The following is extracted from Emmanual Hebey's 1999 book "Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities." It spells out what is going with the ∇ operator in more detail.

Let D be an affine connection on a Riemmanian manifold. (In the seminar I was using ∇ for D to agree with Do Carmo, but D is Aubin's and Hebey's and, I believe, Jerry's notation.) Define ∇_i by

$$\nabla_i = D_{\frac{\partial}{\partial x^i}},$$

so ∇_i maps a vector field into a vector field. Define the Christoffel symbols, $\{\Gamma_{ij}^k\}$ by

$$\nabla_i \left(\frac{\partial}{\partial x^j} \right) = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

If $Y = Y^j \frac{\partial}{\partial x^j}$ is a vector field, then by the Liebniz rule,

$$\begin{aligned} \nabla_i Y &= \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + Y^j \nabla_i \left(\frac{\partial}{\partial x^j} \right) \\ &= \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + Y^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} \\ &= \left(\frac{\partial Y^j}{\partial x^i} + \Gamma_{i\alpha}^j Y^\alpha \right) \frac{\partial}{\partial x^j}, \end{aligned}$$

where we re-indexed in the last step.

We can generalize the operation of ∇_i to a (p, q)-tensor T (see my little spiel on tensors below, if you like) by

$$\begin{aligned} (\nabla_i T)_{i_1 \cdots i_p}^{j_1 \cdots j_q} = \frac{\partial T_{i_1 \cdots i_p}^{j_1 \cdots j_q}}{\partial x^i} &- \sum_{k=1}^p \Gamma_{i i_k}^\alpha T_{i_1 \cdots i_{k-1} \alpha i_{k+1} \cdots i_p}^{j_1 \cdots j_q} \\ &+ \sum_{k=1}^q \Gamma_{i\alpha}^{j_k} T_{i_1 \cdots i_p}^{j_1 \cdots j_{k-1} \alpha j_{k+1} \cdots j_q}. \end{aligned}$$

Our calculation for $Y = Y^j \frac{\partial}{\partial x^j}$ was the special case of a $(0, q)$-tensor—a vector (field).

We now define the ∇ operator in such a way that it maps a (p, q)-tensor into a $(p + 1, q)$ tensor (so it is really a series of operators, one for each p). For a smooth function f on M (a $(0, 0)$-tensor), we define $\nabla f = df$ (a $(1, 0)$-tensor). For a (p, q) tensor T, $p \geq 1$, we define

$$(\nabla T)_{i_1 \cdots i_{p+1}}^{j_1 \cdots j_q} = (\nabla_{i_1} T)_{i_2 \cdots i_{p+1}}^{j_1 \cdots j_q}.$$

It makes sense to compose ∇ operators: $\nabla^k = \nabla \cdots \nabla$ maps a (p, q)-tensor to a $(p + k, q)$-tensor.

For instance, if f is a smooth function on a manifold,

$$(\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k},$$

and is called the Hessian of f.

Hebey defines the norm of $\nabla^k u$ by

$$|\nabla^k u| = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots i_k}.$$

It appears, though, that this is actually $|\nabla^k u|^2$, which I will assume is the case (I don't think it would be a norm, otherwise). Note that this is a definition, not a calculation, since $\nabla^k u$ is a vector field only for $k = 1$.

Hebey's $|\nabla^k u|^2$ (assuming I am right about the square), then, would be the same as Aubin's definition of $|\nabla^k \varphi|^2$ on p. 32 (with φ in place of u) if by $\nabla_{\alpha_1} \nabla_{\alpha_2} \cdots \nabla_{\alpha_k} \varphi$ Aubin means what Hebey means by $(\nabla^k \varphi)_{\alpha_1 \alpha_2 \cdots \alpha_k}$ and if Aubin means to have the indices raised after all the derivatives have been performed (or if raising the indices somehow commutes with the derivatives). Aubin's terse definition 1.14 on p. 4 seems to indicate this may be the case. If so, then Hebey's definition of the Sobolev norm is identical to that of Aubin's.

In any case, when u is a real-valued function on the manifold—a $(0,0)$-tensor—$\nabla^k u$ is a $(k,0)$-tensor. This looks a lot like a k-form, but is not antisymetric; in any case, it is only its coefficients in a coordinate system that enter into the Sobolev norms, and these coefficients contain all the derivatives of order k along with lower-order derivatives. In the special case of a connection with all of $\Gamma^i_{jk} = 0$, the coefficients would consist exactly of all derivatives of order k, and in flat space, the definition of the Sobolev space norm that Aubin gives would coincide with the usual definition.

Tensors

A (p,q)-tensor is a multilinear form that maps, for each $x \in M$ the space

$$T_x(M) \times \cdots \times T_x(M) \times T_x(M)^* \times \cdots \times T_x(M)^*,$$

into $\mathbb{R}$, where there are p products of $T_x(M)$ and q products of $T_x(M)^*$. A basis element for the space of (p,q)-tensors looks like

$$dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_q}},$$

if that helps clear things up. We can write a (p,q)-tensor, T, in coordinates as

$$T^{j_1 \cdots j_q}_{i_1 \cdots i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_q}},$$

but we usually just write the coefficients $T^{j_1 \cdots j_q}_{i_1 \cdots i_p}$ if the coordinate system is understood.

A (p,q)-tensor transforms under changes of coordinates from x to y as follows:

$$\widetilde{T}^{j_1 \cdots j_q}_{i_1 \cdots i_p} = T^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p} \frac{\partial x^{\alpha_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{\alpha_p}}{\partial y^{i_p}} \frac{\partial y^{j_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x^{j_q}}{\partial y^{\beta_q}}.$$