Practice Final
MATH 9C

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1. Determine if the following improper integral converges or diverges:

$$\int_{-1}^{1} \frac{2x^2 - x + 2}{x^2(x-2)}\, dx$$

$$\int_{-1}^{1} \frac{2x^2 - x + 2}{x^2(x-2)}\, dx = \lim_{b \to 0} \int_{-1}^{b} \frac{2x^2 - x + 2}{x^2(x-2)}\, dx + \lim_{c \to 0} \int_{c}^{1} \frac{2x^2 - x + 2}{x^2(x-2)}\, dx.$$

$$\frac{2x^2 - x + 2}{x^2(x-2)} = \frac{-1}{x^2} + \frac{2}{x-2}.$$

Note $\int \frac{-1}{x^2} dx = \frac{1}{x} + C$ and $\lim_{x \to 0} \frac{1}{x} = \infty$. So both integrals diverge.

2. Consider the sequence $a_n = \frac{3^n + \cos(n\pi)}{2^n}$.

   (a) Is a_n bounded above? If so, find an upper bound.

   $a_n \geq \frac{3^n - 1}{2^n}$ and $\lim_{n \to \infty} \frac{3^n - 1}{2^n} = \infty$ as $\frac{3^n}{2^n}$ is a geometric sequence with $r > 1$. So a_n is not bounded.

   (b) Is a_n non decreasing? Explain your answer.

   a_n is non decreasing as $\frac{3^{n+1} + \cos[(n+1)\pi]}{2^{n+1}} = \frac{3^{n+1} + (-1)^{n+1}}{2^{n+1}} \geq \frac{3^n + (-1)^n}{2^n}$
   This is clearly true when n is odd.

   When n is even, note that $3^{n+1} - 1 = 2(3^n + 3^{n-1} + \cdots + 3 + 1)$ so $\frac{3^{n+1} - 1}{2^{n+1}} = \frac{2(3^n + 3^{n-1} + \cdots + 3 + 1)}{2^{n+1}} = \frac{3^n + 3^{n-1} + \cdots + 3 + 1}{2^n} \geq \frac{3^n + (-1)^n}{2^n}$.

   (c) Find $\lim_{n \to \infty} a_n$.

   This was computed in part a.

3. List the first three partial sums of $\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n2^n}$.

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$, $S_3 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} = \frac{5}{12}$.

4. Suppose a_n and b_n are positive sequences and $\sum_{n=1}^{\infty} a_n + b_n$ converges. Is it true that $\sum_{n=1}^{\infty} a_n$ converges? Why or why not?

As $a_n \geq 0$ and $b_n \geq 0$, then $a_n + b_n \geq a_n$. Thus if $\sum_{n=1}^{\infty} a_n + b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges by the direct comparison test.

5. Find the sum of the series $\sum_{n=0}^{\infty} \frac{3(-2)^n}{3^n}$

$$\sum_{n=0}^{\infty} \frac{3(-2)^n}{3^n} = \frac{3}{1+\frac{2}{3}} = \frac{9}{5}.$$

6. Determine if the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

As $\frac{n^n}{(n!)^2}$ contains a factorial, ratio test is the best option.

We compute $\lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{((n+1)!)^2}}{\frac{n^n}{(n!)^2}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n(n+1)^2} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n(n+1)}$.

Set $y = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$. Then take ln of both sides to obtain $\ln y = \lim_{n\to\infty} \ln \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} n \ln \frac{(n+1)}{n}$. Now using L'hopital, we obtain $\ln y = \lim_{n\to\infty} \frac{\frac{n}{n+1}(-\frac{1}{n^2})}{-\frac{1}{n^2}} = 1$, so $y = e$ and $\lim_{n\to\infty} \frac{(n+1)^n}{n^n(n+1)} = 0$. Thus by the ratio test $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ converges.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

We will use the alternating series test. For $n \geq 2$ $\frac{\ln n}{n} \geq 0$, $\frac{\ln n}{n} \geq \frac{\ln(n+1)}{n+1}$ and $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. Thus by the alternating series test $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Using a partial sum decomposition $\frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$ and $\lim_{n\to\infty} \frac{1}{2n} = 0$. Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is a telescoping series which converges to $\frac{3}{4}$.

(d) $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$

$0 \leq \frac{|\sin n|}{3^n} \leq \frac{1}{3^n}$. As $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a converging geometric series, the direct comparison test implies $\sum_{n=1}^{\infty} \frac{|\sin n|}{3^n}$ converges. Thus $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$ is absolutely convergent and hence convergent.

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

We will use alternating series test again. $\frac{1}{n+1} \geq 0$, $\frac{1}{n+1} \geq \frac{1}{n+1+1}$ and $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the alternating series test.

(f) $\sum_{n=1}^{\infty} \ln n - \ln(n+1)$

The partial sums are $S_n = \ln 1 - \ln(n+1)$. But $\lim_{n\to\infty} \ln(n+1) = \infty$ so $\sum_{n=1}^{\infty} \ln n - \ln(n+1)$ diverges.

7. Determine if the following series are conditionally convergent, absolutely convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{8n+1}}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{8n+1}}$ converges by the alternating series test as $\frac{1}{\sqrt[3]{8n+1}} \geq 0$, $\frac{1}{\sqrt[3]{8n+1}} \geq \frac{1}{\sqrt[3]{8(n+1)+1}}$ and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{8n+1}} = 0$. Now $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n+1}}$ can be compared to the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ using the Limit Comparison test. $\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{8n+1}}}{\frac{1}{\sqrt[3]{n}}} = \frac{1}{2}$. Thus the limit comparison test implies that $\sum_{n=1}^{\infty} \frac{(1}{\sqrt[3]{8n+1}}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{8n+1}}$ is conditionally convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n3^n}$

Using the root test on $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$, we observe that $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n3^n}} = \frac{2}{3} < 1$. Thus, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n3^n}$ is absolutely convergent.

(c) $\sum_{n=1}^{\infty} \frac{(4n-1)^n}{(2n+1)^n}$

Using the root test, we observe that $\lim_{n\to\infty} \sqrt[n]{\frac{(4n-1)^n}{(2n+1)^n}} = 2 > 1$. So $\sum_{n=1}^{\infty} \frac{(4n-1)^n}{(2n+1)^n}$ is divergent by the root test.

8. Determine the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n}(2x-1)^n$

We use the root test to determine the radius of convergence. $\lim_{n\to\infty} \sqrt[n]{\frac{|2x-1|^n}{5^n}} = \frac{|2x-1|}{5}$. This converges when $\frac{|2x-1|}{5} < 1$. In other words, when $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2}$ or $-2 < x < 3$.

Pluging in the endpoints we get the series $\sum\limits_{n=1}^{\infty} 1$ and $\sum\limits_{n=1}^{\infty} (-1)^n$ respectively. Neither converge so the interval of convergence is the interval given above.

9. Determine the interval of convergence for the power series $\sum\limits_{n=0}^{\infty} \dfrac{n!}{2^n}(x+4)^n$

   We use the ratio test to determine the radius of convergence. $\lim\limits_{n\to\infty} \dfrac{\frac{(n+1)!|x+4|^{n+1}}{2^{n+1}}}{\frac{n!|x+4|^n}{2^n}} = \lim\limits_{n\to\infty} \dfrac{(n+1)|x+4|}{2} = \infty$ Thus, the radius of convergence is 0 and $\sum\limits_{n=0}^{\infty} \dfrac{n!}{2^n}(x+4)^n$ converges only at $x = -4$.

10. If $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-2)^n$ has a radius of convergence equal to 0, does $f(2)$ necessarily converge?

    A power series always converges at its center, so yes it will converge.

11. Find the 2nd Taylor polynomial for $f(x) = \sin(\pi x)$ centered at $x = \frac{1}{4}$.

    $f'(x) = \pi\cos(\pi x)$ and $f''(x) = -\pi^2\sin(\pi x)$. Thus $f(\frac{1}{4}) = \frac{\sqrt{2}}{2}$ and $f'(\frac{1}{4}) = \frac{\pi\sqrt{2}}{2}$ and $f''(\frac{1}{4}) = -\frac{\pi^2\sqrt{2}}{2}$

    Thus the 2nd Taylor polynomial is $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{2}(x-\frac{1}{4}) - \frac{\pi^2\sqrt{2}}{4}(x-\frac{1}{4})^2$

12. Find the Taylor series for $f(x) = xe^x + 1$ centered at $x = 0$.

    $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, thus $xe^x = \sum\limits_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum\limits_{n=1}^{\infty} \frac{x^n}{(n-1)!}$. Thus $xe^x + 1 = 1 + \sum\limits_{n=1}^{\infty} \frac{x^n}{(n-1)!}$.

13. Find the Maclaurin series for $f(x) = \dfrac{\cos(x)-1}{x^2}$.

    $\cos x = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ Thus $\cos x - 1 = \sum\limits_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and $\dfrac{\cos(x)-1}{x^2} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n-2}}{(2n)!}$.

14. Find the Maclaurin series for $f(x) = \arctan x^3$.

    $\arctan x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Thus $\arctan x^3 = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$.

15. Show that $y = e^x - e^{-x}$ is a solution to the differential equation $y' + y = 2e^x$.

    $y' = e^x + e^{-x}$. Thus $y' + y = e^x + e^{-x} + e^x - e^{-x} = 2e^x$.

16. Sketch the slope fields for $y' = \frac{y}{x}$

    This will be posted on my office door.

17. Find the general solution to $y' = (y^2 - 1)e^x$.

Separating the variables we obtain $\frac{dy}{y^2-1} = e^x dx$ Integrating both sides, we obtain $\frac{\ln y-1}{2} - \frac{\ln y+1}{2} = e^x + C$ or $\ln \frac{y-1}{y+1} = 2e^x + C$. Hence $\frac{y-1}{y+1} = Ce^{2e^x}$. Cross multiplying we obtain $y - 1 = (y+1)Ce^{2e^x}$ or $y = \dfrac{Ce^{2e^x} + 1}{1 - Ce^{2e^x}}$

18. Find the solution to the initial value problem $y' + y = e^{-x} \cos x$, $y(0) = 1$.

    $v = e^{-x}$ and $y = ue^{-x}$. Thus $u'e^{-x} = e^{-x} \cos x$. And $u = \sin x + C$. Hence, $y = e^{-x}(\sin x + C)$. We use the initial condition to solve for C and obtain $1 = C$. Thus $y = e^{-x}(\sin x + 1)$.

19. Find the general solution to $x^2 y' - y = 1$.

    This is an linear differential equation with $p(x) = -\frac{1}{x^2}$ and $q(x) = \frac{1}{x^2}$. $v(x) = e^{\frac{1}{x}}$ and $y = ue^{\frac{1}{x}}$. We solve for u using the separable equation $u'e^{\frac{1}{x}} = \frac{1}{x^2}$ or $u' = \dfrac{e^{-\frac{1}{x}}}{x^2}$ and $u = -e^{\frac{1}{x}} + C$. Thus $y = -1 + Ce^{\frac{1}{x}}$.

20. A 20 gallon container contains 20 gallons of water and 2 pounds of salt. A saline solution with .2 pounds of salt per gallon is poured into the container at a rate of 4 gallons a minute. If the subsequent mixture leaves the container at a rate of 4 gallons a minute, set up a differential equation with initial conditions to solve for the amount of salt in the container at any time t.

    Let $y(t)$ be the salt in the container for any time t. $y(0) = 2$ and $y'(t) = .2(lb/gal) \cdot 4(gal/min) - \frac{y(t)}{20}(lb/gal) \cdot 4(gal/min)$ or $y' = .8 - \frac{y(t)}{5}$ with initial condition $y(0) = 2$.

21. \$10,000 is invested in a bank account which accrues interest at an annual rate of 4 percent compounded continuously. Find an equation to solve for the amount of money in the account at any time t.

    $P = 10,000e^{.4t}$.

22. A cart is coasting down the road at a rate of 50 meters per minute. If the cart weighs 200 kg and the resistance is given by four times the speed it is traveling, find its velocity after 5 minutes.

    $200v'(t) = -4v$. $v = 50e^{-\frac{t}{50}}$ and $v(5) = 50e^{-\frac{1}{10}}$ meters per minute.

23. Find the orthogonal trajectories to $y = cx^2$.

    $y' = 2cx$ or $y' = \frac{2y}{x}$. Thus the orthogonal trajectories will be the solution to $y' = -\frac{x}{2y}$ or $y^2 = C - \frac{x^2}{2}$ or $y = \pm\sqrt{C - \frac{x^2}{2}}$.

24. Find the series solution to $y'' + y = 0$ with initial conditions $y'(0) = 1$ and $y(0) = 0$.

    $y = \sum_{n=0}^{\infty} a_n x^n$ and $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Thus $\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n = 0$ and $a_0 = 0$ and $a_1 = 1$, $a_{2n} = 0$ and $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$. Thus $y = \sin x$.

25. Find the slope of the tangent line to the curve $r = 4\cos(2\theta)$ at the point $(2, \frac{\pi}{6})$.

$\frac{dy}{dx} = \dfrac{4\cos(2\theta)\cos\theta + 8\sin(2\theta)\sin\theta}{-4\cos(2\theta)\sin\theta + 8\sin(2\theta)\cos\theta}$. Thus $\frac{dy}{dx}|_{\frac{\pi}{6}} = \frac{\sqrt{3}+2\sqrt{3}}{-1+6} = \frac{3\sqrt{3}}{5}$ is the slope to the tangent line at $(2, \frac{\pi}{6})$.

26. Find the points (in polar form) where the curve $r = 1 - 2\cos\theta$ has a horizontal tangent line.

    $\frac{dy}{dx} = \dfrac{(1-2\cos\theta)\cos\theta + (2\sin\theta)\sin\theta}{-(1-2\cos\theta))\sin\theta + (2\sin\theta)\cos\theta} = \dfrac{1-2\cos 2\theta}{(4\cos\theta - 1)\sin\theta)}$. Setting the top to 0, we obtain $\cos 2\theta = \frac{1}{2}$ or $2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Thus the points are $(1-\sqrt{3}, \frac{\pi}{6})$, $(1+\sqrt{3}, \frac{5\pi}{6})$, $(1+\sqrt{3}, \frac{7\pi}{6})$, and $(1-\sqrt{3}, \frac{\pi}{6})$.

27. Find the area enclosed by one of the petals of the rose curve $r = 2\cos(3\theta)$.

    Setting $r = 0$, we obtain $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

    Using symmetry about the x-axis, the area will be $A = \int_0^{\frac{\pi}{6}} 4\cos^2(3\theta)d\theta = \int_0^{\frac{\pi}{6}} 2(1+\cos(6\theta))d\theta = 2\theta + \dfrac{1}{3}\sin(6\theta)|_0^{\frac{\pi}{6}} = \dfrac{\pi}{3}$.

28. Find the area outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\sin\theta$.

    Set $r = 0$ to find where the inner loop starts. $\sin\theta = -\frac{1}{2}$ or $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$. Using Symmetry about the y-axis, $A = \int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} (1+2\sin\theta)^2 d\theta - \int_{\frac{7\pi}{6}}^{\frac{\pi}{2}} (1+2\sin\theta)^2 d\theta$.

    $\int (1+2\sin\theta)^2 d\theta = \int 1 + 4\sin\theta + 4\sin^2\theta d\theta = \int 1 + 4\sin\theta + 2 - 2\cos(2\theta)d\theta = 3\theta - 4\cos\theta - \sin(2\theta)$.

    Thus $A = \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} - \frac{3\pi}{2} + 0 - \frac{9\pi}{2} + 0 + \frac{7\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi - 3\sqrt{3}$

29. Set up the integral to find the length of one of the petals on the curve $r = 2\cos(2\theta)$.

    Setting $r = 0$ we see that the petals start and end at $\theta = \frac{\pi}{4} + \frac{k\pi}{2}$. Thus $L = 2\int_0^{\frac{\pi}{4}} \sqrt{4\cos^2(2\theta) + 16\sin^2(2\theta)}d\theta$.