

# Practice Final

## MATH 9B

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1. Write a left sum to approximate the area under  $y = x \sin x$  on the interval  $[0, \pi]$  with  $n = 4$  equal subintervals.

$$LS = 0\left(\frac{\pi}{4}\right) + \frac{\pi\sqrt{2}}{8}\left(\frac{\pi}{4}\right) + \frac{\pi}{2}\left(\frac{\pi}{4}\right) + \frac{3\pi\sqrt{2}}{8}\left(\frac{\pi}{4}\right) = \frac{\pi^2(\sqrt{2}+1)}{8}.$$

2. You and a companion are sailing with a device that records the speed at any given time. Your friend records the speed at 1 minute intervals for 5 minutes. The table below gives the speeds.

Time (in minutes)	Speed (in miles per hour)
0	60
1	55
2	65
3	45
4	50
5	60

What is an lower estimate for the distance you and your friend have traveled in that time?

$$LS = 55\left(\frac{1}{60}\right) + 55\left(\frac{1}{60}\right) + 45\left(\frac{1}{60}\right) + 45\left(\frac{1}{60}\right) + 50\left(\frac{1}{60}\right) = \frac{25}{6}$$

3. Write  $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n e^{2c_k} \Delta x_k$ , where  $P$  is a partition of  $[0, 2]$ , as a definite integral.

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n e^{2c_k} \Delta x_k = \int_0^2 e^{2x} dx$$

4. Find the area under  $f(x) = \ln x$  on the interval  $[1, 4]$

$$A = \int_1^4 \ln x dx. \text{ This is an integral requiring integration by parts with } u = \ln x, \\ (du = \frac{1}{x} dx) \text{ and } dv = dx \text{ (} v = x \text{)}. \text{ Thus } A = x \ln x \Big|_1^4 - \int_1^4 dx = x \ln x - x \Big|_1^4 = \\ 4 \ln 4 - 4 - (0 - 1) = 4 \ln 4 - 3$$

5. Find  $\frac{d}{dx} \int_0^{\tan^{-1} x} \frac{dt}{1 - \tan t}$

$$\frac{d}{dx} \int_0^{\tan^{-1} x} \frac{dt}{1 - \tan t} = \frac{1}{1 - \tan(\tan^{-1} x)} \cdot \frac{1}{1 + x^2} = \frac{1}{(1 - x)(1 + x^2)}$$

6. Find the average value of  $f(x) = \frac{1}{\sqrt{4-x^2}}$  on  $[0, 1]$

$$\text{avg}(f) = \frac{1}{1-0} \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1 = \frac{\pi}{6}$$

7. Find the area between  $f(y) = \frac{1}{y}$  and  $g(y) = 1 - y^2$  on the interval  $[1, 3]$ .

$$f(y) \geq g(y) \text{ on the interval } [1, 3]. \text{ Thus } A = \int_1^3 \left(\frac{1}{y} - (1 - y^2)\right) dy = \ln y - y + \frac{y^3}{3}\Big|_1^3 = \ln 3 - 3 + 9 - \left(0 - 1 + \frac{1}{3}\right) = \ln 3 + \frac{20}{3}.$$

8. Find the area enclosed by  $f(x) = x^2 - 2x$  and  $g(x) = x + 10$ .

We first need to find the points of intersection. Set  $x^2 - 2x = x + 10$  or  $x^2 - 3x - 10 = 0$  or  $x = -2, 5$ .  $x + 10$  lies above  $x^2 - 2x$ , so  $A = \int_{-2}^5 (x + 10 - (x^2 - 2x)) dx = \int_{-2}^5 (3x + 10 - x^2) dx = \frac{3}{2}x^2 + 10x - \frac{1}{3}x^3\Big|_{-2}^5 = \frac{75}{2} + 50 - \frac{125}{3} - \left(3 \cdot \frac{4}{2} - 20 + \frac{8}{3}\right) = \frac{343}{6}$ .

9. Evaluate  $\int_{-1}^1 (2 + 3x)^3 dx$

Using  $u = 2 + 3x$ ,  $du = 3dx$   $u(-1) = -1$ ,  $u(1) = 5$ ,  $\int_{-1}^1 (2 + 3x)^3 dx = \frac{1}{3} \int_{-1}^5 u^3 du = \frac{1}{12} u^4\Big|_{-1}^5 = \frac{625}{12} - \frac{1}{12} = 52$

10. Given  $\int_0^4 f(x) dx = -7$  and  $\int_4^5 f(x) dx = -2$ , what is  $\int_0^5 2f(x) dx$ ?

$$\int_0^5 2f(x) dx = 2(-7) + 2(-2) = -18$$

11. Set up the integral to find the volume of the solid given by revolving the region bounded by  $y = e^x$ , the  $y$ -axis, the  $x$ -axis and  $x = 2$  about the line  $x = 2$ .

As  $x = 2$  is a vertical line, the method to use is shells.  $h(x) = e^x$ ,  $r(x) = 2 - x$  and  $V = \int_0^2 2\pi(2 - x)e^x dx$ .

12. Set up the integral to find the volume of the solid given by revolving the region bounded by  $y = x$  and  $y = \sqrt{x}$  about the line  $y = 1$ .

As  $y = 1$  is a horizontal line, the method to use is washers. The outer radius is  $R(x) = 1 - x$  and the inner radius is  $r(x) = 1 - \sqrt{x}$ . So

$$V = \int_0^1 \pi[(1 - x)^2 - (1 - \sqrt{x})^2] dx = \int_0^1 \pi(x^2 - 3x - 2\sqrt{x}) dx$$

13. Set up the integral to find the volume of the solid given by revolving the region bounded by  $x = \tan^{-1} y$ ,  $x = \frac{\pi}{4}$  and the  $x$ -axis about the  $y$ -axis.

As the  $y$ -axis is a vertical line and the function is given in terms of  $y$ , we can use washers. The outer radius is  $R(y) = \frac{\pi}{4}$  and the inner radius is  $r(y) = \tan^{-1} y$ . Thus

$$V = \int_0^1 \pi \left( \frac{\pi^2}{16} - (\tan^{-1} y)^2 \right) dy.$$

14. Find the length of the curve  $y = \cosh t$  on the interval  $[0, 1]$ .

$$y' = \sinh t, \text{ Thus } L = \int_0^1 \sqrt{1 + \sinh^2 t} dt = \int_0^1 \sqrt{\cosh^2 t} dt = \int_0^1 \cosh t dt = \sinh t \Big|_0^1 = \frac{e - e^{-1}}{2}.$$

15. Set up the integrals to find the center of mass for the lamina of density 1 given by the region bounded by  $y = x^2 - x$  and  $y = 2$ .

$$m = \int_{-1}^2 2 - (x^2 - x) dx$$

$$M_y = \int_{-1}^2 x(2 - (x^2 - x)) dx$$

$$M_x = \int_{-1}^2 4 - (x^2 - x)^2 dx$$

with the center of mass given by  $(\frac{M_y}{m}, \frac{M_x}{m})$ .

16. Set up the integral to find the surface area of the barrel given by revolving the curve  $y = \sin x$  on the interval  $[\frac{\pi}{4}, \frac{3\pi}{4}]$  about the  $x$ -axis.

$$y' = \cos x \text{ and so } SA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\pi \sin x \sqrt{1 + \cos^2 x} dx.$$

17. Find the work done in stretching a spring an additional foot if a force of 10 pounds is used to stretch the spring one foot.

$$10 = k(1) \text{ so } k = 10, W = \int_1^2 10x dx = 5x^2 \Big|_1^2 = 20 - 5 = 15 \text{ foot-pounds.}$$

18. Set up the integral to find the work done in filling an empty cylindrical water tank with radius 5 feet and height 8 feet which is 10 feet above the ground from a source at ground level with water of weight 62.4 lb/ft<sup>3</sup>.

$$W = \int_0^8 (62.4)25\pi(y + 10) dy$$

19. If the half life of a radioactive element is 1500 years, set up an equation to find the amount of this radioactive element in a storage site which contains 100 kg of the element now for any time  $t$ .

$\frac{1}{2} = e^{1500k}$ . Thus  $k = -\frac{\ln 2}{1500}$ . Hence  $A = 100e^{-\frac{t \ln 2}{1500}}$  will give an equation for the amount of this radioactive element present for any time  $t$ .

20. Evaluate the following integrals:

(a)  $\int x^2 e^{x^3} dx$

This integral requires the substitution  $u = x^3$ ,  $du = 3x^2 dx$ .  $\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{e^u}{3} + C = \frac{e^{x^3}}{3} + C$ .

(b)  $\int \frac{\ln x}{x^2} dx$

This integral requires integration by parts.  $u = \ln x$  ( $du = \frac{1}{x} dx$ ) and  $dv = \frac{1}{x^2} dx$  ( $v = -\frac{1}{x}$ ). Thus  $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$ .

(c)  $\int \sqrt{1 - \cos t} dt$

This integral requires multiplication by 1 in the form of  $\frac{\sqrt{1+\cos t}}{\sqrt{1+\cos t}}$ .  $\int \sqrt{1 - \cos t} dt = \int \frac{\sqrt{1 - \cos^2 t}}{\sqrt{1 + \cos t}} dt = \int \frac{\sqrt{\sin^2 t}}{\sqrt{1 + \cos t}} dt = \int \frac{|\sin t|}{\sqrt{1 + \cos t}} dt$ . From here do substitution  $u = 1 + \cos t$ ,  $du = -\sin t dt$ .  $\int \frac{|\sin t|}{\sqrt{1 + \cos t}} dt = \mp \int \frac{1}{\sqrt{u}} du = \mp 2\sqrt{u} + C = \mp 2\sqrt{1 + \cos t} + C$ .

(d)  $\int \frac{4 + 2x}{9 + x^2} dx$

$$\int \frac{4 + 2x}{9 + x^2} dx = \int \frac{4}{9 + x^2} + \frac{2x}{9 + x^2} dx = \frac{4}{3} \tan^{-1}\left(\frac{x}{3}\right) + \ln(9 + x^2) + C$$

(e)  $\int \sec^3 2x \tan 2x dx$

This is a trig integral. Split of a  $\sec 2x$  and the integral becomes  $\int \sec^2 2x \sec 2x \tan 2x dx$ .

Now let  $u = \sec 2x$ ,  $du = 2 \sec 2x \tan 2x dx$ . Thus,  $\int \sec^2 2x \sec 2x \tan 2x dx = \frac{1}{2} \int u^2 du = \frac{1}{6} u^3 + C = \frac{1}{6} \sec^3(2x) + C$ .

(f)  $\int \frac{2x + 1}{x(x^2 + 1)} dx$

This integral is best done with partial fractions.

$$\frac{2x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Now get a common denominator and equate the numerators.  $2x + 1 = A(x^2 + 1) + Bx^2 + Cx$ . Now equate the coefficients of  $x^2$ ,  $x$  and 1 on both sides.  $A + B = 0$ ,  $C = 2$  and  $A = 1$ , and  $B = -1$ . So

$$\int \frac{2x + 1}{x(x^2 + 1)} dx = \int \frac{1}{x} + \frac{-x + 2}{x^2 + 1} dx = \ln|x| - \frac{1}{2} \ln(x^2 + 1) + 2 \tan^{-1} x + C$$

(g)  $\int \frac{2x + 1}{x^2 - 6x + 8} dx$

This is also an integral requiring partial fractions.

$$\frac{2x + 1}{x^2 - 6x + 8} = \frac{A}{x - 2} + \frac{B}{x - 4}$$

Now get a common denominator and equate the numerators.  $2x + 1 = A(x - 4) + B(x - 2)$ . Using the modified heaviside method, plug in 4 to get  $B = \frac{9}{2}$  and plug in 2 to get  $A = -\frac{5}{2}$ .

$$\text{Thus } \int \frac{2x + 1}{x^2 - 6x + 8} dx = \int -\frac{5}{2(x - 2)} + \frac{9}{2(x - 4)} dx = -\frac{5}{2} \ln(x - 2) + \frac{9}{2} \ln(x - 4) + C.$$

(h)  $\int \sin^4 y dy.$

This is a trig integral. Since the power of  $\sin y$  is even, we need to use  $\sin^2 y = \frac{1 - \cos 2y}{2}$ . Thus  $\int \sin^4 y dy = \frac{1}{4} \int (1 - \cos 2y)^2 dy = \frac{1}{4} \int (1 - 2 \cos 2y + \cos^2 2y) dy = \frac{1}{4} \int (1 - 2 \cos 2y + \frac{1 + \cos 4y}{2}) dy = \frac{3}{8} y - \frac{1}{4} \sin 2y + \frac{1}{32} \sin 4y + C$

(i)  $\int \frac{dt}{(9 + t^2)^{\frac{3}{2}}}$

This uses trig substitution with  $t$  on the opposite side, 3 on the adjacent side and  $\sqrt{9 + t^2}$  on the hypotenuse of a right triangle. Thus  $t = 3 \tan \theta$ ,  $dt = 3 \sec^2 \theta d\theta$  and  $\sqrt{9 + t^2} = 3 \sec \theta$ . So  $\int \frac{dt}{(9 + t^2)^{\frac{3}{2}}} = \int \frac{3 \sec^2 \theta d\theta}{(3 \sec \theta)^3} = \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + C = \frac{t}{9\sqrt{9 + t^2}} + C.$

(j)  $\int \frac{t^3 dt}{\sqrt{1 - t^2}}$

This also uses trig substitution with  $t$  on the opposite side, 1 on the hypotenuse and  $\sqrt{1 - t^2}$  on the adjacent side of a right triangle. Thus  $t = \sin \theta$ .  $dt = \cos \theta d\theta$  and  $\sqrt{1 - t^2} = \cos \theta$ . Thus,  $\int \frac{t^3 dt}{\sqrt{1 - t^2}} = \int \frac{\sin^3 \theta \cos \theta d\theta}{\cos \theta} = \int \sin^3 \theta d\theta = \int (1 - \cos^2 \theta) \sin \theta d\theta = \int u^2 - 1 du$  using the substitution  $u = \cos \theta$  and  $du = -\sin \theta d\theta$ .

$$\text{Now } \int u^2 - 1 du = \frac{u^3}{3} - u + C = \frac{\cos^3 \theta}{3} - \cos \theta + C = \frac{(1 - t^2)^{\frac{3}{2}}}{3} - \sqrt{1 - t^2} + C.$$