Practice Midterm
MATH 46

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1. What is the order of the following differential equation: $y^{(4)} - 2y' + xy = y^6$?

   The differential equation is a 4th order differential equation as the highest derivative appearing is 4.

2. Give a rough sketch of the solution of $y' = y^2 - x$ with initial condition $y(2) = 1$? (Hint: Draw a direction field and sketch the integral curve containing the above point.)

   The sketch is posted outside my office.

3. Solve the initial value problem $xy' - y = x \ln x$, $y(1) = 4$.

   As $x \neq 0$, the above differential equation is equivalent to $y' - \frac{y}{x} = \ln x$. This is a first order nonhomogeneous equation with $y_1 = x$ and $y = ux$. Solving $u' = \dfrac{\ln x}{x}$, we see that $u = \dfrac{(\ln x)^2}{2} + C$ and $y = \dfrac{x(\ln x)^2}{2} + Cx$. As $y(1) = 4$, we obtain $C = 4$ and $y = \dfrac{x(\ln x)^2}{2} + 4x$

4. Solve $y' = \dfrac{y^2}{x^2+1}$.

   This is a separable differential equation. Separating the variables we obtain $\dfrac{dy}{y^2} = \dfrac{dx}{x^2+1}$. Integrating both sides we obtain, $-\frac{1}{y} = \arctan x + C$ or $y = \dfrac{1}{C - \arctan x}$. There is also a trivial solution $y = 0$.

5. Solve $y' + y = xe^{2x}y^3$.

   This is a Bernoulli Equation. The solution to the complementary homogeneous equation is $y_1 = e^{-x}$ and $y = ue^{-}x$. Now $\dfrac{u'}{u^3} = x$. Integrating both sides we obtain $-\dfrac{1}{u^2} = \dfrac{x^2}{2} + C$ or $u = \dfrac{1}{\sqrt{C - x^2}}$ or $u = -\dfrac{1}{\sqrt{C - x^2}}$. Thus $y = \dfrac{e^{-x}}{\sqrt{C - x^2}}$ or $y = -\dfrac{e^{-x}}{\sqrt{C - x^2}}$. There is also a trivial solution $y = 0$.

6. Solve $y' = \dfrac{x^2+y^2}{xy}$.

   This is a homogeneous equation with $y' = \frac{1}{u} + u$. As $u = \frac{y}{x}$, we obtain the differential equation $u'x + u = \frac{1}{u} + u$ or $u'x = \frac{1}{u}$. Separating the variables we obtain $udu = \frac{dx}{x}$. Integrating both sides we obtain $\frac{u^2}{2} = \ln|x| + C$ or $u = \sqrt{\ln(x^2)+C}$ or $u = -\sqrt{\ln(x^2)+C}$. Hence $y = x\sqrt{\ln(x^2)+C}$ or $y = -x\sqrt{\ln(x^2)+C}$.

7. Determine if $(4xy + y^2)dx + (2xy + 2x^2 + y^3)dy = 0$ is exact.

   $M_y = 4x + 2y = N_x$. Thus the above differential equation is exact.

8. Find an integrating factor for $(x^2 + y^2)dx + (2xy + 3xy^2 + x^3)dy = 0$.

   $M_y = 2y$, $N_x = 2y + 3y^2 + 3x^2$. $\frac{N_x - M_y}{M} = 3$ is a function of y so $\mu(y) = e^{3y}$ is an integrating factor for the above differential equation.

9. If the amount of a substance in the air changes at a rate which is inversely proportional to the amount present, write a differential equation to find $Q(t)$, the amount of the substance in the air.

   The differential equation used to solve for $Q(t)$ is $Q'(t) = \dfrac{k}{Q(t)}$ as the rate of change of the substance is inversely proportional to the substance present.

10. A radioactive substance decays at a rate proportional to the amount present. If half the original quantity Q_0 is left after 1000 years, in how many years will the substance decay to one fourth of the original quantity?

    Using the radioactive decay model, $Q(t) = Q_0 e^{-\frac{\ln 2}{1000}t}$. $\frac{1}{4} = e^{-\frac{\ln 2}{1000}t}$ implies $t = 1000\frac{\ln 4}{\ln 2} = 2000$ years.

11. A thermometer reading 300 degrees is placed in a room where the temperature is 75 degrees. After two minutes the thermometer reads 200 degrees. What does the thermometer read after 10 minutes?

    Using $T = T_m + (T_0 - T_m)e^{-kt}$, we obtain $T = 75 + 225e^{-kt}$. After two minutes we obtain $200 = 75 + 225e^{-2k}$. Thus $k = \frac{1}{2}\ln\frac{9}{5}$. The temperature equation becomes $T = 75 + 225e^{-\frac{1}{2}\ln\frac{9}{5}t}$ and after 10 minutes the temperature is approximately 87 degrees.

12. An object with mass 98 kg is launched vertically upward with initial velocity of 49 m/s in a medium which exerts a resistive force with magnitude proportional to the square of the speed. Write a differential equation to find the velocity of the object.

    The differential equation for the above situation is $98v' = -\dfrac{98^2}{10} - kv^2$, $v_0 = 49$.

13. Find the orthogonal trajectories of $x^2 + 4y^2 = c^2$.