Practice Final
MATH 46

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1. Find the orthogonal trajectories of $y = cxe^x$

   $y' = c(x+1)e^x$. As $y = cxe^x$, $c = \dfrac{y}{xe^x}$. So we can rewrite $y' = \dfrac{y(x+1)}{x}$. Hence the orthogonal trajectories will have slope $y' = -\dfrac{x}{y(x+1)}$ or $yy' = -1 + \dfrac{1}{x+1}$. Integrating we obtain $\frac{y^2}{2} = -x + \ln|x+1| + C$ or $y = \pm\sqrt{2\ln|x+1| - 2x + C}$.

2. What is the Wronskian of the set$\{e^x, e^x \cos x\}$?

   $W = \det \begin{pmatrix} e^x & e^x \cos x \\ e^x & e^x(\cos x - \sin x) \end{pmatrix} = -e^{2x} \sin x.$

3. Find the solution to the initial value problem $y'' - 3y' - 40y = 0$ with $y(0) = 3$, $y'(0) = -2$.

   The characteristic polynomial is $r^2 - 3r - 40 = 0$ and the roots are $r = 8$ and $r = -5$. The general solution has the form $y = c_1e^{8x} + c_2e^{-5x}$. Using the initial conditions $y(0) = 3$ and $y'(0) = -2$ we get the system

   $$c_1 + c_2 = 3$$
   $$8c_1 - 5c_2 = -2.$$

   The solution to the system is $c_1 = 1$ and $c_2 = 2$. So the solution to the differential equation is $y = e^{8x} + 2e^{-5x}$.

4. Find the particular solution of $y'' + 9y = 1 + x^2$.

   As $F(x) = 1 + x^2$, $y_p = A + Bx + Cx^2$. Taking derivatives we get $y_p' = B + 2Cx$ and $y_p'' = 2C$. Plugging in y_p'' and y_p into the differential equation we obtain $2C + 9A + 9B + 9Cx^2 = 1 + x^2$. Setting the coefficients of x^2 equal, we see that $C = \frac{1}{9}$. Setting the coefficients of x equal, we see $B = 0$ and setting the coefficient of one on both sides equal, we see that $A = \frac{7}{81}$. Thus $y_p = \frac{7}{81} + \frac{1}{9}x^2$.

5. Find the particular solution of $y'' - y' - 6y = 4e^{3x}$.

   As 3 is a root of the characteristic polynomial $r^2 - r - 6 = 0$, $y_p = Axe^{3x}$. Taking derivatives, we get $y_p' = (3Ax + A)e^{3x}$ and $y_p'' = (9Ax + 6A)e^{3x}$. Plugging y_p'', y_p' and y_p back into the differential equation and simplifying we get $5Ae^{3x} = 4e^{3x}$ implying $A = \frac{4}{5}$. So $y_p = \frac{4}{5}xe^{3x}$.

6. Find the particular solution of $y'' - y = 2\sin x - \cos x$.

   As i is not a root to the characteristic polynomial $r^2 - 1 = 0$, we guess that $y_p = A\cos x + B\sin x$. Taking derivatives we obtain $y_p' = B\cos x - A\sin x$ and $y_p'' = -A\cos x - B\sin x$. Substituting these back into the differential equation we get $-2A\cos x - 2B\sin x = 2\sin x - \cos x$ implying $A = \frac{1}{2}$ and $B = -1$. So $y_p = \frac{1}{2}\cos x - \sin x$.

7. Find the general solution of $x^2y'' - xy' - 3y = 0$ if $y_1 = x^3$.

   We will use reduction of order. $y = ux^3$ so $y' = u'x^3 + 3x^2u$ and $y'' = u''x^3 + 6x^2u' + 6xu$. Plugging these into the differential equation, we obtain $x^5u'' + 5x^4u' = 0$. Dividing both sides by x^5 and setting $u' = z$ we obtain the differential equation $z' + \frac{5}{x}z = 0$. Integrating, we obtain $z = \frac{c_1}{x^5}$, but $u' = z$ so integrating again we obtain $u = \frac{c_1}{x^4} + c_2$. Now, $y = \frac{c_1}{x} + c_2x^3$.

8. Find the particular solution of $4x^2y'' + 4xy' - y = x^3$ if $y_1 = \sqrt{x}$ and $y_2 = \dfrac{1}{\sqrt{x}}$.

   As we know y_1 and y_2, we will use variation of parameters, so $y_p = u_1\sqrt{x} + u_2\frac{1}{\sqrt{x}}$. Next we use the system

$$u_1'\sqrt{x} + u_2'\frac{1}{\sqrt{x}} = 0$$

$$u_1'\frac{1}{2\sqrt{x}} - u_2'\frac{1}{2\sqrt{x^3}} = \frac{x}{4}$$

   to solve for u_1 and u_2. Plugging $u_1' = -\frac{1}{x}u_2'$ into the second equation, we obtain $-\frac{1}{\sqrt{x^3}}u_2' = \frac{x}{4}$ or $u_2' = -\frac{\sqrt{x^5}}{4}$, implying $u_2 = -\frac{1}{14}x^{\frac{7}{2}}$. Plugging $u_2' = -xu_1'$ into the second equation, we obtain $\frac{1}{\sqrt{x}}u_1' = \frac{x}{4}$ or $u_1' = \frac{\sqrt{x^3}}{4}$, implying $u_1 = \frac{1}{10}x^{\frac{5}{2}}$.

   Thus $y_p = \frac{1}{35}x^3$.

9. A 4 pound weight stretches a spring 3 inches into equilibrium. If the initial displacement of the spring is 2 inches above the equilibrium and released with a downward velocity of 1 ft/sec, what is the amplitude, the frequency, the period and the phase shift of the system.

   The spring above is not damped and there is no external force so the relevant differential equation is $y'' + \frac{k}{m}y = 0$, with $\frac{k}{m} = \frac{g}{\Delta l} = \frac{32}{.25} = 128$. So $y'' + 128y = 0$. The root of the characteristic polynomial $r^2 + 128 = 0$ are $r = \pm 8\sqrt{2}i$ so $y = c_1\cos(8\sqrt{2}t) + c_2\sin(8\sqrt{2}t)$ is the general solution. Using the initial conditions, we see that $c_1 = \frac{1}{6}$ and $c_2 = -\frac{1}{8\sqrt{2}}$.

   The amplitude is $R = \sqrt{\frac{1}{36} + \frac{1}{128}} = \frac{1}{24}\sqrt{\frac{17}{2}}$.

   The frequency is $\omega = 8\sqrt{2}$.

   The period is $T = \frac{2\pi}{\omega} = \frac{\pi}{4\sqrt{2}}$.

   The phase shift angle is $\phi = \arctan(-\frac{3}{4\sqrt{2}})$.

10. A 2 kg mass is attached to a spring with spring constant 12 N/m. If the spring is attached to a dashpot with damping constant 18 N-s/m and subjected to an external force of $F = 2\cos(2t)$ Find the steady state component of the displacement of the spring.

This spring has damping and an external force, so the differential equation you would use to find the displacement is $my'' + cy' + ky = F(t)$. Reading the constants from above we have $2y'' + 18y' + 12y = 2\cos(2t)$. To find the steady state solution, we need to find y_p. $y_p = A\cos(2t) + B\sin(2t)$ and has derivatives $y_p' = 2B\cos(2t) - 2A\sin(2t)$ and $y_p'' = -4A\cos(2t) - 4B\sin(2t)$. Plugging these back into the differential equation we get the system

$$4B - 36A = 0$$

$$36B + 12A = 2$$

which has solution $A = \frac{1}{164}$ and $B = \frac{9}{164}$. Hence, $y_p = \frac{1}{164}\cos(2t) + \frac{9}{164}\sin(2t)$.

11. Set up a differential equation to find the current in the RLC circuit if $R = 2$ ohms, $L = .05$ henrys, $C = .025$ farads and $Q_0 = 0$ coulombs and $I_0 = 2$ amperes.

Using the model $LQ'' + RQ' + \frac{1}{C}Q = 0$, we get the differential equation

$$\frac{1}{20}Q'' + 2Q' + 40Q = 0.$$

12. Find the inverse Laplace transform of $\dfrac{s^2 + 2s}{(s+1)(s+3)(s+5)}$.

$\dfrac{s^2+2s}{(s+1)(s+3)(s+5)} = \dfrac{A}{s+1} + \dfrac{B}{s+3} + \dfrac{C}{s+5}$. To solve for the constants we note that $s^2 + 2s = A(s+3)(s+5) + B(s+1)(s+5) + C(s+1)(s+3)$ and we plug in $s = -1$ to get $A = -\frac{1}{8}$, $s = -3$ to get $B = -\frac{3}{4}$ and $s = -5$ to get $C = \frac{15}{8}$. So $\mathcal{L}^{-1}\left(\dfrac{s^2+2s}{(s+1)(s+3)(s+5)}\right) = \mathcal{L}^{-1}\left(-\dfrac{\frac{1}{8}}{s+1} - \dfrac{\frac{3}{4}}{s+3} + \dfrac{\frac{15}{8}}{s+5}\right)$
$= -\frac{1}{8}e^{-t} - \frac{3}{4}e^{-3t} + \frac{15}{8}e^{-5t}$.

13. Use the Laplace transform to solve the initial value problem $y'' - 4y' - 5y = e^{-t}$ given $y(0) = 2$ and $y'(0) = -4$.

We will use $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$ and $\mathcal{L}(y'') = s^2\mathcal{L}(y') - y'(0) - sy(0)$. Take the Laplace transform of both sides and use the above to obtain $s^2\mathcal{L}(y) + 4 - 2s - 4s\mathcal{L}(y) + 8 - 5\mathcal{L}(y) = \frac{1}{s+1}$ i.e. $(s^2 - 4s - 5)\mathcal{L}(y) = \frac{1}{s+1} + 2s - 12$ or

$$\mathcal{L}(y) = \frac{(s+1)(2s-12)+1}{(s+1)^2(s-5)} = -\frac{1}{6}\frac{1}{(s+1)^2} + \frac{83}{36}\frac{1}{s+1} - \frac{11}{36}\frac{1}{s-5}$$

Taking the inverse Laplace Transform we obtain $y = -\frac{1}{6}te^{-t} + \frac{83}{36}e^{-t} - \frac{11}{36}e^{5t}$.

14. What is the order of the following differential equation: $x^4\dfrac{d^3y}{dx^3} - x^2\dfrac{dy}{dx} + y = y^3$?

The order is third order.

15. Give a rough sketch of the solution of $y' = \dfrac{3xy}{1+x^2+y^2}$ with initial condition $y(0) = 1$? (Hint: Draw a direction field and sketch the integral curve containing the above point.)

The solution will be posted outside my office.

16. Does $y' = \sqrt[3]{x+y}$, $y(0) = 1$ have a unique solution on the open rectangle $R = \{(x,y) | -2 \leq x \leq 2, 0 \leq y \leq 4\}$?

$f(x,y) = \sqrt[3]{x+y}$ is continuous everywhere on the xy-plane, but $f_y(x,y) = \dfrac{1}{3\sqrt[3]{(x+y)^2}}$ is not continous on $y = -x$. $y = -x$ has nontrivial intersection with R so by the Existence and Uniqueness Theorem, there is not a unique solution at $(1,0)$.

17. Solve the initial value problem $x^2y' - xy = x + 1$, $y(1) = 2$.

Dividing by x^2, we obtain $y' - \frac{1}{x}y = \frac{x+1}{x^2}$. $y_1 = x$ and $y = ux$ with $u' = \frac{x+1}{x^3} = \frac{1}{x^2} + \frac{1}{x^3}$. Integrating we obtain $u = -\frac{1}{x} - \frac{1}{2x^2} + C$ so $y = -1 - \frac{1}{2x} + Cx$. Now using the initial condition we obtain $2 = -1 - \frac{1}{2} + C$ or $C = \frac{7}{2}$. So $y = -1 - \frac{1}{2x} + \frac{7x}{2}$.

18. Solve $y' = \dfrac{x^2(y+1)}{x^3+1}$.

This is a separable differential equation so separating the variables we obtain $\dfrac{y'}{y+1} = \dfrac{x^2}{x^3+1}$. Now integrating both sides we get $\ln|y+1| = \frac{1}{3}\ln|x^3+1| + C$ or $y = -1 + C\sqrt[3]{x^3+1}$.

19. Solve $y' + y = e^x y^4$.

$y_1 = e^{-x}$ and $y = ue^{-x}$ with $\dfrac{u'}{u^4} = e^x \cdot e^{-3x} = e^{-2x}$. Integrating we get $-\frac{1}{3u^3} = -\frac{1}{2}e^{-2x} + C$ or $u = \dfrac{1}{\sqrt[3]{C + \frac{3}{2}e^{-2x}}}$. Thus $y = \dfrac{e^{-x}}{\sqrt[3]{C + \frac{3}{2}e^{-2x}}}$.

20. Solve $y' = \dfrac{x^2 + xy + 4y^2}{x^2}$.

This is a homogeneous equation. Set $u = \frac{y}{x}$ and $u'x + u = y'$. Hence $u'x + u = 1 + u + 4u^2$ or $\dfrac{u'}{1+4u^2} = \dfrac{1}{x}$. Integrating both sides, we get $\frac{1}{2}\arctan 2u = \ln|x| + C$ or $u = \frac{1}{2}\tan(2\ln|x| + C)$ and $y = x\tan(2\ln|x| + C)$.

21. Find a solution to the exact equation $(x^2 + 4xy + y^2)dx + (2xy + 2x^2 + y^3)dy = 0$.

$G(x,y) = \frac{x^3}{3} + 2x^2y + y^2x$. Differentiating with respect to y we get $G_y(x,y) = 2x^2 + 2xy$. Now $\phi'(y) = 2xy + 2x^2 + y^3 - (2x^2 + 2xy) = y^3$. So $\phi(y) = \frac{1}{4}y^4$ and $F(x,y) = G(x,y) + \phi(y)$ so the solution is $\frac{x^3}{3} + 2x^2y + y^2x + \frac{1}{4}y^4 = C$.

22. Show that $\mu = \dfrac{1}{x^2y^2}$ is an integrating factor for $y^2dx + x^2dy = 0$.

$(\mu M)_y = 0 = (\mu N)_x$ so μ is an integrating factor as it makes the differential form exact.

23. If a population of algae in a pool changes at a rate which is proportional to the square root of the amount present, write a differential equation to find $Q(t)$, the amount of algae in the pool.

$Q'(t) = k\sqrt{Q(t)}$.

24. The money in a bank account grows at a rate proportional to the amount present. If twice the original quantity Q_0 is in the bank account after 10 years, in how many years will you have three times the original amount?

$2 = e^{r10}$ or $r = \frac{\ln(2)}{10}$. $3 = e^{\frac{t\ln(2)}{10}}$ or $t = \frac{10\ln(3)}{\ln(2)}$ years.

25. A 50 gallon tank holds 20 gallons of a saline solution containing 4 pounds of salt. Pure water is poured into a 10 gallon tank at a rate of 6 gal/min. and the solution leaves the tank at a rate of 3 gal/min. Find a differential equation to express the amount of salt in the tank.

We need to find a differential equation for $Q'(t) =$ Rate in- Rate out. Rate in $= 0$ as there is no salt in the water being poured into the tank. Rate out $= \dfrac{3Q(t)}{20+3t}$. So $Q'(t) = -\dfrac{3Q(t)}{20+3t}$ with $Q(0) = 4$.

26. An object with mass 256 pounds is launched vertically upward with initial velocity of 72 ft/s in a medium which exerts a resistive force with magnitude proportional to the speed. If the resistive force is 5 pounds when the speed is 2 ft/s, write a differential equation to find the velocity of the object.

Use $mv' = -mg - kv$, $m = \frac{256}{32} = 8$ and $2k = 5$, so $k = \frac{5}{2}$. So the differential equation is $8v' = -256 - \frac{5}{2}v$ with initial condition $v(0) = 72$.