sec^2 x dx$. Integrating both sides we obtain, $\arctan y = \tan x + C$. Or $y = \tan(\tan x + C)$.

11. Find the solution to the initial value problem $xy' - y = x^2e^x$, $y(1) = 2$.

    This is a linear differential equation with $p(x) = -\frac{1}{x}$ and $q(x) = xe^x$. The integrating factor $v(x) = e^{-\int -\frac{1}{x}\, dx} = e^{\ln|x|} = x$. Now $y = ux$. To solve for u we look at $u'x = xe^x$ or $u' = e^x$. Thus $u = e^x + C$ and $y = xe^x + Cx$. To solve for C, $2 = e + C$ or $C = 2 - e$. Thus $y = xe^x + (2 - e)x$.

12. Find the general solution to $y' - 4y = e^{3x}$.

    This is a linear differential equation with $p(x) = -4$ and $q(x) = e^{3x}$. The integrating factor $v(x) = e^{-\int -4\, dx} = e^{4x}$ Now $y = ue^{4x}$. To solve for u we look at $u'e^{4x} = e^{3x}$ or $u' = e^{-x}$. Thus $u = -e^{-x} + C$ and $y = -e^{3x} + Ce^{4x}$.

13. A 20 gallon container contains 10 gallons of water and one half a pound of salt. Pure water is poured into the container at a rate of 2 gallons a minute. If the subsequent mixture leaves the container at a rate of 1 gallon a minute, set up a differential equation with initial conditions to solve for the amount of salt in the container at any time $0 < t < 10$.

    Let $Q(t)$ be the amount of salt in the container at any time $0 < t < 10$. The rate the salt is entering the container is 0 as there is no salt in pure water. The rate at which the salt is exiting the container is $\frac{Q(t)}{10+t}$ pounds per minute. Thus a differential equation to solve for $Q(t)$ is

    $Q'(t) = -\frac{Q(t)}{10+t}$ with initial condition $Q(0) = \frac{1}{2}$.

14. A population of bacteria grows at a rate proportional to the amount present. If the population doubles in two hours, find the general solution for the amount of bacteria present.

    Let $P(t)$ be the bacteria population at any time t. The word problem describes growth at a rate $P'(t) = kP(t)$. The solution to this differential equation is $P(t) = P(0)e^{kt}$.

15. A bicyclist is coasting down the road at a rate of 3 meters per second. If the bicyclist and his bicycle weigh 70 kg and the resistance is given by twice the speed he is traveling, find his velocity after 1 minute.

    We use Newton's second law $F = ma = m\frac{dv}{dt}$ and the resistive force is $F = -kv$ where $k = 2$. Thus $\frac{dv}{dt} = -\frac{2}{70}v$. Solving this differential equation, we obtain $v = v_0e^{-\frac{t}{35}}$. Thus $v(60) = 3e^{-\frac{60}{35}} = 3e^{-\frac{12}{7}}$.

16. Find the series solution to $y'' - y = 0$ with initial conditions $y'(0) = 0$ and $y(0) = 1$.

    Set $y = \sum_{n=0}^{\infty} a_nx^n$. Taking derivatives we obtain $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ and $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Thus $\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0$. Hence, $(n+2)(n+1)a_{n+2} - a_n = 0$ for $n \geq 0$. From the initial conditions we have $a_0 = 1$ and $a_1 = 0$. Using the recurrence relation above we see that $2a_2 - a_0 = 0$ or $a_2 = \frac{1}{2}$. $3 \cdot 2a_3 - a_1 = 0$ or $a_3 = 0$. $4 \cdot 3a_4 - a_2 = 0$ thus $a_4 = \frac{1}{4!}$. In general $a_{2n} = \frac{1}{(2n)!}$ and $a_{2n+1} = 0$. Hence $y = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$. This is equivalent to $y = \frac{e^x+e^{-x}}{2}$.

17. Use series to evaluate $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\sin x}$.

$$\lim_{x\to 0} \frac{\ln(1+x)}{\sin x} = \lim_{x\to 0} \frac{\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n}{\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}} = \lim_{x\to 0} \frac{x + \sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} x^n}{x + \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}} = \lim_{x\to 0} \frac{1 + \sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} x^{n-1}}{1 + \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}}.$$

The terms in both series are all 0; thus the limit is 1.