Practice Final
MATH 9A

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1. What are the dimensions in inches of the largest rectangle which can be inscribed inside the ellipse $\dfrac{x^2}{8}+\dfrac{y^2}{9}=1$?

   As the rectangle must be inscribed in the ellipse $\dfrac{x^2}{8}+\dfrac{y^2}{9}=1$, the base of the rectangle must be of length $2x$, and the height must be $2y$, where (x,y) is a point on the ellipse. (Note the vertices of the rectangle will be (x,y), $(-x,y)$, $(-x,-y)$ and $(x,-y)$.)

   We will solve for y so we get a formula for the area strictly in terms of x.

   $$y=\sqrt{9-\frac{9x^2}{8}}.$$

   So

   $$A=4x\sqrt{9-\frac{9x^2}{8}}.$$

   The domain of A is $[0,\sqrt{8}]$. We know the maximum must occur either at the endpoints or at a critical value.

   $$A'(x)=-\frac{9x^2}{2\sqrt{9-\frac{9x^2}{8}}}+4\sqrt{9-\frac{9x^2}{8}}=\frac{72-18x^2}{2\sqrt{9-\frac{9x^2}{8}}}.$$

   Setting the top to 0, we get the critical values $x=2,-2$. But the only one in the domain is $x=2$. Setting the bottom to 0, we get the critical values $x=\sqrt{8},-\sqrt{8}$. $x=\sqrt{8}$ is the only one in the domain and it is already an endpoint.

   Comparing $A(0)=0$, $A(\sqrt{8})=0$ and $A(2)=12\sqrt{2}$, we see there is a maximum when $x=2$. Hence, the dimensions of the rectangle with a maximum area are 4 inches by $3\sqrt{2}$ inches.

   You may also solve this using the first derivative test.

2. The surface area of a sphere is changing at a rate of 25 square millimeters per second. At what rate is the volume of the sphere changing when the radius is 10 millimeters? (Hint: The surface area and volume of a sphere are given by the formulas $SA=4\pi r^2$ and $V=\frac{4}{3}\pi r^3$.)

   We know $\frac{d(SA)}{dt}=25$ mm^2/sec. Using the surface area formula, we see $\frac{d(SA)}{dt}=8\pi r\frac{dr}{dt}$. When $r=10$ mm., we see that $\frac{dr}{dt}=\frac{25}{80\pi}$ mm/sec.

Now using the volume formula, we see that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.
Hence, when $r = 10$ mm., $\frac{dV}{dt} = 125$ mm^3/sec.

3. The diameter of a tree was 12 inches. During the following year the circumference increased 1 inch. About how much did the tree's cross sectional area increase?

Let the diameter be denoted D. Note the circumference is $C = \pi D$ and the area is $A = \pi D^2/4$. When $D = 12$, we know $dC = 1$. $dC = \pi dD$, so $dD = \frac{1}{\pi}$. Now, $dA = \frac{\pi D}{2} dD$. So $dA = 6\pi \cdot \frac{1}{\pi} = 6$. So the approximate increase in cross sectional area is 6 in^2.

4. It took 10 seconds for a mercury thermometer to rise from 0^oF to 100^oF in boiling water. Show that somewhere along the way the mercury was rising at the rate of 10^oF/sec.

We will denote the temperature function by $T(t)$. $T(10) = 100$ and $T(0) = 0$. Using the Mean Value Theorem, there exists a $c \in (0, 10)$ with $T'(c) = \dfrac{T(10) - T(0)}{10 - 0} = \dfrac{100}{10} = 10$. So there is some time between 0 and 10 seconds where the temperature is rising at a rate of 10^oF/sec.

5. A rocket lifts off the surface of the earth with a constant acceleration of 25 m/sec^2. How fast will the rocket be going after 10 seconds?

Velocity is the antiderivative of acceleration. Thus $v = \int 25dt = 25t + C$. The initial velocity of the rocket was 0 thus $v = 25t$ and the rocket will be traveling at a speed of 250 m/sec after 10 seconds.

6. Find the intervals where the function $f(x) = 3x^2 - 8x^3$ is increasing.

We determine where a function is increasing by analyzing the first derivative. The first derivative of f is $f'(x) = 6x - 24x^2 = 6x(1 - 4x)$. The critical values of f are $x = 0$ and $x = \frac{1}{4}$. Consider the following table

$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
$f'(-1) = -30 < 0$ decreasing	$f'(\frac{1}{8}) = \frac{3}{8} > 0$ increasing	$f'(1) = -18 < 0$ decreasing

Hence, f is increasing on the interval $(0, \frac{1}{4})$.

7. Identify the critical values of $f(x) = \dfrac{x}{4 + x^2}$ as either local maxima, local minima or neither using the first derivative test.

The critical values are the x values where the first derivative is 0 or undefined. $f'(x) = \dfrac{4 + x^2 - 2x^2}{(4 + x^2)^2} = \dfrac{4 - x^2}{(4 + x^2)^2}$. $f'(x)$ is defined for all x but is 0 when $x = -2, 2$. Consider the following table:

$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(-3) = -\frac{5}{169}$ -	$f'(0) = \frac{1}{4}$ +	$f'(3) = -\frac{5}{169}$ -

Now by the first derivative test, f has a local minimum at $x = -2$ and a local maximum at $x = 2$.

8. Find $\lim_{x\to 0} \frac{x-\sin x}{x^2}$.

$\lim_{x\to 0} \frac{x-\sin x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{1-\cos x}{2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sin x}{2} = 0$. Everywhere the equals sign is indicated with the indeterminate form $\frac{0}{0}$ we have applied l'Hopital's rule.

9. Find the antiderivative $F(x)$ of $f(x) = x\sqrt{x^2-9}$ satisfying $F(5) = 3$.

$F(x) = \int x\sqrt{x^2-9}dx$. We must use the substitution $u = x^2 - 9$ and the differential is $du = 2xdx$. Rewrite

$$\int x\sqrt{x^2-9}dx = \frac{1}{2}\int \sqrt{x^2-9}\,2xdx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}\sqrt{u^3}+C = \frac{1}{3}\sqrt{(x^2-9)^3}+C.$$

To solve for C plug in 5. We get $3 = \frac{1}{3}\sqrt{16^3} + C$ and $C = -\frac{55}{3}$. So $F(x) = \frac{1}{3}\sqrt{(x^2-9)^3} - \frac{55}{3}$.

10. Give a rough sketch of f given that $f(x)$ is increasing on $(-\infty,-1)$ and $(3,5)$, decreasing on $(-1,3)$ and $(5,\infty)$, concave up on $(0,4)$ and concave down on $(-\infty,0)$ and $(4,\infty)$ and $f(-1) = 2$, $f(0) = 0$, $f(3) = -2$, $f(4) = 1$ and $f(5) = 4$.

A rough sketch of f will be posted on my door, Surge 276.

11. Find the tangent line to $f(x) = \tan(2x) + (x+1)\cos x$ at the point $(0,1)$.

To find the tangent line we need to determine the slope given by the derivative at $x = 0$. $f'(x) = 2\sec^2(2x) - (x+1)\sin x + \cos x$. $f'(0) = 2+1 = 3$. Hence, the tangent line is $y = 3x + 1$.

12. Using the definition of the limit, show that $\lim_{x\to 2} 7 - 3x = 1$.

By the definition, the limit of $f(x)$ as x approaches c is L if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$.

So starting with

$|7 - 3x - 1| < \epsilon$ for ϵ an arbitrary positive number, we manipulate the left hand side to see that

$3|x-2| < \epsilon$ or $|x-2| < \frac{\epsilon}{3}$.

Setting $\delta = \frac{\epsilon}{3}$, we see that if x is chosen such that $|x-2| < \delta$ we are assured that $|7-3x-1| < \epsilon$. Thus $\lim_{x\to 2} 7 - 3x = 1$ by the definition.

13. For what value of a will $f(x) = \begin{cases} x+a, & x > 2; \\ x^2+2x-3, & x \le 2 \end{cases}$ be continuous.

$f(x)$ is continuous everywhere except possibly at $x = 2$ as $x + a$ and $x^2 + 2x - 3$ are both continuous functions. If $f(x)$ is continuous then $\lim_{x\to 2} f(x) = f(2)$.

Note $f(2) = 5$. As $\lim_{x\to 2^-} x^2 + 2x - 3 = 5$, we need only look at $\lim_{x\to 2^+} x + a = 2 + a$.

Setting $5 = 2 + a$, we see that a must be 3 for f to be continuous.

14. Suppose f and g are continuous functions with $f(4) = 2$, $g(4) = -1$, $f'(4) = \frac{3}{2}$ and $g'(4) = 7$. Compute the following:

   (a) $\left(\frac{f}{g}\right)'(4)$

   (b) $(5f + fg)'(4)$.

   (a) $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$. Now plug in 4 to get
   $\left(\frac{f}{g}\right)'(x) = \dfrac{-\frac{3}{2} - 14}{1} = -\dfrac{31}{2}$.

   (b) $(5f + fg)'(x) = 5f'(x) + f(x)g'(x) + g(x)f'(x)$. Plugging in 4 we see that $(5f + fg)'(4) = \frac{15}{2} + 14 - \frac{3}{2} = 20$.

15. If $\lim_{x\to 3^+} f(x) = -1$ and $\lim_{x\to -3^+} f(x) = 4$, could $f(x)$ be an odd function? Explain your answer.

   Yes, $f(x)$ could be an odd function if

   1) $f(-x) = -f(x)$,

   2) $\lim_{x\to 3^+} f(x) = -\lim_{x\to -3^-} f(x) = -1$ and

   3) $\lim_{x\to -3^+} f(x) = -\lim_{x\to 3^-} f(x) = 4$.

   Note as x approaches -3 from the right, $|x| = -x$ approaches 3 from the left. Also as x approaches -3 from the left, $|x| = -x$ approaches 3 from the right. If the above three conditions do not hold, then $f(x)$ is not odd.