THE STRONG VANISHING VISCOSITY LIMIT WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. We adapt methodology of Tosio Kato to establish necessary and sufficient conditions for the solutions to the Navier-Stokes equations with Dirichlet boundary conditions to converge in a strong sense to a solution to the Euler equations in the presence of a boundary as the viscosity is taken to zero. We extend existing conditions for no-slip boundary conditions to allow for nonhomogeneous Dirichlet boundary conditions and curved boundaries, establishing several new conditions as well. We make some speculations on how the vanishing viscosity limit might hold, and give a brief comparison of various correctors appearing in the literature used for similar purposes.

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Long version

Includes some proofs and speculations not intended for the submitted version. Also includes a proof using the technology developed here of Xiaoming Wang's [65].

1. Introduction

In his seminal paper [26], Tosio Kato established necessary and sufficient conditions for solutions to the Navier-Stokes equations with no-slip boundary conditions to converge as the viscosity goes to zero to a solution to the Euler equations—the so-called *vanishing viscosity* or *inviscid* limit. In the "generic" case in which no special symmetries or partial analyticity of the initial data or geometry is assumed, whether or not this limit holds in even one instance is not known. Most of what has been learned about the generic case fits neatly into Kato's original approach using his original corrector. There have been refinements, most notably those of Xioaming Wang in [65] building on his work with Roger Temam in [60] (these two papers seem to have revived interest in [26]). See also [6, 7, 9, 29, 30, 31, 32].

In this paper, we turn Kato's energy argument, incorporating a fairly recent way of decomposing the nonlinear terms from [7], into a tool (Theorem 4.3) we then apply to obtain, using more uniform methodology, the various existing conditions in [65, 29, 30] for the vanishing viscosity limit to hold. In the process, we develop several novel conditions as well.

The strong vanishing viscosity limit. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, having a C^2 boundary, and define

$$Q := [0, T] \times \Omega$$

for some fixed $T > 0$. We consider solutions to the Navier-Stokes equations,

$$(NS_g) \begin{cases} \partial_t u_g + u_g \cdot \nabla u_g + \nabla p_g = \nu \Delta u_g & \text{in } Q, \\ \operatorname{div} u_g = 0 & \text{in } Q, \\ u_g(0) = u^0 & \text{in } \Omega, \\ u_g = g & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

Here, $\nu > 0$ is the constant viscosity and u^0 is the divergence-free initial velocity with $u^0 \cdot \boldsymbol{n} = 0$ on the boundary, $\partial\Omega$, where $\boldsymbol{n}$ is the outward unit normal vector. The function g is defined on $\partial\Omega$, with $g \cdot \boldsymbol{n} = 0$.

The vector field g induces a type of boundary forcing that influences the solution near the boundary, its effects spreading over time through the body of the fluid. An example is a constant-magnitude g that describes the rotation of a circular boundary, as analyzed in [12, 13]. No-slip boundary conditions, $g \equiv 0$, yield the Navier-Stokes equations in their classical form[1],

$$(NS) \begin{cases} \partial_t u_0 + u_0 \cdot \nabla u_0 + \nabla p_0 = \nu \Delta u_0 & \text{in } Q, \\ \operatorname{div} u_0 = 0 & \text{in } Q, \\ u_0(0) = u^0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

Note that u_0, like u_g, depends upon ν, though, following Kato, we suppress ν in our notation.

When $\nu = 0$, (NS_g), for any g, formally reduces to the Euler equations with no-penetration boundary conditions:

$$(E) \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = 0 & \text{in } Q, \\ \operatorname{div} \overline{u} = 0 & \text{in } Q, \\ \overline{u}(0) = u^0 & \text{in } \Omega, \\ \overline{u} \cdot \boldsymbol{n} = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

A longstanding open question in incompressible fluid mechanics is whether u_0 converges to $\overline{u}$ as $\nu \to 0$ and, if so, in what manner. That u_0 has some weak limit in $L^2(0, T; L^2(\Omega))$ is assured by the uniform-in-ν bound in the space of weak solutions (as in (1.6)). Recently, the work of Constantin and Vicol in [11] and then in conjunction with Lopes Filho and Nussenzveig Lopes in [10] has brought renewed interest in weak convergence to weak solutions. In this paper, however, we will be restrict ourselves to the question of whether or not what we will call the *strong vanishing viscosity* limit,

$$\|u_g(t) - \overline{u}(t)\|^2 + \nu \int_0^t \|\nabla(u_g(s) - \overline{u}(s))\|^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{1.1}$$

[1] Most of the literature that follows in the tradition of Kato assumes $g \equiv 0$. A notable exception is Xiaoming Wang's [65], whose setting is similar to the one we have here, though he assumes a flat boundary.

holds for all $t \in [0,T]$. Here and throughout,

$$
\begin{aligned}
&f \text{ scalar-valued}: && \|f\| := \|f\|_{L^2(\Omega)} = \left(\int_\Omega f^2\right)^{\frac{1}{2}}, \\
&v \text{ vector-valued}: && \|v\| := \||v|\|, \\
&M \text{ matrix-valued}: && \|M\| := \||M|\|,
\end{aligned}
\tag{1.2}
$$

where $|M|^2 = \sum_{ij} M_{ij}^2$. We will write $(\cdot,\cdot)$ for the corresponding inner-product.

We are most interested in (1.1) in the special case of no-slip boundary conditions, in which $g \equiv 0$. It was shown by Tosio Kato in [26] that when $\overline{u}$ is sufficiently regular, (1.1) is equivalent, for $g \equiv 0$, to the weaker condition,

$$
u_0 \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \nu \to 0, \tag{1.3}
$$

which is often referred to as the *classical vanishing viscosity limit*. This equivalence comes from the observation that if (1.3) holds it necessarily follows that

$$
\limsup_{\nu\to 0} \nu \int_0^t \|\nabla u_0\|^2 = 0. \tag{1.4}
$$

(If the limsup is positive, we say the sequence $(u_0)_{\nu>0}$ has an *energy defect*.)

That (1.3) implies (1.4), and hence implies (when $\overline{u}$ is sufficiently regular) (1.1), is clear when $g \equiv 0$: If (1.3) is to hold, then the energy for u_0 must converge to the energy for $\overline{u}$, which is conserved over time. By the classical energy equality for (NS) ((1.6), below) this can only happen if (1.4) holds. The situation for $g \not\equiv 0$ is more complicated, as we will see, because of the more complicated energy bound in (1.8).

We require that the initial velocities be the same for all solutions, so that the vanishing viscosity limit has some chance to hold. (It is also possible to allow $u_g(0) \to u^0$ as $\nu \to 0$.) As a consequence, unless $u^0|_{\partial\Omega} = g(0)$, u_g has an initial boundary layer in that there is an immediate discrepancy in boundary values after the initial time.

Dimension 2. We restrict our arguments to dimension $d = 2$, which yields four related simplifications. First, and most important, the well-posedness and regularity theory for solutions to both the Euler and Navier-Stokes equations are more well-developed in two dimensions than in higher dimensions. Solutions will be global in time, and we will be able to give nearly minimal assumptions on the initial and boundary data required to obtain our results. This also makes it easy to justify all of our energy arguments.

Second, the various energy equalities that we obtain would only be energy inequalities in higher dimension, which would require additional work to properly treat (see Remark 1.6). Third, for $d \geq 3$, weak solutions would have only a type of weak continuity to time zero. Fourth, the vorticity, $\omega_g = \operatorname{curl}(u_g) := \partial_1 u_g^2 - \partial_2 u_g^1$, is a scalar in 2D, which simplifies the form of certain expressions. We do not use the vorticity formulation of the equations, however, so this simplification is more cosmetic than fundamental, as vortex stretching would never be (directly) encountered.

Nonetheless, most of our analyses and results would apply to all $d \geq 3$ up to the time of existence of smooth solutions to the Euler equations, with only minor, technical adaptations.

Well-posedness.

Theorem 1.1 (Theorem 4.1 of [40])**.** *Assume that $u^0 \in C^{k,\alpha}(\Omega) \cap H$ for some integer $k \geq 1$ and $\alpha \in (0,1)$. There exists a unique solution to (E) with $\overline{u} \in C([0,\infty); C^{k,\alpha}(\Omega))$, $\partial_t \overline{u} \in C([0,\infty); C^{k-1,\alpha}(\Omega))$, and*

$$
\|\overline{u}(t)\| = \|u^0\|. \tag{1.5}
$$

We define the classical spaces of fluid mechanics,

$$H := \{v \in L^2(\Omega)^2 \colon \operatorname{div} v = 0, v \cdot \boldsymbol{n} = 0\},$$
$$V := \{v \in H_0^1(\Omega)^2 \colon \operatorname{div} v = 0\},$$

where $u \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined in the sense of a trace.

Theorem 1.2. *Assume that $u^0 \in H$. There exists a unique solution to (NS) with*

$$u \in C([0,\infty); H) \cap L^2(0,\infty; V), \quad \partial_t u \in L^2(0,T;V'),$$

and

$$\|u_0(t)\|^2 + 2\nu \int_0^t \|\nabla u_0\|^2 = \|u^0\|^2. \tag{1.6}$$

Moreover, for any $T > 0$ and $\varphi \in L^2(0,T;V)$,

$$\int_0^T \langle \partial_t u, \varphi \rangle_{V',V} + \int_0^T (u \cdot \nabla u, \varphi) + \nu \int_0^T (\nabla u, \nabla \varphi) = 0. \tag{1.7}$$

Proof. See Theorem II.7.3 of [14], Theorem V.1.4 of [5], the discussion following (V.7) in [5], and Proposition V.1.3 of [5]. □

Note that Theorem 1.1 continues to hold with forcing in $L^2(0,T;H)$.

For (NS_g), we have well-posedess as stated in Theorem 1.4. Its proof is fairly standard, but we include it in Section 10 because of the specific form of the energy inequality that we use. The energy bound in Theorem 1.4 is expressed in terms of the function g extended as in Lemma 1.3, also proved in Section 10.

Lemma 1.3. *Let $g \in L^2(0,\infty; H^{\frac{3}{2}}(\partial\Omega))$, $g \cdot \boldsymbol{n} = 0$ on $[0,\infty) \times \partial\Omega$, with $\partial_t g \in L^2(0,\infty; H^{\frac{1}{2}}(\partial\Omega))$. There exists a divergence-free extension of g to $g \in L^2(0,\infty; H \cap H^2(\Omega)^2)$ (which we continue to call g) with $\partial_t g \in L^2(0,\infty; H \cap H^1(\Omega)^2)$. If $u^0|_{\partial\Omega} = g(0)$ then we can have $g(0) = u^0$.*

By adding g to V, we obtain the affine space $V + g$.

Theorem 1.4. *Assume that $u^0 \in H$ and g is as in Lemma 1.3. There exists a unique solution to (NS_g) with*

$$u_g \in C([0,\infty); H) \cap L^2(0,\infty; V + g), \quad \partial_t u_g \in L^2(0,T;V'),$$

and

$$\begin{aligned} &\|u_g(t)\|^2 + 2\nu \int_0^t \|\nabla u_g\|^2 \\ &\qquad \le 2\left(\|g(t)\|^2 + 2\nu \int_0^t \|\nabla g\|^2\right) + 2\left(2\|u^0\|^2 + C(\nu,t)\right) e^{t + 2\int_0^t (\|\nabla g\|_{L^\infty})}, \end{aligned} \tag{1.8}$$

where

$$C(\nu,t) := 2\|g(0)\|^2 + \int_0^t \|F_g\|^2, \quad F_g := \nu\Delta g - \partial_t g - g \cdot \nabla g.$$

Moreover, (1.7) holds for any $T > 0$ and $\varphi \in L^2(0,T;V)$.

Because g is independent of ν, both (1.6) and (1.8) yield an energy bound that is independent of the viscosity (restricting to, say, $\nu \le 1$ for (1.8)). When $g \equiv 0$, the energy inequality in (1.8) reduces to the inequality arising from (1.6) with an additional factor of $4e^t$. Hence, the bound is not optimal in terms of g, an issue that is closely connected to the strong vanishing viscosity limit itself (see Section 11.3).

For our results, we will make the following assumption on the data for $k = 1$ or 2:

$$(Ass_k) \quad \begin{cases} g \in L^2(0,\infty; H^{\frac{3}{2}}(\partial\Omega)) \text{ with } g \cdot \boldsymbol{n} = 0 \text{ on } [0,\infty) \times \partial\Omega, \\ \partial_t g \in L^2(0,\infty; H^{\frac{1}{2}}(\partial\Omega)), \\ u^0 \in C^{k,\alpha}(\Omega) \cap H, \\ \partial\Omega \text{ is } C^2. \end{cases} \tag{1.9}$$

Remark 1.5. *Because Ω is bounded, $C^{k,\alpha}(\Omega) \subseteq H^1(\Omega)^2$, so if (Ass_1) is satisfied then the hypotheses on the data for Theorems 1.2 and 1.4 are also satisfied.*

Remark 1.6. *As pointed out in the discussion following (V.7) of [5], the ability to apply a test function $\varphi \in L^2(0,T;V)$ in formulating the definition of a weak solution to (NS) as in (1.7) is very much specific to 2D. (These same comments apply to solutions to (NS_g).) This will allow us to easily make the vanishing viscosity energy argument in the proof of Proposition 4.1. In 3D, one avoids (1.7) by using the energy inequality and applying only the corrected Euler velocity as the test function for (NS_g), as Kato did in [26].*

Although we treat a bounded domain in 2D, our results apply as well to a channel periodic in the x_1-direction and to a half-plane, $\{(x_1, x_2) \colon x_2 > 0\}$. (In particular, note that our only use of Poincaré's inequality is through Lemma 2.6 in a boundary layer, which remains valid in these settings.)

Related work. In [26], Kato employs a simple energy argument that almost anyone exploring the vanishing viscosity limit for the first time would attempt. Hence, one cannot say that the use of energy arguments in the vanishing viscosity limit or related singular limits, natural as they are, necessarily means that the author is following in the tradition of Kato. Indeed, some of the most striking results, which make assumptions on the initial data involving some degree of analyticity, such as [47, 48, 43], make only secondary use of energy arguments and do not follow Kato (one might say they follow Prandtl); see also the more recent, [37, 36, 38, 3]. Nonetheless, there is by now a fairly sizeable literature going beyond the study of the strong vanishing viscosity limit, the topic of this paper, that appear very influenced by Kato's approach, adapting his argument and philosophy to a greater or lesser extent. This literature includes papers where the boundary condition is (directly or indirectly) changed [66, 49], the domain is expanded to the whole space or shrunk to a point or points [34, 22], there is some special symmetry to the geometry and initial data [45, 31], or the argument is applied to slightly different equations with sometimes different boundary conditions [42, 50, 67, 2, 39, 41].

Kato's insight was to clearly identify the balance of the two, uncontrollable terms appearing in his energy argument, and to understand that the only feasible thing to do was to create from them a single necessary and sufficient condition to control them both. This balance does not change as long as $g \cdot \boldsymbol{n} = 0$ on the boundary. If we drop this restriction, however, the nature of the problem can change dramatically. This is most clearly seen in [53] (extended in [17] to a bounded domain), in which the vanishing viscosity limit is obtained for inflow, outflow boundary conditions in 3D, in which $\mathbf{g} \cdot \boldsymbol{n} < 0$ on some components, $\mathbf{g} \cdot \boldsymbol{n} > 0$ on others. (We discuss this further in Section 12.2.)

Organization of this paper. We begin in Section 2 by defining the coordinate system we will use in a boundary layer and give some lemmas we will find useful throughout the paper. We define what we call a *fully scalable corrector* (our prime example being that of Kato in [26]) in Section 3, using such a corrector in Section 4 to develop a tool we use in subsequent sections to establish necessary and sufficient conditions for the strong vanishing viscosity limit to hold. We argue in Section 5 that a boundary layer width proportional to the viscosity, as used by Kato in [26], along with an infinitesimally wider one employed by Wang in [65], are the two most useful choices in the context of Kato's argument.

In Sections 6 and 7, we apply, with Kato's width, the tool developed in Section 4 to obtain results in the spirit of Kato's original [26]. We employ the infinitesimally wider layer of Wang in Section 8 to obtain a few simple results. We also reproduce the result of Xiaoming Wang's [65]. We explain in Section 9 how the result from [30] on the formation of a vortex sheet on the boundary continues to hold for nonhomogeneous boundary conditions. We give the proof of Lemma 1.3 and Theorem 1.4 in Section 10.

In Section 11 we make a few speculations and conjectures on the strong vanishing viscosity limit. In particular, we treat the case of zero initial data with $g \not\equiv 0$, demonstrating how the strong vanishing viscosity limit is closely connected to optimizing the energy bound in Theorem 1.4. We close in Section 12 with an overview of other correctors appearing in the literature used to analyze the vanishing viscosity limit, most of them in the tradition of Kato. Appendix A contains proofs some of the curvilinear coordinate expressions stated in Section 2, which are, however, fairly standard. Appendix B proves the estimates on Kato's corrector stated in Section 3.

$u = u_g$

For notational simplicity, until Section 10 we drop the g subscript, writing u for u_g.

2. Coordinates

Let $\boldsymbol{n}, \boldsymbol{\tau}$ be the outward unit normal, tangent vectors to $\partial\Omega$ chosen so that $(\boldsymbol{n}, \boldsymbol{\tau})$ is in the standard orientation of $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Since $\partial\Omega$ is C^∞, there exists a tubular neighborhood (in Ω) of width $\overline{\delta} > 0$. For any $\delta > 0$ we define

$$\Gamma_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Remark 2.1. *Throughout this paper, we assume without comment that* $\delta \in (0, \min\{\overline{\delta}/2, 1\})$.

Each component of $\partial\Omega$ has its own component of Γ_δ. We define coordinates on $\Gamma_{\overline{\delta}}$, and hence on each Γ_δ, component-by-component. Fix an arbitrary point b in a given component of $\partial\Omega$ and let a be any point in the corresponding component of $\Gamma_{\overline{\delta}}$, then let a' be the closest point to a on $\partial\Omega$. We define coordinates (x_1, x_2) for the point a by

$$\begin{aligned} x_1 &= \text{the arc length along } \partial\Omega \text{ from } b \text{ to } a' \text{ in the } \boldsymbol{\tau} \text{ direction}, \\ x_2 &= |a - a'|. \end{aligned}$$

Another way of expressing this is that (x_1, x_2) are coordinate values in the $(\boldsymbol{\tau}, -\boldsymbol{n})$ coordinate frame with $(\boldsymbol{\tau}, -\boldsymbol{n})$ extended from $\partial\Omega$ to $\Gamma_{\overline{\delta}}$ in the natural way—orthogonally to $\partial\Omega$.

We will use coordinates and write vectors in component form only when working with functions or vector fields supported in a tubular neighborhood. Hence, (x_1, x_2) never refers to Cartesian coordinates, but always to the coordinates we just defined, and

$$\partial_j := \partial_{x_j},\ j = 1, 2 \text{ where } x_1, x_2 \text{ are defined on } \Gamma_\delta.$$

In these coordinates, the form of ∇, div, and Δ are distorted because of the curvature of the boundary, with div and Δ also including lower-order terms. For most of our calculations, these will have only a minor effect, but they will impact some of the more delicate estimates. We give the form of these operators in Lemma 2.2. We give the proof of Lemma 2.2 in Appendix A.

In Lemma 2.2, $\nabla^\perp$ is the operator ∇ rotated 90 degrees counterclockwise.

Lemma 2.2. *In Γ_δ, with coordinates defined as above, let $f = f(x_1, x_2)$ be a scalar-valued function and*

$$v = (v^1, v^2) := v^1 \boldsymbol{\tau} + v^2(-\boldsymbol{n}) = v^1 \boldsymbol{\tau} - v^2 \boldsymbol{n},$$

a vector-valued function. Writing $\kappa = \kappa(x_1)$ for the curvature at $(x_1, 0)$,

$$\begin{aligned}
&v^\perp = (-v^2, v^1), && \nabla f = J\partial_1 f \boldsymbol{\tau} - \partial_2 f \boldsymbol{n} = (J\partial_1 f, \partial_2 f),\\
&\nabla^\perp f = -\partial_2 f \boldsymbol{\tau} - J\partial_1 f \boldsymbol{n} = (-\partial_2 f, J\partial_1 f), && \operatorname{div} v = J\partial_1 v^1 + \partial_2 v^2 - \kappa J v^2,\\
&\operatorname{curl} v = J\partial_1 v^2 - \partial_2 v^1 + \kappa J v^1, && \Delta f = J^2 \partial_1^2 f + \partial_2^2 f - \kappa J \partial_2 f + x_2 \kappa' J^3 \partial_1 f,
\end{aligned}$$

where

$$J = J(x_1, x_2) := (1 - \kappa x_2)^{-1} \tag{2.1}$$

is the Jacobian determinant for the map from Cartesian coordinates to (x_1, x_2) coordinates. If $u = (u^1, u^2)$ is also vector-valued in Γ_δ then

$$u \cdot v = u^j v^j,$$

where we use implicit summation notation. Using (x_1, x_2) coordinates,

$$u \cdot \nabla v = \left(J u^1 \partial_1 v^1 + u^2 \partial_2 v^1, J u^1 \partial_1 v^2 + u^2 \partial_2 v^2 \right).$$

When integrating by parts in Γ_δ, we will use Lemma 2.3.

Lemma 2.3. *Assume that f and g are smooth scalar-valued functions on $\overline{\Omega}$ supported in $\overline{\Gamma_\delta}$. Then for $j = 1$, and also for $j = 2$ if fg vanishes on $\partial\Omega$,*

$$(\partial_j f, g) = -(f, \partial_j g) + (f, \alpha_j g),$$

where $\alpha_1 = x_2 \kappa' J$, $\alpha_2 = \kappa J$ (J being as in (2.1)) are smooth and independent of δ. Here, as always, $(\cdot, \cdot)$ is the L^2-inner product on Ω or, because of the supports, on Γ_δ.

Proof. Let Γ_δ^k be one of the finite number of components of Γ_δ, and let ℓ be the arc length of the boundary. Then we can write

$$\int_{\Gamma_\delta^k} \partial_1 f \, g = \int_0^\ell \int_0^\delta \partial_{x_1} f(x_1, x_2) g(x_1, x_2) J(x_1, x_2) \, dx_2 \, dx_1.$$

Integrating by parts in x_1, and noting that f and g are periodic in x_1 so there is no boundary term, we have

$$\begin{aligned}
\int_{\Gamma_\delta^k} \partial_1 f \, g &= - \int_0^\ell \int_0^\delta f(x_1, x_2) \partial_{x_1} (g(x_1, x_2) J(x_1, x_2)) \, dx_2 \, dx_1 \\
&= - \int_0^\ell \int_0^\delta f(x_1, x_2) \partial_{x_1} g(x_1, x_2) J(x_1, x_2) \, dx_2 \, dx_1 \\
&\qquad - \int_0^\ell \int_0^\delta f(x_1, x_2) g(x_1, x_2) \frac{\partial_{x_1} J(x_1, x_2)}{J(x_1, x_2)} J(x_1, x_2) \, dx_2 \, dx_1 \\
&= - \int_{\Gamma_\delta^k} \left(f \partial_1 g + f g \alpha_1 \right),
\end{aligned}$$

where $\alpha_1 = \partial_{x_1} J / J = x_2 J \kappa'$. Summing this expression over each component Γ_δ^k gives the result for $j = 1$. The argument for $j = 2$ is similar, using the vanishing of fg on $\partial\Gamma_\delta$. □

We will also integrate by parts over all of Ω in coordinate-free form. Since we are working with smooth functions, the most basic form is

$$(u, \nabla f) + (\operatorname{div} u, f) = \int_\Omega (u \cdot \boldsymbol{n}) f. \tag{2.2}$$

Here $(f, g) = \int_\Omega fg$ is the L^2-inner product; for vector fields u, v, the L^2-inner product is $(u, v) := \int_\Omega u \cdot v$. This form of integrating by parts leads to Lemmas 2.4 and 2.5.

Lemma 2.4. *Let $v_1, v_2 \in H \cap H^2$ and set $\omega_j = \operatorname{curl} v_j$, $j = 1, 2$. Then,*

$$(\nabla v_1, \nabla v_2) = (\omega_1, \omega_2) + \int_{\partial\Omega} (\omega_2 (v_1 \cdot \boldsymbol{\tau}) - \kappa v_1 \cdot v_2).$$

Proof. We have,

$$(\nabla v_1, \nabla v_2) = -(v_1, \Delta v_2) + \int_{\partial\Omega} (\nabla v_2 \cdot \boldsymbol{n}) \cdot v_1 = -(v_1, \nabla^\perp \omega^2) - \int_{\partial\Omega} \kappa v_1 \cdot v_2,$$

where we used Lemma 4.1 of [28] for the boundary integrand. But,

$$-(v_1, \nabla^\perp \omega^2) = (v_1^\perp, \nabla \omega^2) = -(\operatorname{div} v_1^\perp, \omega^2) - \int_{\partial\Omega} (v_1^\perp \cdot \boldsymbol{n}) \omega_2 = (\omega_1, \omega_2) + \int_{\partial\Omega} \omega_2 (v_1 \cdot \boldsymbol{\tau}). \square$$

The following is adapted from Lemma A.4 of [29]:

Lemma 2.5. *For all vector fields, $u \in H^1(\Omega)$, $v \in H$,*

$$(u \cdot \nabla u, v) = (u^\perp \operatorname{curl} u, v).$$

Proof. We have,

$$(u \cdot \nabla u, v) = (u \cdot (\nabla u - (\nabla u)^T), v) + (u \cdot (\nabla u)^T, v).$$

But,

$$(u \cdot (\nabla u)^T) \cdot v = (u^i \partial_j u^i, v^j) = \frac{1}{2}(v, \nabla |u|^2) = 0,$$

so

$$\begin{aligned}(v, u \cdot \nabla u) &= (u^i(\partial_i u^j - \partial_j u^i), v^j) = (u^1(\partial_1 u^2 - \partial_2 u^1), v^2) + (u^2(\partial_2 u^1 - \partial_1 u^2), v^1) \\ &= \int_\Omega (u^1 v^2 - u^2 v^1) \operatorname{curl} u = (u^\perp \operatorname{curl} u, v).\end{aligned} \qquad \square$$

Lemma 2.6 is the form of Poincaré's inequality that applies to a domain of given width vanishing on one component of the boundary:

Lemma 2.6. *Fix $p \in [1, \infty]$ and assume that $f \in W^{1,p}(\Gamma_\delta)$ with $f = 0$ on $\partial\Omega$. Then*

$$\|f\|_{L^p(\Gamma_\delta)} \le C\delta \|\partial_2 f\|_{L^p(\Gamma_\delta)},$$

where the constant $C = C(\Omega)$ is independent of p and δ (recall Remark 2.1).

Corollary 2.7. *For all $p \in [1, \infty]$,*

$$\begin{aligned}\|u^1\|_{L^p(\Gamma_\delta)} &\le C\delta \|\partial_2 u^1\|_{L^p(\Gamma_\delta)} + C' \delta^{\frac{1}{p}}, \\ \|u^2\|_{L^p(\Gamma_\delta)} &\le C\delta \|\partial_2 u^2\|_{L^p(\Gamma_\delta)},\end{aligned} \tag{2.3}$$

where the constant C is as in Lemma 2.6 and $C' = \|g\|_{W^{1,\infty}(\Omega)}$ is independent of p and δ.

Proof. Since $u^2 = -g \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, the inequality for $\|u^2\|_{L^p(U)}$ follows directly from Lemma 2.6. For the other inequality, we have

$$\begin{aligned}
\|u^1\|_{L^p(\Gamma_\delta)} &\le \|u^1 - g^1\|_{L^p(\Gamma_\delta)} + \|g^1\|_{L^p(\Gamma_\delta)} \\
&\le C\delta \|\partial_2(u^1 - g^1)\|_{L^p(\Gamma_\delta)} + C\delta^{\frac{1}{p}} \|g^1\|_{L^\infty(\Omega)} \\
&\le C\delta \|\partial_2 u^1\|_{L^p(\Gamma_\delta)} + C\delta^{1+\frac{1}{p}} \|\partial_2 g^1\|_{L^\infty(\Gamma_\delta)} + C\delta^{\frac{1}{p}} \|g^1\|_{L^\infty(\Omega)} \\
&\le C\delta \|\partial_2 u^1\|_{L^p(\Gamma_\delta)} + C\|g\|_{W^{1,\infty}} \delta^{\frac{1}{p}},
\end{aligned}$$

where we again applied Lemma 2.6, and used that Ω has finite measure. □

Lemma 2.8 is a version of Hardy's inequality, as in Lemma II.1.10 in [54], which we have combined with Poincaré's inequality.

Lemma 2.8. *Assume that $f \in H^1(\Gamma_{2\delta})$ with $f = 0$ on $\partial\Omega$,*

$$\|f/x_2\|_{L^2(\Gamma_\delta)} \le C_H \|\partial_2 f\|_{L^2(\Gamma_{2\delta})},$$

where we recall that x_2 is the distance from a point to the boundary.

Proof—for long version only. In this proof, we use the cutoff function φ_δ of Definition 3.1. The proof follows that of Lemma II.1.10 in [54]—we include the proof only to show that we can introduce the cutoff function φ_δ without introducing a factor of δ or some positive or negative power of it, and to stress that we only need $\partial_2 f$ not the full gradient. (Though we do not take advantage of that.) Letting Γ_δ^k be a component of Γ_δ and integrating as in the proof of Lemma 2.3,

$$\begin{aligned}
\|f/x_2\|^2_{L^2(\Gamma_\delta^k)} \le \|\varphi_{2\delta} f/x_2\|^2_{L^2(\Gamma_\delta^k)} &= \int_0^\ell \int_0^{2\delta} \frac{[(\varphi_{2\delta} f^2 J^{\frac{1}{2}})(x_1, x_2)]^2}{x_2^2} \, dx_2 \, dx_1 \\
&\le 2 \int_0^\ell \int_0^{2\delta} |\partial_{x_2} [(\varphi_{2\delta} f^2 J^{\frac{1}{2}})(x_1, x_2)]|^2 \, dx_2 \, dx_1.
\end{aligned}$$

Here, we used the classical Hardy inequality in the form,

$$\int_0^\infty \left| \frac{g(x_2)}{x_2} \right|^2 dx_2 \le 2 \int_0^\infty |g'(x_2)|^2 \, dx_2,$$

applied to $g(x_2) := (\varphi_{2\delta} f^2 J^{\frac{1}{2}})(x_1, x_2)$, which we can treat as zero for $x_2 \in [2\delta, \infty)$. Then,

$$g'(x_2) = (\varphi_{2\delta} \, \partial_2 f + \partial_2 \varphi_{2\delta} \, f) J^{\frac{1}{2}} + \varphi_{2\delta} f^2 \partial_2 (J^{\frac{1}{2}}).$$

Since J is C^∞ with $\partial_2 J$ bounded above and below, we can write

$$\partial_2 (J(x_1, x_2)^{\frac{1}{2}}) = h(x_1, x_2) J(x_1, x_2)^{\frac{1}{2}},$$

where h is bounded above by a constant that depends only upon Ω. Hence,

$$\begin{aligned}
|g'(x_2)|^2 &\le 2[(\varphi_{2\delta} \, \partial_2 f + \partial_2 \varphi_{2\delta} \, f)^2 + (\varphi_{2\delta} f \, h)^2] J \\
&\le [4(\varphi_{2\delta} \, \partial_2 f)^2 + 4(\partial_2 \varphi_{2\delta} \, f)^2 + 2(\varphi_{2\delta} f \, h)^2] J \\
&\le [4(\partial_2 f)^2 + C\delta^{-2} f^2 + 2f^2 h^2] J.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|f/x_2\|^2_{L^2(\Gamma_\delta^k)} &= 2 \int_{\Gamma_{2\delta}} [4(\partial_2 f)^2 + C\delta^{-2} f^2 + 2f^2 h^2] \\
&\le 8\|\partial_2 f\|^2_{L^2(\Gamma_{2\delta})} + C[1 + \delta^{-2}] \|f\|^2_{L^2(\Gamma_{2\delta})} \le C\|\partial_2 f\|^2_{L^2(\Gamma_{2\delta})},
\end{aligned}$$

by Lemma 2.6. □

Lemma 2.9. *Let $v \in H$ and let f be supported in Γ_δ, both v and f being smooth. With $\alpha_1(x_1, x_2) = x_2 \kappa'(x_1) J(x_1, x_2)$, as in Lemma 2.3,*

$$|(v^2, f)| \le C\delta \|v^1\|_{L^2(\Gamma_\delta)} \|\partial_1 f - \alpha_1 f\|_{L^2(\Gamma_\delta)}.$$

Proof. Because $v \in H$, it has a stream function ψ, meaning that $v = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$, with ψ constant on each boundary component. Write Σ_j, $j = 1, \dots, N$ for the N components of $\partial\Omega$ and Γ_δ^j for the component of Γ_δ whose outer boundary is Σ_j. Let c_j be the value of ψ on Σ_j. Define a smooth function ξ on Ω such that $\xi \equiv c_j$ on Γ_δ^j. Then on Γ_δ, $v = \nabla^\perp(\psi - \xi)$, so applying Lemmas 2.3 and 2.6,

$$\begin{aligned} |(v^2, f)| &= |(\partial_1(\psi - \xi), f)| = |(\psi - \xi, \partial_1 f - \alpha_1 f)| \le \|\psi - \xi\|_{L^2(\Gamma_\delta)} \|\partial_1 f - \alpha_1 f\|_{L^2(\Gamma_\delta)} \\ &\le C\delta \|\partial_2(\psi - \xi)\|_{L^2(\Gamma_\delta)} \|\partial_1 f - \alpha_1 f\|_{L^2(\Gamma_\delta)} = C\delta \|v^1\|_{L^2(\Gamma_\delta)} \|\partial_1 f - \alpha_1 f\|_{L^2(\Gamma_\delta)}. \end{aligned}$$

In the second inequality we used the vanishing of $\psi - \xi$ on $\partial\Omega$ and in the last equality we used that $\partial_2 \psi = -v^1$ while $\partial_2 \xi = 0$ in Γ_δ. □

3. Fully scalable correctors

Our model corrector is that used by Tosio Kato in [26], which is an example of what we will call a fully scalable corrector. Before stating precisely what we mean by this phrase, let us first describe Kato's corrector and give its key properties. We leave the detailed derivation of these estimates to Appendix B.

Kato's corrector. Let g be as in Lemma 1.3. We define Kato's corrector separately in each component of $\Gamma_{\overline{\delta}}$. Let

$$v := g - \overline{u}, \tag{3.1}$$

so that $\operatorname{div} v = 0$ and $v \cdot \boldsymbol{n} = 0$ on $\partial\Omega$; that is, $v \in H$. Then let ψ be the stream function for v, meaning that $v = \nabla^\perp \psi$, choosing ψ so that $\psi = 0$ on the given component of $\Gamma_{\overline{\delta}}$. Finally, define the corrector z as

$$z(x_1, x_2) = z_\delta(x_1, x_2) := \nabla^\perp(\varphi_\delta(x_2) \psi(x_1, x_2)), \tag{3.2}$$

where φ_δ is as in Definition 3.1:

Definition 3.1. *Define the cutoff function $\varphi\colon [0, \infty) \to [0, 1]$ to be a C^∞ function with $\varphi \equiv 1$ on $[0, 1/2]$ and $\varphi \equiv 0$ on $[1, \infty]$. Define $\varphi_\delta(\cdot) = \varphi(\cdot/\delta)$.*

Then z is supported in Γ_δ and

$$\operatorname{div} z = 0, \quad z = g - \overline{u} \text{ on } \partial\Omega, \quad z \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \tag{3.3}$$

(Actually, in [26], Kato used a matrix-valued M for which $v = \operatorname{div} M$, an approach that easily extends to higher dimension as well, as in [27, 30, 33]. In 3D, one could equivalently use $v = \operatorname{curl} \psi$, for a vector-valued stream function ψ vanishing on the boundary (for simply connected Ω), as developed, for instance, in [4, 64].)

Boundary layer width. Kato defined his corrector to have a support of width δ that was constant in time, shrinking only in viscosity. We will also allow δ to vary with time. For clarity, we make an explicit definition:

Definition 3.2. *Assume that either*

(1) $\delta = \delta(\nu)$ is continuous at $\nu = 0$ with $\delta(0) = 0$ or
(2) $\delta = \delta(t, \nu)$ is continuous at $\nu = 0$ with $\delta(t, 0) = 0$ and δ increasing in ν.

Remark 3.3. *Definition 3.2 (2) is a generalization of (1), though only when we assume that $\delta(0, \nu) = 0$ does it extend (1) in a meaningful way. Also, we do not assume in (2) any regularity of δ beyond continuity at $\nu = 0$. This will be sufficient to take time derivatives of δ, however, as we note in the derivation of (3.6), below. Although in practice one would typically choose δ to be increasing in ν, this is not strictly needed in (1).*

Remark 3.4. *As mentioned in Remark 2.1, we always assume that $\delta(\nu)$ or $\delta(t, \nu)$ lies in $(0, \min\{\overline{\delta}/2, 1\})$ without explicitly commenting on that fact. In practice, this means that ν must be sufficiently small, how small depending upon the choice of the δ function.*

Proposition 3.5. *Assume that δ is independent of time (though it may depend upon viscosity, for instance, as in Definition 3.2 (1)). We have the following estimates for the Kato corrector as defined in (3.2):*

$$\begin{aligned} &\|\partial_1^j \partial_2^k \partial_t^m z^1\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}-k}, \quad \|\partial_1^j \partial_2^k \partial_t^m z^2\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}+1-k}, \\ &\|z \cdot \nabla z\|_{L^p(\Omega)} \le C_z \delta^{\frac{1}{p}} \end{aligned} \tag{3.4}$$

for any $p \in [1, \infty]$, $j, k \ge 0$, $m = 0, 1$, any $t \in [0, T]$. The constants are independent of p and depend only upon the initial data, T, j, k, and m.

Let δ be as in Definition 3.2 (2). The estimates in (3.4) for $m = 0$ (no time derivative) continue to hold. We also have, for all $p \in [1, \infty]$ and $t \in [0, T]$,

$$\begin{aligned} &\|\partial_t z^1\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}} + C\partial_t \delta\, \delta^{\frac{1}{p}-1}, \quad \|\partial_t z^2\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}+1} + C\partial_t\delta\, \delta^{\frac{1}{p}}, \\ &\|\partial_t z\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}-1}(\delta + \partial_t \delta). \end{aligned} \tag{3.5}$$

Each of the constants above depend upon Ω, v, and T; in particular, they increase with T.

Proof. We defer the proof to Appendix B. □

Fully scalable corrector. We can now define what we mean by a fully scalable corrector, of which Kato's corrector is our prime example.

Definition 3.6. *A corrector is a vector field satisfying (3.3). We call a corrector fully scalable if it can be defined for any parameter $\delta > 0$, has support lying in the closure of $\Gamma_{c\delta}$ for c independent of δ, and satisfies the same bounds as those on the Kato corrector in Proposition 3.5.*

Remark 3.7. *An even simpler fully scalable corrector can be defined by $z = \nabla^\perp \alpha$, where*

$$\alpha = -\delta v^1(t, x_1, 0) f(x_2/\delta),$$

where f is any function in $C^\infty([0, \infty)$ chosen so that $f(0) = 0$, $f'(0) = 1$, and f supported in $[0, 1]$. Then $\operatorname{div} z = \operatorname{div} \nabla^\perp \alpha = 0$ and,

$$z = (-\partial_2 \alpha, \partial_1 \alpha) = (v^1(t, x_1, 0) f'(x_2/\delta), -J(x_1, x_2)\partial_1 v^1(t, x_1, 0) f(x_2/\delta)).$$

Then $z|_{\partial\Omega} = (v^1(t, x_1, 0), 0) = v|_{\partial\Omega}$ and z is supported in Γ_δ. Since α is product form (for a flat boundary only, because of the J factor), the estimates in Proposition 3.5 are as easily obtained as they for the Kato corrector. As we will see in Section 12.1, Wang employed this type of corrector in [65].

This corrector is one derivative less regular than that of Kato, which has no effect on our analysis, since we are assuming C^∞ initial data.

A few observations regarding fully scalable correctors are in order, as they will help guide our strategy in employing one:

(1) Because z is supported on a set of Lebesgue measure $C\delta$, the bounds in L^p for $p < \infty$ would follow from bounds in L^∞.

(2) Because z^2 vanishes on the boundary and grows linearly away from it, it is small compared to z^1, which is merely bounded.
(3) Derivatives in x_1 (tangential direction) are benign, having no effect on the estimates beyond changing values of constants, while each derivative in x_2 (normal direction) increases the bound by a factor of δ^{-1}.
(4) Time derivatives have no effect when δ is independent of time, and even when δ varies, they are benign as long as we integrate the estimates in time.

As an application of observation (4), the final bound in (3.5) gives

$$\begin{aligned}\int_0^t \|\partial_s z(s,\nu)\, ds\| &\le C\int_0^t \delta(s,\nu)^{\frac{1}{2}}\, ds + C\int_0^t \partial_s(\delta(s,\nu)^{\frac{1}{2}})\, ds \\ &\le Ct\delta(t,\nu)^{\frac{1}{2}} + C\left[\delta(t,\nu)^{\frac{1}{2}} - \delta(0,\nu)^{\frac{1}{2}}\right] \le C(1+t)\delta(t,\nu)^{\frac{1}{2}},\end{aligned} \tag{3.6}$$

where we used that $\delta(\cdot,\nu)$ is increasing. We also used that for any increasing function, $f\colon [a,b]\to\mathbb{R}$, $f'\ge 0$ exists almost everywhere, and

$$\int_a^b f'(s)\, ds \le f(b) - f(a).$$

The bound in (3.6), which we apply in (4.8), is the only bound on $\partial_t z$ that we will need.

Boundary vortex sheet. Let $\mathcal{M}(\overline{\Omega})$ be the space of finite Borel signed measures on $\overline{\Omega}$: $\mathcal{M}(\overline{\Omega})$ is the dual space of $C(\overline{\Omega})$. Let μ in $\mathcal{M}(\overline{\Omega})$ be the measure supported on Γ for which $\mu|_\Gamma$ corresponds to Lebesgue measure on Γ (arc length, since $d=2$). For any fully scalable corrector, we have a kind of convergence to a vortex sheet on the boundary in $H^1(\Omega)'$, as we show in Proposition 3.8. (Note that μ is also a member of $H^1(\Omega)'$.)

Proposition 3.8. *Let z be any fully scalable corrector. Assuming that δ is time-independent as in (1) of Definition 3.2,*

$$\operatorname{curl} z \to ((g-\overline{u})\cdot\boldsymbol{\tau})\mu \text{ in } H^1(\overline{\Omega})' \text{ uniformly on } [0,T] \text{ as } \nu\to 0.$$

Proof. Let $h\in H^1(\Omega)$. Then

$$\begin{aligned}(\operatorname{curl} z, h) &= -(\operatorname{div} z^\perp, h) = (z^\perp, \nabla h) - \int_{\partial\Omega}(z^\perp\cdot\boldsymbol{n})\, h = (z^\perp,\nabla h) + \int_{\partial\Omega}(z\cdot\boldsymbol{\tau})\, h \\ &\to ((g-\overline{u})\cdot\boldsymbol{\tau})\mu, h),\end{aligned}$$

since $|(z^\perp,\nabla h)| \le \|z\|\|\nabla h\| \to 0$ by (3.4) and $z = g-\overline{u}$ on $\partial\Omega$. □

Remark 3.9. *The space $H^1(\overline{\Omega})'$ is not a distribution space, so convergence in it must be used cautiously. Though it requires more effort to show, Kato's corrector also converges as a measure supported on the boundary, in the sense that*

$$\operatorname{curl} z \to ((g-\overline{u})\cdot\boldsymbol{\tau})\mu \text{ in } \mathcal{M}(\overline{\Omega}) \text{ uniformly on } [0,T] \text{ as } \nu\to 0.$$

Such convergence does not follow from being a fully scalable corrector, though it does hold for the corrector of Remark 3.7. Having such convergence should probably be viewed more as a limitation than an advantage of the corrector, for such strong convergence should not, in general, be expected of the difference, $u-\overline{u}$.

4. Kato's energy argument

The starting point for almost all of our analysis will be the energy inequality we obtain in Proposition 4.1 for

$$w := u - \overline{u}.$$

Proposition 4.1. *Make the assumption* (Ass_1) *of (1.9). Let* δ *be as in Definition 3.2 and let* z *be a fully scalable corrector as in Definition 3.6. Then*

$$\frac{1}{2}\|w(t)\|^2 + \frac{\nu}{2}\int_0^t \|\nabla w\|^2 = A(t,\nu) + B(t,\nu) + C\int_0^t \|w\|^2, \tag{4.1}$$

where

$$A(t,\nu) := -\int_0^t (u^1 u^2, \partial_2 z^1) + \nu \int_0^t (\nabla u, \nabla z) \tag{4.2}$$

and

$$B(t,\nu) \le C(1+t)\delta^{\frac{1}{2}}.$$

The constants C *depend upon* T*,* u^0*, and* g*, though not upon* $\nu \le 1$*.*

Proof. Recalling Remark 3.3, we will assume that $\delta = \delta(t,\nu)$ is time varying as in (2) of Definition 3.2.

Let

$$\widetilde{w} := w - z = u - \overline{u} - z,$$

and note that $\operatorname{div}\widetilde{w} = 0$ with $\widetilde{w} = 0$ on $\partial\Omega$. Observe that from (1.8) and Proposition 3.5, we know up front that at least

$$\|\widetilde{w}(t)\|, \|w(t)\| \le C(T)$$

for all $t \in [0,T]$.

Subtracting the Euler equations from the Navier-Stokes equations gives

$$\partial_t w + \nabla(p - \overline{p}) = \nu\Delta u - u\cdot\nabla w - w\cdot\nabla\overline{u}. \tag{4.3}$$

(Section 5.3 explains why we start with the equation for w rather than for $\widetilde{w}$.)

By Theorems 1.1 and 1.4 with Remark 1.5, u and $\overline{u}$ (and so z) have sufficient regularity that $\widetilde{w} \in L^2(0,T;V)$. Hence, we can use $\widetilde{w}$ as a test function for (NS_g) as in (1.7). This allows us to pair (4.3) with $\widetilde{w}$. Then, using

$$\begin{aligned}
(\partial_t w, \widetilde{w}) &= \frac{1}{2}\frac{d}{dt}\|w\|^2 - (\partial_t w, z),\\
\nu(\Delta u, \widetilde{w}) &= -\nu(\nabla u, \nabla\widetilde{w}) = -\nu(\nabla u, \nabla w) + \nu(\nabla u, \nabla z)\\
&= -\nu(\nabla w, \nabla w) - \nu(\nabla\overline{u}, \nabla w) + \nu(\nabla u, \nabla z)\\
&\le -\nu\|\nabla w\|^2 + \frac{\nu}{2}\|\nabla\overline{u}\|^2 + \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z)\\
&\le C\nu - \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z),\\
(\nabla(p-\overline{p}), \widetilde{w}) &= 0,\\
-(u\cdot\nabla w, \widetilde{w}) &= -(u\cdot\nabla w, w) + (u\cdot\nabla w, z) = (u\cdot\nabla w, z)\\
&= (u\cdot\nabla u, z) - (u\cdot\nabla\overline{u}, z) = -(u\cdot\nabla z, u) - (u\cdot\nabla\overline{u}, z)\\
&\le -(u\cdot\nabla z, u) + \|\nabla\overline{u}\|_{L^\infty}\|u\|\|z\|\\
&\le -(u\cdot\nabla z, u) + C\|z\| \le -(u\cdot\nabla z, u) + C\delta^{\frac{1}{2}},\\
-(w\cdot\nabla\overline{u}, \widetilde{w}) &= -(w\cdot\nabla\overline{u}, w) + (w\cdot\nabla\overline{u}, z)\\
&\le \|\nabla\overline{u}\|_{L^\infty}\left(\|w\|^2 + \|w\|\|z\|\right) \le C\|w\|^2 + C\delta^{\frac{1}{2}},
\end{aligned}$$

we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 \le (\partial_t w, z) + C\nu + C\delta^{\frac{1}{2}} + C\|w\|^2 - (u\cdot\nabla z, u) + \nu(\nabla u, \nabla z). \tag{4.4}$$

We now examine $-(u \cdot \nabla z, u)$. By virtue of Lemma 2.2, we can divide $-(u \cdot \nabla z, u)$ into parts as in [7], writing

$$\begin{aligned} -(u \cdot \nabla z, u) &= -((Ju^1 \partial_1 z^1 + u^2 \partial_2 z^1), u^1) - ((Ju^1 \partial_1 z^2 + u^2 \partial_2 z^2), u^2) \\ &= -(J\partial_1 z^1, (u^1)^2) - (\partial_2 z^1, u^1 u^2) - (J\partial_1 z^2, u^1 u^2) - (\partial_2 z^2, (u^2)^2). \end{aligned} \tag{4.5}$$

One term in (4.5) is easily bounded:

$$-(J\partial_1 z^2, u^1 u^2) \le C\|\partial_1 z^2\|_{L^\infty} \|u\|^2 \le C\delta.$$

For two of the other terms, we use that

$$w^i w^j = u^i u^j - \overline{u}^i u^j - u^i \overline{u}^j + \overline{u}^i \overline{u}^j \tag{4.6}$$

so that

$$u^i u^j = w^i w^j + \overline{u}^i u^j + u^i \overline{u}^j - \overline{u}^i \overline{u}^j.$$

Hence,

$$\begin{aligned} -(J\partial_1 z^1, (u^1)^2) &\le \|J\partial_1 z^1\|_{L^\infty} \|w\|^2 + 2\|\overline{u}\|_{L^\infty} \|J\partial_1 z^1\| \|u\| + \|\overline{u}\|_{L^\infty}^2 \|J\partial_1 z^1\|_{L^1} \\ &\le C\|w\|^2 + C\delta^{\frac{1}{2}} + C\delta \le C\delta^{\frac{1}{2}} + C\|w\|^2 \end{aligned}$$

and, since $\partial_2 z^2$ has the same bounds as those on $\partial_1 z^1$ above,

$$-(\partial_2 z^2, (u^2)^2) \le C\delta^{\frac{1}{2}} + C\|w\|^2.$$

We see, then, that

$$-(u \cdot \nabla z, u) \le C\delta^{\frac{1}{2}} + C\|w\|^2 - (u^1 u^2, \partial_2 z^1). \tag{4.7}$$

Returning to (4.4), then, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 \le (\partial_t w, z) + C\nu + C\delta^{\frac{1}{2}} + C\|w\|^2 - (u^1 u^2, \partial_2 z^1) + \nu(\nabla u, \nabla z).$$

Integrating in time and using (3.6), we have

$$\begin{aligned} \int_0^t (\partial_t w, z) &= \int_\Omega \int_0^t \partial_t w \cdot z = \int_\Omega \left[w(t) \cdot z(t) - \int_0^t w \partial_t z \right] \\ &\le \|w(t)\| \|z(t)\| + \int_0^t \|w\| \|\partial_t z\| \le C\|z(t)\| + C\int_0^t \|\partial_t z\| \le C\delta^{\frac{1}{2}}. \end{aligned} \tag{4.8}$$

Then,

$$\begin{aligned} &\frac{1}{2}\|w(t)\|^2 + \frac{\nu}{2}\int_0^t \|\nabla w\|^2 \\ &\quad \le C(1+t)\delta^{\frac{1}{2}} + C\nu t - \int_0^t (u^1 u^2, \partial_2 z^1) + \nu \int_0^t (\nabla u, \nabla z) + C\int_0^t \|w\|^2, \end{aligned}$$

which can be re-expressed in the form of (4.1). We used here that

$$\int_0^t \delta(s,\nu)^{\frac{1}{2}}\, ds \le \delta(t,\nu)^{\frac{1}{2}} t = \delta^{\frac{1}{2}} t,$$

since $\delta(s, \nu)$ is increasing in s. □

Proposition 4.1 leads to Theorem 4.3, which gives general necessary and sufficient criteria for the vanishing viscosity limit to hold. But we will need first the following lemma, also useful in its own right:

Lemma 4.2. *If $g \equiv 0$ and (1.3) holds then (1.1) holds. If (1.1) holds then*
$$\nu \int_0^T \|\nabla u\|^2,\ \nu \int_0^T \|\nabla w\|^2 \to 0 \text{ as } \nu \to 0.$$

Proof. First assume $g \equiv 0$. That (1.3) implies (1.1) is proved in [26] using only the energy *inequality* for the Navier-Stokes equations. The argument in 2D, where the energy equality holds is slightly simpler: We have, from (1.6),
$$\|u(t)\|^2 - \|\overline{u}(t)\|^2 + 2\nu \int_0^T \|\nabla u\|^2 = 0.$$
If (1.3) then $\|u(t)\|^2 - \|\overline{u}(t)\|^2 \to 0$, hence, $\nu \int_0^T \|\nabla u\|^2 \to 0$. But also $\nu \int_0^T \|\nabla \overline{u}\|^2 \to 0$, and we conclude that $\nu \int_0^T \|\nabla w\|^2 \to 0$. From this, (1.1) follows.

Now assume that (1.1) holds. Then $\nu \int_0^T \|\nabla w\|^2 \to 0$ as $\nu \to 0$ follows directly, and then
$$\nu \int_0^T \|\nabla u\|^2 \le \nu \int_0^T \|\nabla w\|^2 + \nu \int_0^T \|\nabla \overline{u}\|^2 \to 0. \qquad \square$$

Theorem 4.3. *Make the assumption (Ass_1) of (1.9). If there exists some δ as in Definition 3.2 (1) or (2) for which $A(\cdot, \nu) \to 0$ in $L^\infty([0,T])$ as $\nu \to 0$, with A as defined in (4.2), then the strong vanishing viscosity limit as in (1.1) holds.*

Conversely, if (1.1) holds (when $g \equiv 0$ we only require (1.3)) then $A(\cdot, \nu) \to 0$ in $L^\infty([0,T])$ as $\nu \to 0$ for any δ as in Definition 3.2 (1) or (2).

Furthermore, we can equivalently define $A = A_1^j + A_2^k$, $j, k \in \{1, 2\}$, where
$$\begin{aligned} A_1^1 &:= -\int_0^t (u^1 u^2, \partial_2 z^1), & A_1^2 &:= -\int_0^t (u \cdot \nabla z, u) \\ A_2^1 &:= \nu \int_0^t (\nabla u, \nabla z), & A_2^2 &:= \nu \int_0^t (\operatorname{curl} u, \operatorname{curl} z). \end{aligned} \tag{4.9}$$
Finally, we can add to A either
$$a_1 \nu \int_0^t \|\nabla u\|^2 + a_2 \nu \|w\|^2 \text{ or } a_1 \nu \int_0^t \|\nabla w\|^2 + a_2 \nu \|w\|^2 \tag{4.10}$$
for any $a_1 < \frac{1}{2}$ and any $a_2 \in \mathbb{R}$ without affecting the conclusions of the theorem.

Remark 4.4. *The function δ appears implicitly in this theorem through A, which contains the δ-dependent corrector, z.*

Proof of Theorem 4.3. Assume that $A(\cdot, \nu) \to 0$ in $L^\infty([0,T])$ as $\nu \to 0$, with A as defined in (4.2), for some choice of δ as in Definition 3.2. Applying Gronwall's inequality to (4.1), we conclude that
$$\frac{1}{2}\|w(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla w\|^2 \le \left[\|A(\cdot, \nu)_{L^\infty([0,T])}\| + C(1+t)t\delta^{\frac{1}{2}} + C\nu t^2 \right] e^{Ct},$$
which vanishes as $\nu \to 0$ since $\delta(\nu) \to 0$ or $\delta(t, \nu) \to 0$ as $\nu \to 0$. This gives (1.1).

Either of the terms in (4.10) can be added to A since they can be absorbed in the energy inequality in (4.1).

Conversely, assume that the vanishing viscosity limit holds. Then by Lemma 4.2, we know that $\left(t \mapsto \nu \int_0^t \|\nabla w\|^2\right) \to 0$ in $L^\infty([0,T])$ as $\nu \to 0$. For *any* δ as in Definition 3.2, $B(\cdot, \nu) \to 0$ in $L^\infty([0,T])$, with B as in Proposition 4.1, since $\delta(\nu) \to 0$ or $\delta(t, \nu) \to 0$ as $\nu \to 0$. This leaves only the term $A(\cdot, \nu)$ in (4.1), which therefore must vanish as $\nu \to 0$ as well.

Note also that the terms in (4.10) also vanish if (1.3) holds by Lemma 4.2.

The equivalence of A_1^1 and A_1^2 follow from the bounds on the term $-(u \cdot \nabla z, u)$ in the proof of Proposition 4.1. For the equivalence of A_2^1 and A_2^2, we apply Lemma 2.4, which gives

$$\nu(\nabla u, \nabla z) = \nu(\operatorname{curl} u, \operatorname{curl} z) + \nu \int_{\partial\Omega} (\operatorname{curl}(z)(z \cdot \boldsymbol{\tau}) - \kappa z \cdot u).$$

Then,

$$\nu \int_{\partial\Omega} (\operatorname{curl}(z)(z \cdot \boldsymbol{\tau}) - \kappa z \cdot u) = -\nu \int_{\partial\Omega} (\operatorname{curl} \overline{u}((g - \overline{u}) \cdot \boldsymbol{\tau}) - \kappa(g - \overline{u}) \cdot g),$$

which is bounded by $C\nu$, since $\operatorname{curl} \overline{u}$, g, and $\overline{u}$ are each bounded independently of ν on the boundary. Hence, A_2^1 and A_2^2 are interchangeable. □

Remark 4.5. *Since the converse in Theorem 4.3 holds for any δ it follows that so, too, does the forward direction of the theorem in the sense that if $A(\cdot, \nu) \to 0$ in $L^\infty([0,T])$ for one choice of δ then A vanishes in the same manner for any other choice of δ. (All δ's must be as in Definition 3.2, of course.) A priori, however, the forward direction is stronger with "there exists δ" rather than "for all δ."*

Remark 4.6. *Lemma 2.2 gives* $\operatorname{curl} z = J\partial_1 z^2 - \partial_2 z^1 + \kappa J z^1$. *Now, $\|J\partial_1 z^2\|_{L^\infty} \le C\delta$, and when the boundary is flat, there is no $\kappa J z^1$ term (and $J \equiv 1$). We seen, then, that in a half-plane or a periodic channel, A_1^1 and A_1^2 are also equivalent to*

$$A_1^3 := -\int_0^t (u^1 u^2, \operatorname{curl} z).$$

It is not clear how to effectively bound $\kappa J z^1$ with a curved boundary, however, making the equivalence of A_1^3 uncertain in that case.

5. Boundary layer widths

In applying Theorem 4.3, the key is the control of the two terms A_1^j and A_2^k, as in (4.9), that make up A, regardless of which form is used. The term A_1^j originates in the convective or nonlinear terms in the Navier-Stokes and Euler equations, A_2^k from the effect of the boundary on the viscous term in the Navier-Stokes equations. Either term can be controlled individually: Without the convective term we have the Stokes equation (the Euler equations becoming steady) and the vanishing viscosity limit holds as shown, for instance, in [16]. Without the boundary, the vanishing viscosity limit holds as shown in many contexts ([51, 24, 25, 8, 44], for instance). Ideally, one could handle the combined effect of these terms, but no such technique is currently available. We have little choice, then, but to handle the two terms separately.

Thus, if we wish to establish a sufficient condition for the vanishing viscosity limit to hold, we require that

$$\int_0^T (u^1 u^2, \partial_2 z^1) \to 0 \text{ as } \nu \to 0 \text{ and} \tag{5.1}$$

$$\nu \int_0^T (\nabla u, \nabla z) \to 0 \text{ as } \nu \to 0. \tag{5.2}$$

5.1. **Kato layer.** In his seminal paper [26], Tosio Kato chose to set (with $g \equiv 0$) $\delta = C\nu$. In this case, (5.1) and (5.2) are both critical in the sense that they can be shown to be bounded by the basic energy inequality for the Navier-Stokes equations, but the energy inequality is insufficient to show that these integrals vanish with viscosity. Kato shows that both of these conditions can be replaced by

$$\nu \int_0^T \|\nabla u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0.$$

Following in this same spirit, [29] gives two other ways to find a common condition that applies to (5.1) and (5.2). These are the conditions in (6.1) and (6.2) that we discuss in Section 6, along with an improvement that comes from dividing $(u \cdot \nabla z, u)$ as in [7].

Definition 5.1. *We call the boundary layer, $\Gamma_{C\nu}$, the Kato (boundary) layer and $C\nu$ the Kato width or scaling.*

5.2. **Wang layer.** Alternately, we can allow δ to be infinitesimally larger than ν, though still vanishing as $\nu \to 0$. This approach, in the full generality in which we will use it (except for being time-independent), was first taken by Xiaoming Wang in [65] (see [60] for an earlier, less general version of this idea). We define it as follows:

Definition 5.2. *Let δ be as in Definition 3.2 (2) with the additional property that*

$$\int_0^T \frac{\nu}{\delta(s,\nu)}\, ds \to 0 \text{ as } \nu \to 0. \tag{5.3}$$

The resulting boundary layer, Γ_δ, we call a Wang (boundary) layer and such a δ we call a Wang width or scaling.

If, like a Wang layer, the corrector has width larger than that of Kato then (5.2) follows very easily (see the proof of Theorem 8.1). This is because the factor of ν in (5.2) came from the diffusion term in the Navier-Stokes equations, while the bound on ∇z improves as δ increases. This leaves only the condition in (5.1) or an equivalent condition to be treated. Alternately, if the width is narrower than that of Kato, then (5.1) is easily controlled; this would seem to be of no advantage, however, since even for the linearized fluid equations, (5.2) would not be controllable with such a width.

5.3. **Using the corrected difference.** In (4.1), as well as in (1.1), the gradient of the uncorrected difference, w, appears, not the corrected difference, $\widetilde{w}$. For the Kato layer, one cannot obtain convergence with the corrected difference. This is because we know from Kato's original conditions in [26] (for no-slip conditions) that if (1.3) holds then $\nu \int_0^t \|\nabla u\|^2 \to 0$. Then because $\overline{u} \in C^1(Q)$, we also have $\nu \int_0^t \|\nabla \overline{u}\|^2 \to 0$. But the inequality, $\|\nabla z\| \le C\delta^{-\frac{1}{2}}$ is easily seen to be tight, serving also as a lower bound. Hence, if (1.3) holds then asymptotically for small ν,

$$\nu \int_0^t \|\nabla \widetilde{w}\|^2 \sim C\frac{\nu}{\delta} t.$$

Hence, an energy inequality obtained using $\nabla \widetilde{w}$ in place of $\widetilde{w}$ is not possible for the Kato layer, where $\delta = C\nu$, or any smaller layer. It is possible, however, for a Wang layer, as is, in fact, done in [65]. It is also possible for inflow, outflow boundary conditions, as we see in [53, 17], though there other issues arise.

6. Using the Kato layer

The use of the Kato layer of width proportional to ν leads naturally to Theorem 6.1, the result for (6.1) and (6.2) (for $g \equiv 0$) appearing in [29].

Theorem 6.1. *Make the assumption* (Ass_1) *of (1.9). The strong vanishing viscosity limit in (1.1) holds if*

$$\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0 \text{ or} \tag{6.1}$$

$$\frac{1}{\nu} \int_0^t \|u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0, \text{ or} \tag{6.2}$$

or, if (Ass_2) *of (1.9) holds,*

$$\frac{1}{\nu} \int_0^t \int_{\Gamma_\nu} ((u^1)^2 + |u^1 u^2|) \to 0 \text{ as } \nu \to 0. \tag{6.3}$$

As a partial converse, if (1.1) holds (or simply (1.3) when $g \equiv 0$*) then (6.1) holds, as do*

$$\begin{aligned} &\frac{1}{\nu} \int_0^t \|u - g\|^2_{L^2(\Gamma_\nu)} \to 0 && \text{as } \nu \to 0, \\ &\frac{1}{\nu} \int_0^t \int_{\Gamma_\nu} ((u^1 - g^1)^2 + |(u^1 - g^1) u^2|) \to 0 && \text{as } \nu \to 0. \end{aligned} \tag{6.4}$$

Proof. We prove first the partial converse. The simple bound,

$$\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} \le C\nu \int_0^t \|\nabla u\|^2_{L^2(\Gamma_\nu)} \le C\nu \int_0^t \|\nabla u\|^2,$$

shows the necessity of (6.1).

For the necessity of $(6.4)_1$, we have

$$\frac{1}{\nu} \int_0^t \|u - g\|^2_{L^2(\Gamma_\nu)} \le \frac{1}{\nu} \int_0^t C\nu^2 \|\partial_2 (u - g)\|^2_{L^2(\Gamma_\nu)} \le C\nu \int_0^t \|\nabla u\|^2 + C\nu \int_0^t \|\nabla g\|^2.$$

We applied Lemma 2.6, using that $u - g$ vanishes on $\partial\Omega$. The two terms on the right vanish by Lemma 4.2 and by the independence of ∇g on ν.

For the necessity of $(6.4)_2$, we write,

$$\begin{aligned} (u^1 - g^1)^2 + |(u^1 - g^1) u^2| &\le (u^1 - g^1)^2 + |(u^1 - g^1)(u^2 - g^2)| + |(u^1 - g^1) g^2| \\ &\le 2|u - g|^2 + \frac{(g^2)^2}{2}, \end{aligned}$$

where we used Young's inequality. Then the necessity of $(6.4)_2$ follows from the necessity of $(6.4)_1$ and the bound,

$$\frac{1}{2\nu} \int_0^t \|g^2\|^2_{L^2(\Gamma_\nu)} \le \frac{1}{2\nu} \int_0^t C\nu^2 \|\partial_2 g^2\|^2_{L^2(\Gamma_\nu)} \le C\nu.$$

Here, we were able to apply Lemma 2.6, because g^2 vanishes on $\partial\Omega$.

For the sufficiency of the conditions, it is clear that (6.2) implies (6.3). It remains, then, to show the sufficiency of (6.1) and (6.3).

First assume (6.1). With A_1^2, A_2^2 as in (4.9), we bound A_1^2 by

$$\begin{aligned} |A_1^2| &= \left| \int_0^t (u \cdot \nabla z, u) \right| = \left| \int_0^t (u \cdot \nabla u, z) \right| = \left| \int_0^t (u^\perp \operatorname{curl} u, z) \right| \\ &\le \|z\|_{L^\infty([0,T] \times \Omega)} \int_0^t \|u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\ &\le C\nu \int_0^t \|\nabla u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} + C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}. \end{aligned}$$

In the second equality we applied Lemma 2.5 to exchange ∇u for $\operatorname{curl} u$, and in the last inequality we applied Corollary 2.7.

For the first term,

$$\begin{aligned}
C\nu \int_0^t \|\nabla u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} &\le C \left(\nu \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 \, ds\right)^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}} \\
&\le C(T) \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}}.
\end{aligned}$$

In the last inequality we applied the energy inequality in (1.8). Also,

$$C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \le t^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}},$$

so

$$\Big| \int_0^t (u \cdot \nabla z, u) \Big| \le C(T) \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}}.$$

We then bound A_2^2 by

$$\begin{aligned}
|A_2^2| = \nu \Big| \int_0^t (\operatorname{curl} u, \operatorname{curl} z) \Big| &\le \nu \int_0^t \|\nabla z\| \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\
&\le C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \le Ct^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Then (1.1) follows from Theorem 4.3.

Now assume (6.3). Integrating by parts using (2.2) and applying Lemma 2.2, we see that

$$\begin{aligned}
(\nabla u, \nabla z) &= -(u, \Delta z) + \int_{\partial\Omega} (\nabla z \cdot \boldsymbol{n}) g \\
&= -(u, J^2 \partial_1^2 z) - (u^1, \partial_2^2 z^1) - (u^2, \partial_2^2 z^2) + (u, \kappa J \partial_2 z) \\
&\qquad - (u, x_2 \kappa' J^3 \partial_1 z) + \int_{\partial\Omega} (\nabla z \cdot \boldsymbol{n}) g.
\end{aligned}$$

To bound Δz, here, we required (Ass_2). Using Proposition 3.5, we have

$$\begin{aligned}
&- \nu(u, J^2 \partial_1^2 z) \le C\nu \|u\| \|\partial_1^2 z\| \le C\nu\nu^{\frac{1}{2}} = C\nu^{\frac{3}{2}}, \\
&- \nu(u^2, \partial_2^2 z^2) \le \nu \|u\| \|\partial_2^2 z^2\| \le C\nu\nu^{-\frac{1}{2}} = C\nu^{\frac{1}{2}}, \\
&\nu\,(u, \kappa J \partial_2 z) \le C\nu \|u\| \|\partial_2 z\| \le C\nu\nu^{-\frac{1}{2}} = C\nu^{\frac{1}{2}}, \\
&- \nu(u, x_2 \kappa' J^3 \partial_1 z) \le C\nu \|u\| \|\partial_1 z\| \le C\nu\nu^{\frac{1}{2}} = C\nu^{\frac{3}{2}}. \\
&\nu \int_{\partial\Omega} (\nabla z \cdot \boldsymbol{n}) g \le C\nu.
\end{aligned}$$

Therefore, we can write

$$A(t, \nu) = f(t, \nu) - \int_0^t \left((u^1 u^2, \partial_2 z^1) + \nu(u^1, \partial_2^2 z^1)\right),$$

where $f(\cdot, \nu) \to 0$ in $L^\infty(0, T; L^2(\Omega))$ as $\nu \to 0$. But, applying Proposition 3.5 with $\delta = \nu$,

$$-(u^1 u^2, \partial_2 z^1) \le \int_{\Gamma_\nu} \|\partial_2 z^1\|_{L^\infty} |u^1 u^2| \le \int_{\Gamma_\nu} \frac{C}{\nu} |u^1 u^2| \tag{6.5}$$

and

$$\begin{aligned}\nu\Big|\int_0^t (u^1, \partial_2^2 z^1)\Big| &\le C\nu \int_0^t \|u^1\|_{L^2(\Gamma_\nu)} \|\partial_2^2 z^1\| \le \frac{C}{\sqrt{\nu}} \int_0^t \|u^1\|_{L^2(\Gamma_\nu)} \\ &\le C \left(\int_0^t 1\right)^{\frac{1}{2}} \left(\frac{1}{\nu} \int_0^t \|u^1\|^2_{L^2(\Gamma_\nu)}\right)^{\frac{1}{2}}.\end{aligned}$$

Then (1.1) follows from Theorem 4.3. □

We might hope to extend Kato's conditions and the Kato-like conditions in Theorem 6.1 to use a layer of width νt. We should expect the effect of the initial layer of vorticity forming at the boundary to take some time to move into the fluid, so the width of the layer should increase with time. The heat equation solution depends only upon νt with simple geometries for instance (though its weak boundary layer is of "width" $\sqrt{\nu t}$), so such a scaling would seem reasonable. It is not, however, possible.

To see this, let us consider the condition,

$$\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_{\nu s})}\, ds \to 0 \text{ as } \nu \to 0 \tag{6.6}$$

in place of (6.1). Certainly this is a necessary condition, being weaker than the condition in (6.1). To adapt the proof of sufficiency of (6.1) above, we need only change the width of the layer. Note that this brings powers of the time into the time integrals. For bounding the convective term in A, we find (including only the key steps) that

$$\begin{aligned}\Big|\int_0^t (u\cdot\nabla z, u)\Big| &\le \int_0^t \|u\|_{L^2(\Gamma_{\nu s})} \|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})} \|z\|_{L^\infty}\, ds \\ &\le C\int_0^t \nu s \|\nabla u\|_{L^2(\Gamma_{\nu s})} \|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})}\, ds + C\nu^{\frac{1}{2}} \int_0^t s^{\frac{1}{2}} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\ &\le Ct\left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2([0,T];L^2(\Gamma_{\nu s}))}\, ds\right)^{\frac{1}{2}}.\end{aligned}$$

Here, Poincare's inequality via Corollary 2.7 brings an additional factor of s into the integral, which we bound above by t and bring outside the integral. The end result is a harmless additional factor of t.

The boundary term, however, has a significant problem. To see this, let us treat this term for a general δ as in Definition 3.2, a bound we will find useful later in the proof of Theorem 8.1. We have, using A_2^2,

$$\begin{aligned}|A_2^2| &= \nu\Big|\int_0^t (\operatorname{curl} u, \operatorname{curl} z)\Big| \le \nu \int_0^t \|\operatorname{curl} z\| \|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})}\, ds \\ &\le C\nu \int_0^t \frac{\|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})}}{\delta(s,\nu)^{\frac{1}{2}}}\, ds \le C\left(\int_0^t \frac{\nu}{\delta(s,\nu)}\, ds\right)^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_{\nu t})}\right)^{\frac{1}{2}}.\end{aligned} \tag{6.7}$$

So the first time integral above must at least be finite for $A(t,\nu)$ to have a chance to vanish with ν. When $\delta(s,\nu) = \nu s$, however, the integral is infinite.

In estimating the convective term, we integrated by parts in the first step, removing the gradient on $z = z_\delta$ ($\delta = \nu$ or νs, here). The estimate for $\|z_\delta\|_{L^\infty}$ is independent of δ, so this simply leads to an additional factor of t in the estimate. There appears to be no way to avoid leaving at least part of the derivative on z in estimating the boundary term, however; in particular, $\partial_1 z^2$, which dominates ∇z, seems unavoidable.

It is clear from these estimates that for any $\alpha \in [0,1)$ we could use a boundary layer of width νt^α in (6.1), replacing that condition with

$$\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_{\nu s^\alpha})}\, ds \to 0 \text{ as } \nu \to 0,$$

in place of (6.1), though such a boundary layer would fail for (6.2) and (6.3).

7. A little more with Kato's layer

In [65], Wang gives necessary and sufficient conditions for the vanishing viscosity limit to hold based upon the magnitude of the tangential derivatives of either the tangential components of the velocity or of the normal component of the velocity. The penalty is that the boundary layer considered must be infinitesimally larger than that of Kato (as in (5.3)). We discuss [65] in detail in Section 8, but first we derive in a simpler manner a result using Kato's original boundary layer. The conditions required are stronger (less satisfactory as a sufficient condition) than those of [65] in that they each involve a derivative normal to the boundary. They apply, however, to the thinner boundary layer of Kato.

Theorem 7.1. *Make the assumption* (Ass_1) *of (1.9). If*

(1) $\nu \int_0^T \|\partial_2 u\|^2_{L^2(\Gamma_\nu)} = \nu \int_0^T \|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^2\|^2_{L^2(\Gamma_\nu)} \to 0$ *as* $\nu \to 0$

or

(2) $\nu \int_0^T \|\nabla u^1\|^2_{L^2(\Gamma_\nu)} = \nu \int_0^T \|\partial_1 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} \to 0$ *as* $\nu \to 0$

then the strong vanishing viscosity limit in (1.1) holds. Conversely, if (1.1) holds (or simply (1.3) when $g \equiv 0$*) then* (1) *and* (2) *hold.*

Proof. First observe that (1) and (2) are equivalent since u is divergence-free, so by Lemma 2.2, $\partial_2 u^2 = -J\partial_1 u^1 + \kappa J u^2$, and $\nu\|\kappa J u^2\|_{L^2(\Gamma_\nu)} \le C\nu$.

That (1.3) $\implies$ (1), (2) follows from Lemma 4.2.

For the forward implications, assume (1). We will apply Theorem 4.3 to A using A_1^1.

Setting $\delta = \nu$, we have,

$$\begin{aligned}
|A_1^1| &= |(u^1 u^2, \partial_2 z^1)| \le \|\partial_2 z^1\|_{L^\infty} \|u^1\|_{L^2(\Gamma_\nu)} \|u^2\|_{L^2(\Gamma_\nu)} \\
&\le \frac{C}{\nu}\left(\nu\|\partial_2 u^1\|_{L^2(\Gamma_\nu)} + \nu^{\frac12}\right)\nu\|\partial_2 u^2\|_{L^2(\Gamma_\nu)} \\
&= C\nu\|\partial_2 u^1\|_{L^2(\Gamma_\nu)}\|\partial_2 u^2\|_{L^2(\Gamma_\nu)} + C\nu^{\frac12}\|\partial_2 u^2\|_{L^2(\Gamma_\nu)} \\
&\le C\nu\left(\|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^2\|^2_{L^2(\Gamma_\nu)}\right) + C\nu^{\frac12}\|\partial_2 u^2\|_{L^2(\Gamma_\nu)},
\end{aligned} \tag{7.1}$$

where we used Corollary 2.7.

Letting $f_1(x_1, x_2) = J$, $f_2(x_1, x_2) = 1$, we can use Lemma 2.2 to write

$$\begin{aligned}
-\nu(\nabla u, \nabla z) &= -\nu f_i \partial_i z^j f_j \partial_i u^j \le \nu \sum_{(i,j)\ne(2,1)} \|f_i\partial_i z^j\| \|f_i \partial_i u^j\|_{L^2(\Gamma_\nu)} + \nu\|\partial_2 z^1\|\|\partial_2 u^1\|_{L^2(\Gamma_\nu)} \\
&\le C\nu\nu^{\frac12}\|\nabla u\| + C\nu\nu^{-\frac12}\|\partial_2 u^1\|_{L^2(\Gamma_\nu)} \le C\nu + \frac{\nu^2}{2}\|\nabla u\|^2 + C\nu^{\frac12}\|\partial_2 u^1\|_{L^2(\Gamma_\nu)}.
\end{aligned}$$

Integrating in time, we have

$$\begin{aligned}
A(t,\nu) \le C\nu \int_0^t \left(\|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^2\|^2_{L^2(\Gamma_\nu)}\right) + C\nu^{\frac12}\int_0^t \|\partial_2 u^2\|_{L^2(\Gamma_\nu)} \\
+ C\nu t + \frac{\nu}{2}\left(\nu\int_0^t \|\nabla u\|^2\right) + C\nu^{\frac12}\int_0^t \|\partial_2 u^1\|_{L^2(\Gamma_\nu)}
\end{aligned}$$

$$\leq C\nu \int_0^t \sum_{j=1}^2 \|\partial_2 u^j\|^2_{L^2(\Gamma_\nu)} + C(T)\nu + \sum_{j=1}^2 t^{\frac{1}{2}} \left(\nu \int_0^t \|\partial_2 u^j\|^2_{L^2(\Gamma_\nu)} \right)^{\frac{1}{2}},$$

where we used (1.8). The assumption (1) insures that $A(t, \nu) \to 0$ as $\nu \to 0$, which gives (1.1) by Theorem 4.3. □

8. Using a Wang layer

Theorem 4.3 applied to a Wang layer easily yields sufficient conditions for the vanishing viscosity limit to hold for such a layer, leading to Theorem 8.1.

Theorem 8.1. *Make the assumption* (Ass_1) *of (1.9). Let* δ *be a Wang width as in Definition 5.2. If*

$$\int_0^t \int_{\Gamma_\delta} \frac{1}{\delta} |u^1 u^2| \to 0 \text{ or } \int_0^t ((u^1 u^2, \partial_2 z^1) \to 0 \text{ as } \nu \to 0 \tag{8.1}$$

then (1.1) holds.

Proof. Since (8.1) holds, it follows from (6.7) that $\nu \int_0^t |(\operatorname{curl} u, \operatorname{curl} z)| \to 0$ as $\nu \to 0$. (Note that since $\delta(\cdot, \nu)$ is increasing, $\delta(\cdot, \nu) \to 0$ in $L^\infty(0, T)$.) Hence, by Theorem 4.3, if the vanishing viscosity limit holds then the second condition in (8.1) holds. But (6.5) shows that the second condition in (8.1) is bounded by the first condition; hence if either condition in (8.1) holds then the vanishing viscosity limit holds. □

A simple and direct use of a Wang layer yields Theorem 8.2.

Theorem 8.2. *Make the assumption* (Ass_1) *of (1.9). Let* δ *be a Wang width as in Definition 5.2. If*

$$\frac{1}{\nu} \int_0^t \|u^1\|^2_{L^2(\Gamma_\delta)} \to 0 \text{ or } \frac{1}{\nu} \int_0^t \|u^2\|^2_{L^2(\Gamma_\delta)} \to 0 \text{ as } \nu \to 0 \tag{8.2}$$

then (1.1) holds.

Proof. We have,

$$\begin{aligned} |(u^1 u^2, \partial_2 z^1)| &\leq \|\partial_2 z^1\|_{L^\infty} \|u^1 u^2\|_{L^1(\Gamma_\delta)} \leq \frac{C}{\delta} \|u^1 u^2\|_{L^1(\Gamma_\delta)} \leq \frac{C}{\delta} \|u^1\|_{L^2(\Gamma_\delta)} \|u^2\|_{L^2(\Gamma_\delta)} \\ &\leq \frac{C}{\delta} \|u^1\|_{L^2(\Gamma_\delta)} C\delta \|\partial_2 u^2\|_{L^2(\Gamma_\delta)} = C \|u^1\|_{L^2(\Gamma_\delta)} \|\partial_1 u^1\|_{L^2(\Gamma_\delta)}, \end{aligned}$$

where we used $(2.3)_2$ of Corollary 2.7. Hence,

$$\begin{aligned} \int_0^t (u^1 u^2, \partial_2 z^1) &\leq C \left(\int_0^t \|u^1\|^2_{L^2(\Gamma_\delta)} \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_1 u^1\|^2_{L^2(\Gamma_\delta)} \right)^{\frac{1}{2}} \\ &= C \left(\nu^{-1} \int_0^t \|u^1\|^2_{L^2(\Gamma_\delta)} \right)^{\frac{1}{2}} \left(\nu \int_0^t \|\partial_1 u^1\|^2_{L^2(\Gamma_\delta)} \right)^{\frac{1}{2}}. \end{aligned}$$

The second factor on the right-hand side is bounded by (1.6) or (1.8). The result for the first condition in (8.2) thus follows from Theorem 8.1.

For the second condition in (8.2), we interchange the roles of u^1 and u^2, which we see gives

$$\begin{aligned} |(u^1 u^2, \partial_2 z^1)| &\leq \frac{C}{\delta} \|u^2\|_{L^2(\Gamma_\delta)} C \left(\delta \|\partial_2 u^1\|_{L^2(\Gamma_\delta)} + \delta^{\frac{1}{2}} \right) \\ &= C \|u^2\|_{L^2(\Gamma_\delta)} \|\partial_2 u^1\|_{L^2(\Gamma_\delta)} + C\delta^{-\frac{1}{2}} \|u^2\|_{L^2(\Gamma_\delta)}. \end{aligned}$$

Hence,

$$\int_0^t (u^1u^2, \partial_2 z^1) \le C\left(\nu^{-1}\int_0^t \|u^2\|^2_{L^2(\Gamma_\delta)}\right)^{\frac{1}{2}}\left(\nu\int_0^t \|\partial_1 u^2\|^2_{L^2(\Gamma_\delta)}\right)^{\frac{1}{2}} + C\int_0^t \frac{\|u^2\|_{L^2(\Gamma_\delta)}}{\delta(s,\nu)^{\frac{1}{2}}}\,ds$$
$$\le C\left(\frac{1}{\nu}\int_0^t \|u^2\|^2_{L^2(\Gamma_\delta)}\right)^{\frac{1}{2}} + C\left(\int_0^t \frac{\nu}{\delta(s,\nu)}\,ds\right)^{\frac{1}{2}}\left(\frac{1}{\nu}\int_0^t \|u^2\|_{L^2(\Gamma_\delta)}\,ds\right)^{\frac{1}{2}},$$

which vanishes by the second condition in (8.2). □

Long version only: A more subtle use of the infinitesimally thicker boundary layer leads to the result of Xiaoming Wang [65] in Theorem 8.4, below. Though we allow a time-varying boundary layer, we restrict ourselves, as does Wang, to a 2D channel periodic in the x_1 direction (cf. Remark 8.7.) Hence, x_1 and x_2 reduce to Cartesian coordinates (though with opposite orientation), the usual formula for the divergence holds, and there is no lower-order term when integrating by parts, as there is in Lemma 2.3.

The proof of Theorem 8.4 is based upon the following estimates:

Lemma 8.3. *Assume that Ω is a 2D channel periodic in the x_1 direction. Let δ as in Definition 3.2 be a the width of a boundary layer. Then*

$$|(u^1u^2, \partial_2 z^1)| \le \frac{\nu}{4}\|\nabla u\|^2_{L^2(\Gamma_\delta)} + \frac{C\nu}{\delta} + C\left(\frac{\delta}{\nu}\right)^2\left(\nu\|\partial_1 u^1\|^2_{L^2(\Gamma_\delta)}\right) \tag{8.3}$$

and

$$|(u^1u^2, \partial_2 z^1)| \le \frac{\nu}{4}\|\nabla u\|^2_{L^2(\Gamma_\delta)} + C\|w\|^2 + C\delta^{\frac{1}{2}} + \left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}} + C\left(\frac{\delta}{\nu}\right)^4\left(\nu\|\partial_1 u^2\|^2_{L^2(\Gamma_\delta)}\right), \tag{8.4}$$

Proof. To prove (8.3), we start with (6.5):

$$|(u^1u^2, \partial_2 z^1)| \le \frac{C}{\delta}\int_{\Gamma_\delta} |u^1u^2| \le C\delta^{-1}\|u^1\|_{L^2(\Gamma_\delta)}\|u^2\|_{L^2(\Gamma_\delta)}$$
$$\le \frac{C}{\delta}\left(\delta\|\partial_2 u^1\|_{L^2(\Gamma_\delta)} + \delta^{\frac{1}{2}}\right)\delta\|\partial_2 u^2\|_{L^2(\Gamma_\delta)}$$
$$= C\nu^{\frac{1}{2}}\|\partial_2 u^1\|_{L^2(\Gamma_\delta)}\frac{\delta}{\nu^{\frac{1}{2}}}\|\partial_1 u^1\|_{L^2(\Gamma_\delta)} + C\frac{\delta}{\nu^{\frac{1}{2}}}\left(\frac{\nu}{\delta}\right)^{\frac{1}{2}}\|\partial_1 u^1\|_{L^2(\Gamma_\delta)}$$
$$\le \frac{\nu}{4}\|\partial_2 u^1\|^2_{L^2(\Gamma_\delta)} + C\frac{\delta^2}{\nu}\|\partial_1 u^1\|^2_{L^2(\Gamma_\delta)} + C\frac{\nu}{\delta},$$

where we paralleled the argument in (7.1), but using $\partial_2 u^2 = -\partial_1 u^1$ and applying Young's inequality asymmetrically.

The proof of (8.4) is more involved. We first make the decomposition,

$$-(u^1u^2, \partial_2 z^1) = (u^1\partial_2 u^2, z^1) + (\partial_2 u^1 u^2, z^1),$$

where we integrated by parts, using that $u^2 = 0$ on $\partial\Omega$. For the first term in $-(u^1u^2, \partial_2 z^1)$, we use that $\operatorname{div} u = 0$ to obtain

$$(u^1\partial_2 u^2, z^1) = -(u^1\partial_1 u^1, z^1) = -\frac{1}{2}(\partial_1(u^1)^2, z^1) = \frac{1}{2}((u^1)^2, \partial_1 z^1)$$
$$= \frac{1}{2}((w^1)^2, \partial_1 z^1) + (u^1\overline{u}^1, \partial_1 z^1) - \frac{1}{2}((\overline{u}^1)^2, \partial_1 z^1),$$

where, since we integrated by parts in the tangential variable, we needed no boundary condition. Hence,

$$\begin{aligned}|(u^1\partial_2 u^2, z^1)| &\le \frac{1}{2}\|w\|^2\|\partial_1 z^1\|_{L^\infty} + \|u\|\|\overline{u}\|_{L^\infty}\|\partial_1 z^1\| + \frac{1}{2}\|\overline{u}\|_{L^\infty}\|\overline{u}\|\|\partial_1 z^1\| \\ &\le C\|w\|^2 + C\delta^{\frac{1}{2}}.\end{aligned}$$

For the second term in $-(u^1u^2, \partial_2 z^1)$, we have

$$|(\partial_2 u^1 u^2, z^1)| \le \|\partial_2 u^1\|_{L^2(\Gamma_\delta)}\|u^2 z^1\|. \tag{8.5}$$

Defining β by

$$\beta(t, x_1, x_2) := -\int_{x_2}^{\delta(t,\nu)} (z^1(t, x_1, y))^2\, dy, \tag{8.6}$$

we see that

$$\partial_2\beta = (z^1)^2$$

and

$$\begin{aligned}\|\beta\|_{L^\infty(\Gamma_\delta)} &\le \delta\|z^1\|_{L^\infty}^2 \le C\delta, \\ \|\partial_1\beta\|_{L^\infty(\Gamma_\delta)} &\le \delta\|\partial_1 z^1\|_{L^\infty}^2 \le C\delta.\end{aligned}$$

Then,

$$\begin{aligned}\|u^2z^1\|^2 &= \int_{\Gamma_\delta}(u^2)^2(z^1)^2 = \int_{\partial\Omega}\int_0^\delta (u^2(t,x_1,x_2))^2\partial_{x_2}\beta(t,x_1,y)\,dx_2\,dx_1 \\ &= -\int_{\partial\Omega}\int_0^\delta \partial_{x_2}(u^2(t,x_1,x_2))^2\beta(t,x_1,x_2)\,dx_2\,dx_1 \\ &= -\int_{\Gamma_\delta}\partial_2(u^2)^2\beta = -2\int_{\Gamma_\delta}u^2\partial_2u^2\beta = 2\int_{\Gamma_\delta}u^2\partial_1u^1\beta \\ &= -2\int_{\Gamma_\delta}u^1\partial_1(u^2\beta) = -2\int_{\Gamma_\delta}u^1\partial_1u^2\beta - 2\int_{\Gamma_\delta}u^1u^2\partial_1\beta.\end{aligned}$$

In both integrations by parts, we used that $u^2 = 0$ on $\partial\Omega$, the outer component of $\partial\Gamma_\delta$, while $\beta = 0$ on the inner component of $\partial\Gamma_\delta$.

Proceeding,

$$-2\int_{\Gamma_\delta}u^1u^2\partial_1\beta \le 2\|u^1\|\|u^2\|\|\partial_1\beta\|_{L^\infty(\Gamma_\delta)} \le C\delta^2\|\partial_2u^2\|_{L^2(\Gamma_\delta)},$$

$$-2\int_{\Gamma_\delta}u^1\partial_1u^2\beta \le 2\|u^1\|\|\partial_1u^2\|_{L^2(\Gamma_\delta)}\|\beta\|_{L^\infty(\Gamma_\delta)} \le C\delta^2\|\partial_2u^1\|_{L^2(\Gamma_\delta)}\|\partial_1u^2\|_{L^2(\Gamma_\delta)}.$$

Thus,

$$\|u^2z^1\| \le C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}}\left(\|\partial_1u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}} + 1\right) \qquad \le C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}}\left(\|\partial_1u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}} + 1\right)$$

and therefore,

$$\begin{aligned}|(\partial_2u^1u^2, z^1)| &\le \|\partial_2u^1\|_{L^2(\Gamma_\delta)}\|u^2z^1\| \le C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}}\left(\|\partial_1u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}} + 1\right) \\ &= C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}}\|\partial_1u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}} + C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}}.\end{aligned}$$

Applying Young's inequality,

$$
\begin{aligned}
C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}}\|\partial_1 u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}} &= C\left(\nu^{\frac{3}{4}}\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}}\right)\left(\frac{\delta}{\nu}\nu^{\frac{1}{4}}\|\partial_1 u^2\|_{L^2(\Gamma_\delta)}^{\frac{1}{2}}\right) \\
&\le \frac{\nu}{8}\|\nabla u\|_{L^2(\Gamma_\delta)}^2 + \frac{\delta^4}{\nu^4}\left(\nu\|\partial_1 u^2\|_{L^2(\Gamma_\delta)}^2\right)
\end{aligned}
$$

and

$$
C\delta\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}} = C\frac{\delta}{\nu^{\frac{1}{4}}}\nu^{\frac{1}{4}}\|\nabla u\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}} \le \frac{\nu}{8}\|\nabla u\|_{L^2(\Gamma_\delta)}^2 + C\left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}}.
$$

Collecting these bounds gives (8.4). □

Theorem 8.4. *[Wang [65]] Assume that Ω is a 2D channel periodic in the x_1 direction. Let δ be a Wang width as in Definition 5.2 with*

$$
\nu\int_0^T \|\partial_1 u^1\|_{L^2(\Gamma_\delta(s,\nu))}^2\,ds \to 0 \text{ as } \nu \to 0 \tag{8.7}
$$

or

$$
\nu\int_0^T \|\partial_1 u^2\|_{L^2(\Gamma_\delta(s,\nu))}^2\,ds \to 0 \text{ as } \nu \to 0. \tag{8.8}
$$

Then the strong vanishing viscosity limit in (1.1) holds. Conversely, if (1.1) holds (or simply (1.3) when $g \equiv 0$) then (8.7) and (8.8) hold for any Wang width.

Proof. For each of (8.7) and (8.8), the converse follows immediately from Lemma 4.2 or the assumption in (1.1), which implies (1.4).

For the forward direction, we know that (5.2) holds simply because δ is a Wang width (see the comment following Definition 5.2). It remains to show that (5.1) holds, for it will follow that $A \to 0$ as in Theorem 4.3.

Assume, first, that (8.7) holds. Integrating (8.3) over time gives

$$
\int_0^T |(\partial_2 u^1 u^2, z^1)| \le \frac{\nu}{4}\int_0^t \|\nabla u\|_{L^2(\Gamma_\delta)}^2 + C\nu\int_0^t \frac{ds}{\delta(s,\nu)} + C\frac{\delta^2}{\nu^2}F_\nu(\delta) \tag{8.9}
$$

(we used here that $\delta(s,\nu) \le \delta(t,\nu)$), where

$$
F_\nu(t,\delta) := \nu\int_0^t \|\partial_1 u^1\|_{L^2(\Gamma_\delta)}^2.
$$

Note that even in 2D, we cannot say that $F_\nu(t,\delta)$ is increasing in ν even for *fixed* δ; we would be hard pressed even to show that it is continuous.

Let us agree to call the function δ for which the condition in (8.7) is assumed to hold, δ_0; this means that we are given that $F_\nu(t,\delta_0(t,\nu)) \to 0$ as $\nu \to 0$. We will show that there exists a possibly smaller Wang width, which we will relabel δ, for which $\frac{\delta^2(t,\nu)}{\nu^2}F_\nu(t,\delta(t,\nu)) \to 0$ as $\nu \to 0$.

As long as $\delta \le \delta_0$ (as functions of ν), we will have

$$
\frac{\delta^2}{\nu^2}F_\nu(t,\delta) \le \frac{\delta^2}{\nu^2}F_\nu(t,\delta_0).
$$

So let

$$
\delta(t,\nu) = \min\left\{\delta_0(t,\nu),\ \inf_{s\in[t,T]}\frac{\nu}{F_\nu(s,\delta_0(s,\nu))^{\frac{1}{4}}}\right\}, \tag{8.10}
$$

which we note is continuous at $\nu = 0$ with $\delta(t, 0) = 0$, and is increasing in t. Then,

$$\frac{\nu}{\delta(t,\nu)} \le \max\left\{\frac{\nu}{\delta_0(t,\nu)}, F_\nu(t,\delta_0(t,\nu))^{\frac{1}{4}}\right\} \to 0,$$

$$\begin{aligned}\frac{\delta(t,\nu)^2}{\nu^2}F_\nu(t,\delta(t,\nu)) &\le \frac{\delta(t,\nu)^2}{\nu^2}F_\nu(t,\delta_0(t,\nu)) \le \frac{\frac{\nu^2}{\sqrt{F_\nu(t,\delta_0(t,\nu))}}}{\nu^2}F_\nu(t,\delta_0(t,\nu)) \\ &= \sqrt{F_\nu(t,\delta_0(\nu))} \to 0\end{aligned}$$

as $\nu \to 0$, and the convergence is uniform in time. Also,

$$\int_0^T \frac{\nu}{\delta(t,\nu)}\,dt \le \max\left\{\int_0^T \frac{\nu}{\delta_0(t,\nu)}\,dt, \int_0^T F_\nu(\delta_0(t,\nu))^{\frac{1}{4}}\,dt\right\}.$$

As $\nu \to 0$, the first integral on the right-hand side vanishes because δ_0 is a Wang width, while the second integral vanishes because $F_\nu(\delta_0(t,\nu)) \le F_\nu(\delta_0(T,\nu)) \to 0$. Hence, we see that δ is a Wang width, so we can apply Theorem 4.3 to the bound in (8.9) using (4.10) to conclude that (1.1) holds.

Now assume that (8.8) holds. Integrating (8.4) over time, we have

$$\int_0^T |(\partial_2 u^1 u^2, z^1)| \le \frac{\nu}{4}\int_0^T \|\nabla u\|^2_{L^2(\Gamma_\delta)} + C\left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}} T + C\frac{\delta^4}{\nu^4}\int_0^T \left(\nu\|\partial_1 u^2\|^2_{L^2(\Gamma_\delta)}\right).$$

We can absorb the first term above by virtue of (4.10), and, if needed, we can always decrease δ to be less than $\nu^{\frac{1}{4}}$ while still keeping the conditions in (5.3) and in Definition 3.2 (2), insuring that the second term above vanishes with ν. The final term we treat in the same manner as we treated the final term in (8.9), writing it in the form, $C\frac{\delta^4}{\nu^4}F_\nu(\delta)$, where now

$$F_\nu(\delta) := \nu\int_0^T \|\partial_1 u^2\|^2_{L^2(\Gamma_\delta)}.$$

Applying Theorem 4.3 using (4.10) to conclude that (1.1) holds, the proof of sufficiency of (8.8) is complete. □

Remark 8.5. *The construction in (8.10) is a little easier to understand when δ is time-independent. We set*

$$\delta(\nu) = \min\left\{\delta_0(\nu), \frac{\nu}{F_\nu(T,\delta_0(\nu))^{\frac{1}{4}}}\right\}.$$

Then δ is continuous at zero with $\delta(0) = 0$. Then since $F_\nu(T,\delta_0(\nu)) \to 0$ by assumption, $\delta(\nu)$ is a Wang width, and

$$\begin{aligned}\frac{\delta(\nu)^2}{\nu^2}F_\nu(t,\delta(\nu)) &\le \frac{\delta(\nu)^2}{\nu^2}F_\nu(t,\delta_0(\nu)) \le \frac{\frac{\nu^2}{\sqrt{F_\nu(t,\delta_0(\nu))}}}{\nu^2}F_\nu(t,\delta_0(\nu)) \\ &= \sqrt{F_\nu(t,\delta_0(\nu))} \le \sqrt{F_\nu(T,\delta_0(\nu))} \to 0.\end{aligned}$$

Remark 8.6. *In [65], Wang uses an energy argument that starts with the equation for what we are calling $\widetilde{w}$ (rather than w, as we did) then multiplies by $\widetilde{w}$ and integrates over time and space. The introduction of F_ν and the use of β, which are at the heart of the proof, are adopted from [65]. Also, Wang uses a different corrector (see Section 12.1 and Remark 3.7), though all the necessary estimates hold for the Kato corrector we are using.*

Remark 8.7. *If we allowed a curved boundary in Theorem 8.4, the lower-order terms in Lemmas 2.2 and 2.3 would lead to the additional term,*

$$I_2 := 2(u^1u^2, (-\kappa(x_2) + \alpha_2)\beta) + ((u^2)^2, \alpha_2\beta),$$

in the estimate of $\|u^2 z^1\|^2$ in the proof of Lemma 8.3. This, in turn would add the additional term,

$$\|\partial_2 u^1\|_{L^2(\Gamma_\delta)} |I_2|^{\frac{1}{2}}$$

to the bound in (8.4). The "trick" of forcing $w = u - \overline{u}$ to appear in this estimate, as was done to obtain (4.6), is of no use here, and the direct estimate,

$$|I_2| \le C\|u\|^2 \|\beta\|_{L^\infty(\Gamma_\delta)} \le C\delta$$

is also insufficient. A more subtle definition of the function β in (8.6) perhaps might circumvent this difficulty, but that issue we leave unexplored.

9. Vortex sheet on the boundary

As in in Proposition 3.8, let μ be arc length measure. We have the following simple extension of a result in [30]:

Theorem 9.1. *Make the assumption (Ass_1) of (1.9). Assume that Ω is simply connected and δ is time-independent, as in (1) of Definition 3.2. The following conditions are equivalent:*

(1) (1.1) holds,
(2) $\omega \to \overline{\omega} + ((g - \overline{u}) \cdot \boldsymbol{\tau})\mu$ in $(H^1(\Omega))'$ uniformly on $[0, T]$,
(3) $\omega \to \overline{\omega}$ in $H^{-1}(\Omega)$ uniformly on $[0, T]$.

Proof. The proof of this theorem for $g \equiv 0$ is given in [30]. Its proof for a general g requires only the trivial replacement of $\overline{u}$ by $\overline{u} - g$ in the arguments in [30]. Note that the presence or absence of an energy defect as in (1.4) does not affect the arguments in [30]. (In some sense, this is because a corrector is not employed in [30].) □

In [12, 13] it is shown that for radially symmetric initial vorticity in a disk, *(2)* of Theorem 9.1 holds in the more classical sense of a vortex sheet, in that

$$\omega \to \overline{\omega} + ((g - \overline{u}) \cdot \boldsymbol{\tau})\mu \text{ in } \mathcal{M}(\overline{\Omega}) \text{ uniformly on } [0, T]. \tag{9.1}$$

The following gives a simple condition for this type of convergence to hold:

Theorem 9.2. *Make the assumption (Ass_1) of (1.9). Let z be the Kato corrector. The convergence in (9.1) holds if and only if $\omega - \overline{\omega} - \operatorname{curl} z \to 0$ in $\mathcal{M}(\overline{\Omega})$ uniformly on $[0, T]$, and both hold if $\omega - \operatorname{curl} z \to \overline{\omega}$ in $L^\infty(0, T; L^1(\Omega))$.*

Proof. Let $\varphi \in C(\overline{\Omega})$. Then by Remark 3.9,

$$(\operatorname{curl} z, \varphi) \to \int_{\partial\Omega} ((g - \overline{u}) \cdot \boldsymbol{\tau})\varphi \text{ uniformly on } [0, T],$$

meaning that $\operatorname{curl} z \to ((g - \overline{u}) \cdot \boldsymbol{\tau})\mu$ in $\mathcal{M}(\overline{\Omega})$ uniformly on $[0, T]$. Hence, convergence in (9.1) holds if and only if $\omega - \overline{\omega} - \operatorname{curl} z \to 0$ in $\mathcal{M}(\overline{\Omega})$ uniformly on $[0, T]$.

Now assume that $\omega - \overline{\omega} - \operatorname{curl} z \to 0$ in $L^\infty(0, T; L^1(\Omega))$. Then

$$|(\omega - \overline{\omega} - \operatorname{curl} z, \varphi)| \le \|\omega - \overline{\omega} - \operatorname{curl} z\|_{L^1} \|\varphi\|_{L^\infty} \to 0$$

uniformly over time, meaning that $\omega - \overline{\omega} - \operatorname{curl} z \to 0$ in $\mathcal{M}(\overline{\Omega})$ uniformly on $[0, T]$. □

10. Well-posedness of (NS_g)

We now give the proof of Lemma 1.3 and use it to prove the existence of solutions to (NS_g), Theorem 1.4. We return to writing u_g rather than simply u, as we did in Sections 4 to 9.

Proof of Lemma 1.3. For any fixed time $t \in [0, \infty)$ let $(\overline{g}(t), q(t))$ solve the stationary Stokes problem,

$$\begin{cases} \Delta \overline{g}(t) = \nabla q(t) & \text{in } \Omega, \\ \operatorname{div} \overline{g}(t) = 0 & \text{in } \Omega, \\ \overline{g}(t) = g(t) & \text{on } \partial\Omega. \end{cases}$$

It follows from Theorem IV.6.1 part (a) of [15] that $\overline{g} \in L^2(0, \infty; H \cap H^2(\Omega)^2)$. We see also that $\partial_t \overline{g}$ satisfies the stationary Stokes problem, $\Delta \partial_t \overline{g}(t) = \nabla \partial_t q(t), \operatorname{div} \partial_t \overline{g}(t) = 0$ in Ω, $\partial_t \overline{g}(t) = \partial_t g(t)$ on $\partial\Omega$, so from Theorem IV.6.1 part (b) of [15] we have $\partial_t \overline{g} \in L^2(0, \infty; H \cap H^1(\Omega)^2)$.

If, in addition, $u^0|_{\partial\Omega} = g(0)$, then $\overline{g} + u^0 - \overline{g}(0) \in C^\infty([0, \infty) \times \overline{\Omega})$, is divergence-free, equals g on $\partial\Omega$ and equals u^0 at time zero.

Relabeling by setting $g = \overline{g}$ or $g = \overline{g} + u^0 - \overline{g}(0)$ completes the proof. □

Proof of Theorem 1.4. With g as in Lemma 1.3, we can rewrite (NS_g) as

$$\partial_t r + \partial_t g + r \cdot \nabla r + r \cdot \nabla g + g \cdot \nabla r + g \cdot \nabla g + \nabla p_g = \nu \Delta r + \nu \Delta g, \tag{10.1}$$

where $r := u_g - g$, noting that $r = 0$ on $\partial\Omega$. Hence, we look for a weak solution to

$$\begin{cases} \partial_t r + r \cdot \nabla r + r \cdot \nabla g + g \cdot \nabla r + \nabla p_g = \nu \Delta r + F_g & \text{on } \Omega, \\ \operatorname{div} r = 0 & \text{on } \Omega, \\ r(0) = u^0 - g(0) & \text{on } \Omega, \\ r = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.2}$$

We define the weak solution by pairing $(10.2)_1$ with a test function $\varphi \in \mathcal{V} = V \cap C_C^\infty(\Omega)^2$. As in the discussion following (V.7) in [5], and Proposition V.1.3 of [5], we can, equivalently, use a test function in $\varphi \in L^2(0, T; V)$. Transforming back to $u_g = r + g$ leads to (1.7).

This is a linear perturbation of the Navier-Stokes equations with the forcing term, F_g. Existence and, in 2D, uniqueness, is standard (see, for instance, [23], where a similar perturbation is worked out in detail). This leads to $r \in C([0, T]; H) \cap L^2(0, T; V)$ with $\partial_t r \in L^2(0, T; V')$, giving the stated membership in function spaces of $u_g = r + g$ and $\partial_t u_g = \partial_t r + \partial_t g$.

Applying (1.7) with $\varphi = r \in L^2(0, T; V)$ is equivalent to pairing $(10.2)_1$ with r. This give

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|r\|^2 + \nu \|\nabla r\|^2 &= -(r \cdot \nabla g, r) + (F_g, r) \\ &\le \|\nabla g\|_{L^\infty} \|r\|^2 + \|F_g\| \|r\| \le \frac{\|F_g\|^2}{2} + \left(\|\nabla g\|_{L^\infty} + \frac{1}{2} \right) \|r\|^2 \end{aligned}$$

so that

$$\frac{d}{dt} \|r\|^2 + 2\nu \|\nabla r\|^2 \le \|F_g\|^2 + (2\|\nabla g\|_{L^\infty} + 1) \|r\|^2.$$

Integrating in time, we see that

$$\|r(t)\|^2 + 2\nu \int_0^t \|\nabla r\|^2 \le \|r(0)\|^2 + \int_0^t \|F_g\|^2 + \int_0^t (2\|\nabla g\|_{L^\infty} + 1) \|r\|^2.$$

Applying Gronwall's lemma gives

$$\|r(t)\|^2 + 2\nu \int_0^t \|\nabla r\|^2 \le \left(\|r(0)\|^2 + \int_0^t \|F_g\|^2 \right) e^{\int_0^t (2\|\nabla g\|_{L^\infty} + 1)}. \tag{10.3}$$

Using (10.3) with $\|r(0)\|^2 \le 2\|u^0\|^2 + 2\|g_0\|^2$ and

$$\|u_g(t)\|^2 + 2\nu \int_0^t \|\nabla u_g\|^2 \le 2 \left(\|r(t)\|^2 + 2\nu \int_0^t \|\nabla r\|^2 + \|g(t)\|^2 + 2\nu \int_0^t \|\nabla g\|^2 \right)$$

yields the bound in (1.8). □

11. How might convergence happen?

11.1. **A very special case.** If we choose to set $g = \overline{u}|_{\partial\Omega}$, we see that $u_g = u_{\overline{u}} = \overline{u}$ on $\partial\Omega$. This eliminates the boundary term in the basic energy argument, giving $u_{\overline{u}} \to \overline{u}$ as in the boundary-free case (though only in the $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$-norm, not in higher norms, since there is still no control of vorticity production of $u_{\overline{u}}$ on the boundary). Thus, we easily obtain Theorem 11.1.

Theorem 11.1. *We have*

$$u_{\overline{u}} \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

with

$$\|u_{\overline{u}}(t) - \overline{u}(t)\| \le C\nu e^{Ct}, \qquad \int_0^t \|\nabla(u_{\overline{u}}(s) - \overline{u}(s))\|^2 \, ds \le C\nu t^{\frac{1}{2}} e^{Ct}.$$

Proof. We can use $g = \overline{u}$ on all of Ω in constructing the corrector z—this gives $z \equiv 0$, as we can see from (3.2). But then any of various conditions in Theorem 4.3 give the vanishing viscosity limit. (Or one can make a direct energy argument, since the boundary integral disappears. It is easier to obtain the convergence rate that way.) □

A simple corollary of Theorem 11.1 is the following:

Corollary 11.2. *We have,*

$$u_g \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \nu \to 0$$

if and only if

$$u_g - u_{\overline{u}} \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \nu \to 0.$$

Proof. By the triangle inequality,

$$\begin{aligned} \|u_g - \overline{u}\| &\le \|u_g - u_{\overline{u}}\| + \|u_{\overline{u}} - \overline{u}\|, \\ \|u_g - u_{\overline{u}}\| &\le \|u_g - \overline{u}\| + \|u_{\overline{u}} - \overline{u}\|, \end{aligned}$$

and the result follows from Theorem 11.1. □

Now consider the issue of the convergence of $u_g - u_{\overline{u}}$ to 0. Let $w = u_g - u_{\overline{u}}$. Then

$$(\partial_t w, w) + (w \cdot \nabla u_{\overline{u}}, w) + (u_g \cdot \nabla w, w) + (\nabla(p - q), w) = \nu(\Delta w, w).$$

The third and fourth terms on the left-hand side vanish after integrating by parts. We integrate the right-hand side by parts to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu \int_0^T \|\nabla w\|^2 &= -(w \cdot \nabla u_{\overline{u}}, w) + \nu \int_{\partial\Omega} (\nabla w \cdot \boldsymbol{n}) \cdot w \\ &= -(w \cdot \nabla u_{\overline{u}}, w) - \nu \int_{\partial\Omega} (\nabla w \cdot \boldsymbol{n}) \cdot (g - \overline{u}). \end{aligned}$$

Now, to obtain convergence we need control both on $\nabla u_{\overline{u}}$ in something close to $L^1(0,T;L^\infty)$, as well as control on the boundary term. So proving $u_g - u_{\overline{u}} \to 0$ appears to be even more difficult than proving $u_g \to \overline{u}$.

11.2. **Speculations and a conjecture (long version only).** Moving into the realm of speculation, consider the following opposed possibilities:

- *Positive*: The vanishing viscosity limit in (1.1) holds for all smooth u_0 and smooth g.
- *Negative*: The vanishing viscosity limit in (1.1) fails to hold for generic u_0 and g.

The qualification "generic" is not meant in any precise technical way, but is to rule out, for instance, initial data for which $\overline{u}$ vanishes on the boundary or which has some degree of analyticity.

Whether one or the other of these possibilities holds (they are not exhaustive, so neither may hold) is related to the question, "Is the solution to (NS) at low viscosity indifferent to the boundary value g, or is it sensitive to it?" Indifference would support the positive possibility, sensitivity would support the negative (or at least non-positive) possibility. We can give some support for each position:

> *Indifferent*: As $\nu \to 0$, the imposition of $u = g$ on the boundary should become less important, since as the fluid becomes less viscous, the boundary forcing should have less effect on it, so less vorticity should be shed off the boundary and transported into the bulk of the fluid. Nonetheless, there is enough shedding of vorticity for a vortex sheet to form at the boundary. Indeed, this is shown to be the case for radially symmetric solutions in [12, 13], and is likely the case for other scenarios in which the non-linearity is weakened or eliminated (though such examples do not seem to have been worked out explicitly in the literature, since $g = 0$ is generally assumed).
>
> *Sensitive*: Theorem 9.1 tells us that when $g = \overline{u}|_{\partial\Omega}$, the shedding of vorticity off the boundary is shut down, no vortex sheet forms on the boundary, and the vanishing viscosity limit holds. On the other hand, if $g \not\equiv \overline{u}|_{\partial\Omega}$ then a vortex sheet must form on the boundary for the vanishing viscosity limit to hold. But perhaps as a vortex sheet begins to form, there is an underlying physical mechanism that pushes back against the convergence of the velocities, and hence also against the continued formation of the vortex sheet.

The key difficulty with using Theorem 9.1 as evidence for or against the vanishing viscosity limit is that its proof is simply a mathematical observation not based on any underlying physical mechanism. And the convergence of the vortex sheet in Theorem 9.1 is weak-$*$ in a non-distribution space, complicating even its mathematical interpretation. An avenue of exploration here is to try to determine whether some stronger type of convergence is compatible with the vanishing viscosity limit. We know that convergence of the vorticity as a finite Borel measure is compatible in certain cases by [12, 13] and we know that convergence in L^p for $p > 1$ is incompatible by [32]. But the former result is for very specialized initial data, and the second result is simply based upon the need for the L^p norms of the Navier-Stokes vorticity to blow up as $\nu \to 0$. Again, no (deep) physical mechanism is involved.

Weaker than either of the two positions is the following conjecture:

Conjecture 1. *Generically, (1.1) holds for u_0 if and only if (1.1) holds for any function $g \in (C^\infty([0,T] \times \partial\Omega))^d$ with $g \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.*

This conjecture is saying, in effect, that except in very special circumstances, the vanishing viscosity limit can hold only if the indifferent position is correct, though it takes no position on whether the vanishing viscosity limit holds generically at all. A motivation for this conjecture is that, as we have seen, the form of the Kato and Kato-like conditions are all indifferent to the choice of g (for those involving derivatives of the velocity fields; for those involving the velocity fields directly, we naturally subtract g).

11.3. **An initial layer only.** In studying the vanishing viscosity limit for no-slip boundary conditions, one often assumes compatible initial data, meaning that (at least) u^0 vanishes on the boundary. This eliminates the added complication of dealing with an initial layer due to incompatible data, putting the focus on the nature of the development of layers for positive time as vorticity is shred from the boundary.

But we can do just the opposite, working only with an initial layer by considering the special case where $u^0 \equiv 0$, so $\overline{u} \equiv 0$ is a (steady) solution to the Euler equations. There is an incompatibility in the boundary conditions for (NS_g) at time zero when $g \not\equiv 0$, so the solution to the Navier-Stokes equations does not vanish. This leads to a special case of the vanishing viscosity limit not included in the classical setting (where $g \equiv 0$ would trivialize to $u_0 \equiv \overline{u} \equiv 0$).

There are only two possibilities:

- *Positive*: $u_g \to 0$ as $\nu \to 0$ for all smooth g.
- *Negative*: there exists smooth g such that $u_g \not\to 0$ as $\nu \to 0$.

A route to a positive answer would be to find a more optimum bound on the energy of u_g than that in (1.8), one that would lead to $\|u_g(t)\| \to 0$ as $\nu \to 0$. But this is entirely equivalent, as we can see from Theorem 4.3, to obtaining a bound on $A(t, \nu)$ that insures it vanishes with ν. Even in simple geometries such as a disk with constant $g \cdot \boldsymbol{\tau}$, then, and even in this simplified form, resolving the vanishing viscosity limit question seems out of reach.

To gain a little insight, though, let us consider a linearized version of (NS_g) in which we drop the term $u_g \cdot \nabla u_g$ in (NS_g): that is, the time-dependent Stokes problem, $\partial_t u_g + \nabla p_g = \nu \Delta u_g$. We will assume, however, that g is time-independent.

We begin by making the same energy argument as in the proof above of Theorem 1.4, but instead of using g itself, we use a "corrector," z. We define z as in Section 3, using $v = g$ in place of (3.1), and with δ to be chosen below. (Hence, the corrector is "correcting" only the boundary value of g.) We can see from Lemma 1.3 and Proposition 3.5 that

$$\|z\| \le C\delta^{\frac{1}{2}}, \quad \nu \|\nabla z\|^2 \le C\frac{\nu}{\delta}.$$

Set $r = u_g - z$ and choose $\delta = \nu^{1/2}$. Because $\partial_t z$ vanishes, in place of (10.1) we have

$$\partial_t r + \nabla p_g = \nu \Delta r + \nu \Delta z.$$

Multiplying by r and integrating over the domain, we have

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 + \nu\|\nabla r\|^2 = \nu(\nabla z, \nabla r) \le \frac{\nu}{2}\|\nabla z\|^2 + \frac{\nu}{2}\|\nabla r\|^2.$$

We conclude that

$$\frac{d}{dt}\|r\|^2 + \nu\|\nabla r\|^2 \le \nu\|\nabla z\|^2 \le C\frac{\nu}{\delta}.$$

Integrating in time, we see that

$$\|r(t)\|^2 + \nu\int_0^t \|\nabla r\|^2 \le \|r(0)\|^2 + C\frac{\nu}{\delta}t = \|z\|^2 + C\frac{\nu}{\delta}t \le C\delta + C\frac{\nu}{\delta}t \le C(t\nu)^{\frac{1}{2}},$$

where in the last step we chose $\delta = (\nu t)^{\frac{1}{2}}$ to balance the two terms. From Grönwall's lemma, then, $u_g \to \overline{u} \equiv 0$ in $L^\infty([0, T]; L^2)$ as $\nu \to 0$.

Hence, for this linearized problem, at least in the special case in which the boundary data is constant in time, we obtain the positive possibility. Of course, this linear situation should not dominate our intuition: the question is whether the nonlinear, convective term disrupts this linear behavior sufficiently to obtain a negative answer.

12. On correctors

A fully scalable corrector, such as Kato's, which we used to obtain all our results, corrects only for the value of $u - \overline{u} = g - \overline{u} = v$ (see (3.1)) on the boundary, while being of a size in the boundary layer, as measured by certain key norms, that allows at least conditional control of each term in the resulting energy argument. In this regard, we view it as a purely *size-based* corrector, meeting what are pretty much the minimal requirements for any usable corrector.

Another approach to obtaining a corrector is to start with the equation satisfied by the difference between a solution and its intended limiting value—$u - \overline{u}$ in our case—and reduce the complexity of the equation by performing formal asymptotics based on assuming certain scaling laws, themselves typically based on (unproven) physical assumptions. Often, an approximate, but explicit solution to the corrector equation is used as the actual corrector.

This approach originates in the work of Prandtl [46], who did not, however, express it in terms of a corrector, but rather by performing formal asymptotics derived by scaling a thin boundary layer; an approach to such problems using a corrector was pioneered by Vishik and Ljusternik [63, 62] (in a linear setting).

There are many correctors in the literature for problems closely related to our own. We restrict ourselves here to a brief discussion of those used to treat the vanishing viscosity limit, primarily for no-slip boundary conditions for the full or linearized Navier-Stokes equations in the spirit of Kato.

Correctors may differ, but they cannot differ too much in size in the $L^\infty(0, T; L^2)$ and $L^2(0, T; H^1)$ norms if they are to be used to investigate the vanishing viscosity limit in (1.1). It is primarily the hope that the structure of some given corrector might more closely match the underlying physical problem for certain situations, however, that motivates the choice of correctors not exclusively based on size.

In defining the correctors in the subsections that follow, we define $v = g - \overline{u}$, as in (3.1) and let

$$U(t, x_1) := \overline{u}^1(t, x_1, 0) - g^1(t, x_1) = -v^1(t, x_1, 0).$$

We note that, like Kato's corrector, all of these correctors satisfy Proposition 3.8.

12.1. **Wang's corrector in** [65]**.** Let $\rho \in C^\infty([0, \infty))$ taking values in $[-1, 1]$ satisfy $\rho(0) = 1$, $\rho'(0) = 0$, $\operatorname{supp} \rho \subseteq [0, 1]$, $\int_0^1 \rho = 0$, and $|\rho'| \le 2$. Working with a flat boundary (a periodic channel), define

$$\alpha = U(t, x_1) \int_0^{x_2} \rho\left(\frac{s}{\delta}\right) ds = \delta U(t, x_1) \int_0^{\frac{x_2}{\delta}} \rho(s)\, ds,$$

and let $z = \nabla^\perp \alpha$. Then we see that the corrector is of the form described in Remark 3.7 with

$$z = \left(-U(t, x_1) \rho\left(\frac{x_2}{\delta}\right), \partial_1 U(t, x_1) \int_0^{x_2} \rho\left(\frac{s}{\delta}\right) ds\right).$$

12.2. **Corrector in** [53, 17] **for inflow, outflow boundary conditions.** In [53] the authors consider solutions to the Navier-Stokes and Euler equations in a 3D periodic channel, in which fluid enters from the top boundary and exits from the bottom. Letting $g = (0, 0, -V)$ for some constant $V > 0$, the boundary condition for Navier-Stokes is $u = g$ on $[0, T] \times \partial\Omega$ (as in (NS_g), though now $g \cdot \boldsymbol{n} \neq 0$) and for the Euler equations they also set $u = g$ on the top boundary, but only $u \cdot \boldsymbol{n} = g \cdot \boldsymbol{n}$ on the bottom boundary. The setup is generalized in [17] to treat a bounded domain in $\mathbb{R}^3$ and to allow V to vary over the boundary, but the essential nature of the problem is unchanged.

Allowing inflow, outflow introduces a number of complications, not least of which is proving the well-posedness (established in Chapter 4 of [1]) and higher regularity of solutions to the Euler equations (only recently established in [21, 20]). Moreover, although the energy argument that establishes the vanishing viscosity limit is akin to that in Section 4, it departs substantially from it, and so is not strictly in the tradition of Kato. The nonlinear terms are handled differently than we did in Section 4, allowing Hardy's inequality in the form of Lemma 2.8 to be advantageously applied to the corrected difference, which is not possible when $g \cdot \boldsymbol{n} = 0$ (see Section 5.3). Moreover, the authors do not integrate by parts to change Δz to ∇z as we did, and extra terms appear because of the inflow, outflow.

The equations are first "homogenized" by subtracting g from the solutions, so that the solution to the Navier-Stokes equations vanishes on the boundary. The key extra term that appears in the energy inequality is $-g \cdot \nabla z$, where g is extended to $\overline{Q}$ as in Lemma 1.3. This term cannot by itself be controlled, but the combination $\nu \Delta z - g \cdot \nabla z$ can be if one uses a corrector that approximately satisfies the 1D elliptic equation,

$$\nu \frac{\partial^2 z^1}{\partial x_2^2} - V \frac{\partial z^1}{\partial x_2} = 0. \tag{12.1}$$

(This would be the 2D version; in [53, 17] x_2 is x_3 and (12.1) applies to z_{tan}.)

The dominant factor in the corrector that results is $e^{-V x_2/\nu}$. The corrector is more complicated, as a cutoff function is required along with other complicating issues, but this dominant factor forces the specific scaling, $\delta = V^{-1}\nu$. This in turn forces a compatibility condition to be assumed on the initial velocity to control one critical term coming from the nonlinearity for (NS), resulting in short-time convergence. (Given that in 3D there is only finite-time existence of the solutions to the Euler equations, this is a minor limitation.)

12.3. **Corrector in [9].** In Section 3 of [9], the authors define a nonnegative smooth cutoff function, ψ, to be supported in $[1/2, 4]$ and to have total mass 1, "approximating $\chi_{[1,2]}$." (We interpret this to mean that $\psi = 1 - \epsilon$ on $[1, 2]$ for some small $\epsilon > 0$ so that the total mass can add to 1.) The corrector as it appears in (3.1, 3.2) of [9] we can write as

$$\begin{aligned} z^1(t, x_1, x_2) &:= -U(t, x_1) \left(e^{-\frac{x_2}{\delta}} - \delta \psi(x_2) \right), \\ z^2(t, x_1, x_2) &:= \delta \partial_1 U(t, x_1) \left(1 - \int_0^{x_2} \psi(y) \, dy - e^{-\frac{x_2}{\delta}} \right), \end{aligned}$$

working explicitly with a flat boundary (the upper half-plane). In [9], the authors use $\delta = \alpha\tau(t)$, where $\tau(t) = \min\{t, 1\}$ and, ultimately, α is set to ν. Observe that $z = \nabla^\perp \alpha$ where

$$\alpha = \delta U(t, x_1) \left(1 - \int_0^{x_2} \psi(y) \, dy - e^{-\frac{x_2}{\delta}} \right).$$

Then α and z are of magnitude δ in a fixed-width boundary layer outside of which they decay exponentially fast. Like the simple corrector of Remark 3.7, the stream function α is product form and vanishes on the boundary, but it does not (purely) scale like $\delta f(x_2/\delta)$ in the x_2 variable.

12.4. **Heat equation-based correctors.** The idea of using the solution to the 1D heat equation to correct for a 2D PDE (heat equation or Stokes equation) with a divergence-free corrector goes back to Temam and Wang in [55]. In the context of Kato-like arguments, such correctors appear in a simple form in [16, 7]. In [16] Gie uses the corrector in a 3D bounded domain with curved boundary, while in [7] it is used in a half-plane. In [16], the system studied is linear, the Stokes equations, but both [16, 7], in effect, apply formal asymptotics to the equation for $w = u - \overline{u}$, and focus on controlling the terms of the form, $\partial_t w - \nu \Delta w$. We present the corrector in [7], which works specifically in the half-plane, $x_2 > 0$, the technical complexities being lessened over those in [16]. (Or see [55] for the construction in a 2D periodic channel.)

The corrector z is required to satisfy $z = v = g - \overline{u}$ on the boundary ($g = 0$ in [16, 7]) and be divergence-free. The tangential component $z^1(t, x_1, \cdot)$ satisfies the 1D heat equation (as would also follow from appropriate formal asymptotics) with $z(t, x_1, 0) = v(t, x_1, 0)$ and then z^2 is chosen uniquely to enforce the divergence-free condition and vanish on the boundary. This yields a corrector in which (see (3.2) of [7])

$$z^1 = -U(t, x_1) \left(\operatorname{erfc}(x_2/\delta) - \delta\eta(x_2)\right),$$

where $\delta = \sqrt{4\nu t}$, $\operatorname{erfc}(r) = 1 - \operatorname{erf}(r) = \frac{2}{\sqrt{\pi}} \int_r^\infty e^{-y^2}\, dy$, and $\eta \in C^\infty([0, \infty))$ with $\operatorname{supp} \eta \in [1, 2]$ and $\int_0^\infty \eta(r)\, dr = \pi^{-\frac{1}{2}}$. Hence, $z = \nabla^\perp \alpha$, where

$$\begin{aligned} \alpha &= U(t, x_1) \left(\int_0^{x_2} \operatorname{erfc}(s/\delta)\, ds - \delta \int_0^{x_2} \eta(s)\, ds \right) \\ &= \delta U(t, x_1) \left(\int_0^{\frac{x_2}{\delta}} \operatorname{erfc}(s)\, ds - \delta \int_0^{x_2} \eta(s)\, ds \right), \end{aligned}$$

which we note vanishes on the boundary. Then

$$z^2 = \partial_1 \alpha = -\partial_1 U(t, x_1) \left(\int_0^{\frac{x_2}{\delta}} \operatorname{erfc}(s)\, ds - \delta \int_0^{x_2} \eta(s)\, ds \right),$$

which also vanishes on the boundary. The condition $\int_0^\infty \eta(r)\, dr = \pi^{-\frac{1}{2}}$ allows sufficient decay of z^2 as $x_2 \to \infty$, as shown in [7].

The key distinction between the use of this corrector and that of Kato (Section 3) or of Wang (Section 12.1) is that it is designed to control the term, $(\partial_t z - \nu\Delta z, \widetilde{w})$, which is shown in [7] to be bounded by $C(\nu^{\frac{1}{2}} t^{-\frac{1}{2}} + (\nu t)^{\frac{1}{4}})$. Integrating in time, this gives a $C(T)\nu^{\frac{1}{4}}$ bound. By contrast, in Kato's energy argument, $(\partial_t z, \widetilde{w})$ and $(\nu\Delta z, \widetilde{w})$ are controlled separately, by integrating by parts, in time or in space. In place of $(\nu\Delta z, \widetilde{w})$ one has $\nu(\nabla u, \nabla z)$, which is easily controlled, since the boundary layer is wider than that of Kato's.

As it turns out, Kato's approach would work to obtain the results in [7]—as it would in Gie's [16] to obtain convergence of Stokes solutions to the inviscid solution in $L^\infty([0, T]; L^2)$. The control on $\partial_t z - \nu\Delta z$ in [16], however, is critical in demonstrating that the Stokes solutions remain bounded in $L^\infty(0, T; H^1)$. This is something that cannot happen for solutions to the Navier-Stokes equations if the vanishing viscosity limit is to hold (as shown in [32]).

That controlling the H^1 norm is the critical use of such correctors is already apparent in [55], where it is explicitly stated. The use of similar correctors to control or even obtain convergence of the corrected difference of solutions in the H^1 norm is apparent in much of the subsequent work of Temam and Wang and those following their general approach; this includes, but is hardly limited to, [56, 61, 57, 59, 52, 58, 60, 53, 35, 19, 18].

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Appendix A. Curvilinear coordinates(long version only)

Proof of Lemma 2.2. First, we obtain the expression for $v^\perp$ by the simple calculation,

$$v^\perp = v^1 \boldsymbol{\tau}^\perp + v^2(-\boldsymbol{n})^\perp = v^1(-\boldsymbol{n}) - v^2 \boldsymbol{\tau} = (-v^2, v^1).$$

Now let

$$\gamma = \gamma(x_1, x_2) := \frac{1}{1 - \kappa x_2}.$$

That is, γ is the same as the expression for J in (2.1), though we do not yet know that it is, in fact, the Jacobian determinant.

Let $P = (x_1, x_2) \in \Gamma_\delta$ in coordinates as used above and let $y = (y_1, y_2)_C$ be P in Cartesian coordinates. Let $\xi(t)$ be a local parameterization of $\partial\Omega$ by arc length, expressed in Cartesian coordinates. Then x_1 is arc length along image ξ and

$$y = \xi(x_1) + x_2(\xi'(x_1))^\perp = \xi(x_1) - x_2\boldsymbol{n}(x_1)$$

or

$$\begin{aligned} y_1 &= \xi^1(x_1) - x_2 n^1(x_1) = \xi^1(x_1) - x_2 n^1(x_1), \\ y_2 &= \xi^2(x_1) - x_2 n^2(x_1) = \xi^2(x_1) - x_2 n^2(x_1). \end{aligned}$$

Here, and in what follows, we write the vectors $\boldsymbol{n}$ and later $\boldsymbol{\tau}$ in *Cartesian* coordinates as

$$\boldsymbol{n} = (n^1, n^2), \quad \boldsymbol{\tau} = (\tau^1, \tau^2) = (-n^2, n^1).$$

Hence,

$$\begin{aligned} \partial_{x_1} &= \frac{\partial y_1}{\partial x_1}\partial_{y_1} + \frac{\partial y_2}{\partial x_1}\partial_{y_2} \\ &= \left[(\xi^1)'(x_1) - x_2(n^1)'(x_1)\right]\partial_{y_1} + \left[(\xi^2)'(x_1) - x_2(n^2)'(x_1)\right]\partial_{y_2} \\ &= \boldsymbol{\tau}(x_1)\cdot\nabla_y - x_2\boldsymbol{n}'(x_1)\cdot\nabla_y = (1 - \kappa(x_1)x_2)\boldsymbol{\tau}(x_1)\cdot\nabla_y = \gamma^{-1}\boldsymbol{\tau}(x_1)\cdot\nabla_y, \\ \partial_{x_2} &= \frac{\partial y_1}{\partial x_2}\partial_{y_1} + \frac{\partial y_2}{\partial x_2}\partial_{y_2} = -n^1(x_1)\partial_{y_1} - n^2(x_1)\partial_{y_2} = -\boldsymbol{n}(x_1)\cdot\nabla_y, \end{aligned}$$

where we used that

$$\boldsymbol{n}'(x_1) = \frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\tau}} = \kappa\boldsymbol{\tau},$$

κ being the scalar curvature on the boundary. We can write this as $\nabla_x = A\nabla_y$, where

$$A = \begin{pmatrix} \gamma^{-1}\tau^1 & \gamma^{-1}\tau^2 \\ -n^1 & -n^2 \end{pmatrix} = \begin{pmatrix} \gamma^{-1}\tau^1 & \gamma^{-1}\tau^2 \\ \tau^2 & -\tau^1 \end{pmatrix}.$$

Noting that $\det A = -\gamma^{-1}$ we see that A is invertible with

$$A^{-1} = -\gamma\begin{pmatrix} -\tau^1 & -\gamma^{-1}\tau^2 \\ -\tau^2 & \gamma^{-1}\tau^1 \end{pmatrix} = \begin{pmatrix} \gamma\tau^1 & \tau^2 \\ \gamma\tau^2 & -\tau^1 \end{pmatrix}.$$

Hence,

$$\nabla_y = A^{-1}\nabla_x = \begin{pmatrix} \gamma\tau^1 & \tau^2 \\ \gamma\tau^2 & -\tau^1 \end{pmatrix}\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} = \begin{pmatrix} \gamma\tau^1\partial_{x_1} + \tau^2\partial_{x_2} \\ \gamma\tau^2\partial_{x_1} - \tau^1\partial_{x_2} \end{pmatrix} = \begin{pmatrix} \gamma\tau^1\partial_{x_1} - n^1\partial_{x_2} \\ \gamma\tau^2\partial_{x_1} - n^2\partial_{x_2} \end{pmatrix},$$

so that

$$\nabla f = \gamma\partial_1 f\boldsymbol{\tau} - \partial_2 f\boldsymbol{n}.$$

From this we obtain the expression for the Jacobian determinant in (2.1),

$$J(x_1, x_2) = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = |\det A^{-1}| = |-\gamma(\tau^1)^2 - \gamma(\tau^2)^2| = \gamma.$$

Now,

$$\operatorname{div} v = \partial_{y_1}(v\cdot\boldsymbol{e}_1) + \partial_{y_2}(v\cdot\boldsymbol{e}_2),$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are the standard basis vectors in $\mathbb{R}^2$. But,

$$\begin{aligned} v \cdot \boldsymbol{e}_1 &= v^1\tau^1 - v^2 n^1 = -v^1 n^2 - v^2 n^1, \\ v \cdot \boldsymbol{e}_2 &= v^1\tau^2 - v^2 n^2 = v^1 n^1 - v^2 n^2, \end{aligned}$$

so

$$\begin{aligned} \operatorname{div} v = -\partial_{y_1} v^1 n^2 - \partial_{y_1} v^2 n^1 - v^1 \partial_{y_1} n^2 - v^2 \partial_{y_1} n^1 \\ + \partial_{y_2} v^1 n^1 - \partial_{y_2} v^2 n^2 + v^1 \partial_{y_2} n^1 - v^2 \partial_{y_2} n^2. \end{aligned}$$

Now, $\boldsymbol{n}$ does not change with changes in x_2, so applying the chain rules gives

$$\partial_{y_j} n^k = \partial_{x_1} n^k \partial_{x_1} y_j$$

for each j, k. And,

$$\begin{aligned} \partial_{x_1} y_1 &= (\xi^1)' - x_2 (n^1)' = \tau^1 - x_2 \kappa \tau^1 = J\tau^1 = -Jn^2, \\ \partial_{x_1} y_2 &= (\xi^2)' - x_2 (n^2)' = \tau^2 - x_2 \kappa \tau^2 = J\tau^2 = Jn^1. \end{aligned}$$

Also,

$$\partial_{x_1} n^1 = (n^1)' = \kappa\tau^1 = -\kappa n^2, \quad \partial_{x_1} n^2 = (n^2)' = \kappa\tau^2 = \kappa n^1$$

Thus,

$$\begin{aligned} \partial_{y_1} n^1 &= \partial_{x_1} n^1 \partial_{x_1} y_1 = (-\kappa n^2)(-Jn^2) = \kappa J (n^2)^2, \\ \partial_{y_1} n^2 &= \partial_{x_1} n^2 \partial_{x_1} y_1 = \kappa n^1 (-Jn^2) = -\kappa J n^1 n^2, \\ \partial_{y_2} n^1 &= \partial_{x_1} n^1 \partial_{x_1} y_2 = (-\kappa n^2) J n^1 = -\kappa J n^1 n^2, \\ \partial_{y_2} n^2 &= \partial_{x_1} n^2 \partial_{x_1} y_2 = \kappa n^1) J n^1 = \kappa J (n^1)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{div} v &= -\partial_{y_1} v^1 n^2 - \partial_{y_1} v^2 n^1 + v^1 \kappa J n^1 n^2 - v^2 \kappa J (n^2)^2 \\ &\qquad + \partial_{y_2} v^1 n^1 - \partial_{y_2} v^2 n^2 - v^1 \kappa J n^1 n^2 - v^2 \kappa J (n^1)^2 \\ &= -\partial_{y_1} v^1 n^2 - \partial_{y_1} v^2 n^1 + \partial_{y_2} v^1 n^1 - \partial_{y_2} v^2 n^2 - v^2 \kappa J. \end{aligned}$$

From our expression for ∇_y, we have

$$\begin{aligned} \partial_{y_1} v^1 &= J\tau^1 \partial_{x_1} v^1 - n^1 \partial_{x_2} v^1 = -Jn^2 \partial_{x_1} v^1 - n^1 \partial_{x_2} v^1, \\ \partial_{y^2} v^1 &= J\tau^2 \partial_{x_1} v^1 - n^2 \partial_{x_2} v^1 = Jn^1 \partial_{x_1} v^1 - n^2 \partial_{x_2} v^1, \\ \partial_{y_1} v^2 &= -Jn^2 \partial_{x_1} v^2 - n^1 \partial_{x_2} v^2, \\ \partial_{y^2} v^2 &= Jn^1 \partial_{x_1} v^2 - n^2 \partial_{x_2} v^2. \end{aligned}$$

So

$$\begin{aligned} \operatorname{div} v &= (Jn^2 \partial_{x_1} v^1 + n^1 \partial_{x_2} v^1) n^2 + (Jn^2 \partial_{x_1} v^2 + n^1 \partial_{x_2} v^2) n^1 \\ &\qquad + (Jn^1 \partial_{x_1} v^1 - n^2 \partial_{x_2} v^1) n^1 - (Jn^1 \partial_{x_1} v^2 - n^2 \partial_{x_2} v^2) n^2 - v^2 \kappa J \\ &= \partial_{x_1} v^1 (J(n^2)^2 + J(n^1)^2) + \partial_{x^2} v^2 ((n^1)^2 + (n^2)^2) + \partial_{x_2} v^1 (n^1 n^2 - n^2 n^1) \\ &\qquad + \partial_{x_1} v^2 (Jn^2 n^1 - Jn^1 n^2) - v^2 \kappa J \\ &= J \partial_{x_1} v^1 + \partial_{x^2} v^2 - v^2 \kappa J = J \partial_1 v^1 + \partial_2 v^2 - v^2 \kappa J, \end{aligned}$$

which gives our expression for $\operatorname{div} v$. The expression for $\operatorname{curl} v$ follows from the identity $\operatorname{curl} v = -\operatorname{div} v^\perp$. For Δf, we calculate,

$$\Delta f = \operatorname{div} \nabla f = \operatorname{div}(J\partial_1 f, \partial_2 f) = J\partial_1 (J \partial_1 f)) + \partial_2^2 f - \kappa J \partial_2 f$$

$$= J^2\partial_1^2 f + \partial_2^2 f - \kappa J\partial_2 f + J\partial_1 J\partial_1 f.$$

But,

$$\partial_1 J = -\frac{1}{(1-\kappa x_2)^2}(-x_2\partial_1\kappa) = x_2 J(x_1, x_2)^2\kappa'(x_1),$$

which gives the expression for Δf.

Because $(\boldsymbol{\tau}, \boldsymbol{n})$ are orthonormal, the expression for $u \cdot v$ is the same in (x_1, x_2) coordinates as it is in Cartesian coordinates; that is, $u \cdot v = u^j v^j$.

To calculate $u \cdot \nabla v$, we go to the raw definition of $u \cdot \nabla v = (u \cdot \nabla)v$ as a directional derivative.:

$$\begin{aligned}
u \cdot \nabla v(x) &= \lim_{h\to 0}\frac{v(x+hu)-v(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{v(x+hu)-v(x)}{h}\cdot\boldsymbol{\tau}\right]\boldsymbol{\tau} + \lim_{h\to 0}\left[\frac{v(x+hu)-v(x)}{h}\cdot(-\boldsymbol{n})\right](-\boldsymbol{n}) \\
&= \left(\lim_{h\to 0}\frac{v^1(x+hu)-v^1(x)}{h}, \lim_{h\to 0}\frac{v^2(x+hu)-v^2(x)}{h}\right) \\
&= \nabla v^1 \cdot u + \nabla v^2 \cdot u = (J\partial_1 v^1, \partial_2 v^1)\cdot(u^1, u^2) + (J\partial_1 v^2, \partial_2 v^2)\cdot(u^1, u^2) \\
&= \left(Ju^1\partial_1 v^1 + u^2\partial_2 v^1, Ju^1\partial_1 v^2 + u^2\partial_2 v^2\right).
\end{aligned}$$

Or, in somewhat more detail, let $\alpha(s)$ be any smooth path for which $\alpha(0) = x$, $\alpha'(0) = u$. Then

$$\begin{aligned}
u\cdot\nabla v(x) &= \lim_{s\to 0}\frac{v(\alpha(s))-v(\alpha(0))}{s} \\
&= \lim_{s\to 0}\left[\left[\frac{v(\alpha(s))-v(\alpha(0))}{s}\cdot\boldsymbol{\tau}\right]\boldsymbol{\tau}\right] + \lim_{s\to 0}\left[\left[\frac{v(\alpha(s))-v(\alpha(0))}{s}\cdot(-\boldsymbol{n})\right](-\boldsymbol{n})\right] \\
&= \left[\lim_{s\to 0}\frac{v^1(\alpha(s))-v^1(\alpha(0))}{s}\right]\boldsymbol{\tau} + \left[\lim_{s\to 0}\frac{v^2(\alpha(s))-v^2(\alpha(0))}{s}\right](-\boldsymbol{n}).
\end{aligned}$$

In the second line, the $\boldsymbol{\tau}$ and $\boldsymbol{n}$ are evaluated at $x + hu$, which in the next line become evaluated at x by the product rule for limits. Now,

$$\begin{aligned}
\lim_{s\to 0}\frac{v^1(\alpha(s))-v^1(\alpha(0))}{s} &= \left.\frac{dv^1(\alpha(s))}{ds}\right|_{s=0} \\
&= \left.\frac{\partial v^1}{\partial\alpha_1}\right|_{x_1=x_1(s)}\left.\frac{d\alpha^1(s)}{ds}\right|_{s=0} + \left.\frac{\partial v^1}{\partial\alpha_2}\right|_{x_2=x_2(s)}\left.\frac{d\alpha^2(s)}{ds}\right|_{s=0} \\
&= \nabla v^1(x)\cdot\alpha'(0) = \nabla v^1(x)\cdot u(x) \\
&= Ju^1\partial_1 v^1 + u^2\partial_2 v^1,
\end{aligned}$$

using our expression for ∇ and for the scalar product of two vector fields. Similarly,

$$\lim_{s\to 0}\frac{v^2(\alpha(s))-v^2(\alpha(0))}{s} = Ju^1\partial_1 v^2 + u^2\partial_2 v^2,$$

which yields

$$u \cdot \nabla v(x) = \left(Ju^1\partial_1 v^1 + u^2\partial_2 v^1, Ju^1\partial_1 v^2 + u^2\partial_2 v^2\right). \qquad \square$$

Appendix B. Proof of corrector estimates(long version only)

In the section, we give the proofs of Propositions 3.5 and 3.8.

Proof of Proposition 3.5. Working on a single component of $\Gamma_{\overline{\delta}}$ using Lemma 2.2, we have,

$$\begin{aligned} z(x_1, x_2) &= \nabla^\perp(\varphi_\delta(x_2)\psi(x_1, x_2)) = (-\partial_2(\varphi_\delta(x_2)\psi(x_1, x_2)), J\partial_1(\varphi_\delta(x_2)\psi(x_1, x_2))) \\ &= -\left(\varphi_\delta'(x_2)\psi(x_1, x_2), 0\right) + (-\varphi_\delta(x_2)\partial_2\psi(x_1, x_2), J\varphi_\delta(x_2)\partial_1\psi(x_1, x_2)) \\ &= -\left(\varphi_\delta'(x_2)\psi(x_1, x_2), 0\right) + \varphi_\delta(x_2)\nabla^\perp\psi(x_1, x_2) \\ &= -\left(\varphi_\delta'(x_2)\psi(x_1, x_2), 0\right) + \varphi_\delta(x_2)v(x_1, x_2). \end{aligned} \tag{B.1}$$

Hence,

$$\begin{aligned} \partial_1 z^1 &= -\varphi_\delta'(x_2)\partial_1\psi(x_1, x_2) + \varphi_\delta(x_2)\partial_1 v^1(x_1, x_2) \\ &= -\varphi_\delta'(x_2)v^2(x_1, x_2) + \varphi_\delta(x_2)\partial_1 v^1(x_1, x_2), \\ \partial_2 z^1 &= -\varphi_\delta'(x_2)\partial_2\psi(x_1, x_2) - \varphi_\delta''(x_2)\psi(x_1, x_2) \\ &\qquad + \varphi_\delta'(x_2)v^1(x_1, x_2) + \varphi_\delta(x_2)\partial_2 v^1(x_1, x_2) \\ &= 2\varphi_\delta'(x_2)v^1 - \varphi_\delta''(x_2)\psi(x_1, x_2) + \varphi_\delta(x_2)\partial_2 v^1(x_1, x_2), \\ \partial_1 z^2 &= \varphi_\delta(x_2)\partial_1 v^2(x_1, x_2), \\ \partial_2 z^2 &= -\partial_1 z^1 + \kappa J z^2. \end{aligned}$$

In the last equality, we used $\operatorname{div} z = 0$ and the form of $\operatorname{div} z$ given in Lemma 2.2.

Now,

$$\begin{aligned} |\psi(x_1, x_2)| &\le \|v\|_{L^\infty} x_2 = Cx_2, \\ |v^2(x_1, x_2)| &\le \|\partial_2 v^2\|_{L^\infty} x_2 \le Cx_2, \\ |\partial_1 v^2(x_1, x_2)| &\le \|\partial_2\partial_1 v^2\|_{L^\infty} x_2 \le Cx_2, \\ |\varphi_\delta'(x_2)x_2| &\le C, \qquad |\varphi_\delta''(x_2)x_2| \le C\delta^{-1}, \end{aligned}$$

so we have the pointwise bounds (for all $\delta \le \delta_0$, for some fixed $\delta_0 > 0$),

$$\begin{array}{ll} |z^1(x_1, x_2)| \le C, & |z^2(x_1, x_2)| \le Cx_2, \\ |\partial_1 z^1(x_1, x_2)| \le C, & |\partial_2 z^1(x_1, x_2)| \le C\delta^{-1}, \\ |\partial_1 z^2(x_1, x_2)| \le Cx_2, & |\partial_2 z^2(x_1, x_2)| \le C \end{array} \tag{B.2}$$

with all quantities supported in Γ_δ. These bounds lead directly to the bounds in Proposition 3.5 given in (3.4).

We explicitly calculate the final bound in (3.4), which is the most involved. We have,

$$\begin{aligned} \|z \cdot \nabla z\|_{L^p}^p &= \sum_{i,j} \|z^i \partial_i z^j\|_{L^p}^p = \int_{\Gamma_\delta} \sum_{i,j} |z^i(x_1, x_2)|^p |\partial_i z^j(x_1, x_2)|^p \, dx_1 \, dx_2 \\ &= \int_{\Gamma_\delta} (|z^1(x_1, x_2)|^p |\partial_1 z^1(x_1, x_2)|^p + |z^2(x_1, x_2)|^p |\partial_2 z^1(x_1, x_2)|^p \\ &\qquad + |z^1(x_1, x_2)|^p |\partial_1 z^2(x_1, x_2)|^p + |z^2(x_1, x_2)|^p |\partial_2 z^2(x_1, x_2)|^p) \\ &\qquad\qquad dx_1 \, dx_2 \\ &\le \int_{\Gamma_\delta} \left(C^p C^p + x_2^p C^p \delta^{-p} + C^p x_2^p + C x_2^p C^p\right) \, dx_1 \, dx_2 \\ &\le C^p \int_{\Gamma_\delta} dx_1 \, dx_2 = C^p \delta. \end{aligned}$$

This yields the final bound in (3.4). (The point, here, is that $z \cdot \nabla z$ is bounded on $\overline{\Omega}$.)

Because

$$\partial_t z(x_1, x_2) = \nabla^\perp(\varphi_\delta(x_2)\partial_t \psi(x_1, x_2))$$

and $\partial_t \psi$ is bounded in the same manner as ψ (just with different constants), the estimates in (B.2) and so in (3.4) hold as well for $\partial_t z$ in place of z.

This establishes (3.4) for $j, k = 0$; $j = 1, k = 0$; $j = 0, k = 1$. Because additional derivatives in x_1 of z^1 or z^2 affect only ψ and v, which are C^∞, we also obtain the result for any value of j. Each additional derivative of z^1 or z^2 in x_2 has the same effect on ψ and v, but also adds one additional derivative on φ_δ, introducing an additional factor of δ. This leads to an additional factor of δ^{-k} for ∂_2^k. Since, however, $\partial_2 z^2 = -\partial_1 z^1$, there is one less factor of δ^{-1} for $\partial_2^k z^2$ than there is for $\partial_2^k z^1$. Similar considerations apply to $\partial_1^j \partial_2^k$, completing the proof of (3.4).

We now turn to the proof of (3.5). The estimates in (3.4) continue to hold unchanged when $m = 0$. If δ also varies with time, however, the cutoff function, φ_δ, has an additional dependence on time thorough δ, so that

$$\partial_t \varphi_\delta(x_2) = \partial_t \varphi\left(\frac{x_2}{\delta}\right) = \varphi'\left(\frac{x_2}{\delta}\right)\frac{\partial}{\partial t}\frac{x_2}{\delta} = -x_2 \frac{\partial_t \delta}{\delta^2}\varphi'\left(\frac{x_2}{\delta}\right).$$

Hence,

$$\begin{aligned}\partial_t z(x_1, x_2) &= \nabla^\perp(\varphi_\delta(x_2)\partial_t \psi(x_1, x_2)) - \nabla^\perp\left(x_2 \frac{\partial_t \delta}{\delta^2}\varphi'\left(\frac{x_2}{\delta}\right)\psi(x_1, x_2)\right) \\ &=: v_1 + v_2.\end{aligned}$$

To obtain the estimates in (B.2) for $\partial_t z$ in place of z, v_1 is bounded as before, so that, in particular,

$$\begin{aligned}\|v_1^1(x_1, x_2)\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}}, \\ \|v_1^2(x_1, x_2)\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}+1}.\end{aligned}$$

In bounding v_2, $-\varphi'(x_2/\delta)$ plays the role that $\varphi_\delta(x_2)$ played in bounding z, and is bounded in the same manner (the vanishing of φ' in a neighborhood of the boundary does not improve any estimates), but there is an additional factor of $x_2 \frac{\partial_t \delta}{\delta^2}$ that is included in each of the corresponding bounds in (3.4) for v_2. We need only the first two bounds,

$$\begin{aligned}|v_2^1(x_1, x_2)| &\le C x_2 \frac{\partial_t \delta}{\delta^2}, \\ |v_2^2(x_1, x_2)| &\le C x_2^2 \frac{\partial_t \delta}{\delta^2}.\end{aligned} \tag{B.3}$$

Hence (assuming that $\partial_t \delta > 0$),

$$\begin{aligned}\|v_2^1\|_{L^p(\Omega)} &\le C\frac{\partial_t \delta}{\delta^2}\left(\int_0^\delta x_2^p\right)^{\frac{1}{p}} \le C\frac{\partial_t \delta}{\delta^2}\delta^{1+\frac{1}{p}}, \\ \|v_2^2\|_{L^p(\Omega)} &\le C\frac{\partial_t \delta}{\delta^2}\left(\int_0^\delta x_2^{2p}\right)^{\frac{1}{p}} \le C\frac{\partial_t \delta}{\delta^2}\delta^{2+\frac{1}{p}}.\end{aligned}$$

(We have suppressed the Jacobian in these integrals, which is bounded above as in the proof of Lemma 2.3, and so only changes the values of the constants.) From this, $(3.5)_{1,2}$ follow directly. Then

$$\|\partial_t z\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}}(1 + \delta) + C\partial_t \delta\, \delta^{\frac{1}{p}-1}(1 + \delta)$$

$$\le C\delta^{\frac{1}{p}-1}(\delta + \partial_t \delta)(1+\delta) \le C\delta^{\frac{1}{p}-1}(\delta + \partial_t \delta)$$

for δ less than any fixed $\delta_0 > 0$, which is $(3.5)_3$. □

Proof of Proposition 3.8. Let $\phi \in C(\overline{\Omega})$. Since $v = g - \overline{u}$ on $\partial\Omega$, what we must show is that

$$\int_\Omega \operatorname{curl} z \, \phi = (\operatorname{curl} z, \phi) \to \int_\Gamma (v \cdot \boldsymbol{\tau})\phi \text{ as } \nu \to 0 \text{ uniformly on } [0, T]. \tag{B.4}$$

Recalling the definition of φ_δ, z, v, and ψ in (3.1) and (3.2), since $\varphi_\delta \psi$ is the stream function for z, we have

$$\begin{aligned}
\operatorname{curl} z &= \Delta(\varphi_\delta \psi) = \Delta\varphi_\delta \psi + \varphi_\delta \Delta\psi + 2\nabla\varphi_\delta \cdot \nabla\psi \\
&= (\varphi_\delta''(x_2) - \kappa J \varphi_\delta'(x_2))\psi + \varphi_\delta \operatorname{curl} v - 2\varphi_\delta'(x_2) v^1.
\end{aligned}$$

In switching to coordinates in the third equality, we used Lemma 2.2, noting in particular, that

$$\nabla\varphi_\delta \cdot \nabla\psi = \nabla^\perp \varphi_\delta \cdot \nabla^\perp \psi = (-\varphi_\delta'(x_2), 0) \cdot (v^1, v^2) = -\varphi_\delta'(x_2) v^1.$$

One term in $(\operatorname{curl} z, \phi)$ can be easily bounded by

$$|(\varphi_\delta \operatorname{curl} v, \phi)| \le \int_{\Gamma_\delta} \|\operatorname{curl} v\|_{L^\infty} \|\phi\|_{L^\infty} \le C\delta,$$

which vanishes in the limit as $\nu \to 0$, since $\delta \to 0$ by Definition 3.2. Using the definitions in the proof of Lemmas 2.3 and 2.9, to treat the other terms in $(\operatorname{curl} z, \phi)$, we integrate separately over each component of Γ_δ (which allows us to assume that ψ vanishes on the intersection of the boundary of that component with $\partial\Omega$). We have,

$$\begin{aligned}
&\int_{\Gamma_\delta^k} (-2\varphi_\delta' v^1 - \kappa J \varphi_\delta' \psi)\phi \\
&\qquad = \frac{1}{\delta} \int_0^\ell \int_0^\delta (-2v^1(x_1, x_2) - \kappa J \psi(x_1, x_2)) \varphi'\left(\frac{x_2}{\delta}\right) \phi(x_1, x_2) J(x_1, x_2) \, dx_1 \, dx_2 \\
&\qquad = \int_0^\ell \int_0^1 (-2v^1(x_1, \delta y) - \kappa J(\delta y)\psi(x_1, \delta y))\varphi'(y)\phi(x_1, \delta y) J(x_1, \delta x_2) \, dx_1 \, dy \\
&\qquad \to \int_0^\ell (-2v^1(x_1, 0) - \kappa J(0)\psi(x_1, 0))\phi(x_1, 0) J(x_1, 0) \, dx_1 \int_0^1 \varphi'(y) \, dy \\
&\qquad = \int_0^\ell 2v^1(x_1, 0)\phi(x_1, 0) \, dx_1 = 2\int_{\Sigma_k} (v \cdot \boldsymbol{\tau})\phi.
\end{aligned}$$

Convergence holds because v^1, J, and ϕ are each continuous on $\overline{\Omega}$. The equality following convergence holds because

$$\int_0^1 \varphi'(y) \, dy = [\varphi(1) - \varphi(0)] = -1$$

and because $J(x_1, 0) = 1$.

For the final term in $(\operatorname{curl} z, \phi)$, we have

$$\begin{aligned}
\int_{\Gamma_\delta^k} \varphi_\delta'' \psi \, \phi &= \frac{1}{\delta^2} \int_0^\ell \int_0^\delta \varphi''\left(\frac{x_2}{\delta}\right) \psi(x_1, x_2)\phi(x_1, x_2) J(x_1, x_2) \, dx_1 \, dx_2 \\
&= \int_0^\ell \int_0^1 \varphi''(y) \frac{\psi(x_1, \delta y)}{\delta} \phi(x_1, \delta y) J(x_1, \delta y) \, dx_1 \, dy.
\end{aligned}$$

But,

$$\frac{\psi(x_1, \delta y)}{\delta} = y \frac{\psi(x_1, \delta y) - \psi(x_1, 0)}{\delta y} \to y\partial_2 \psi(x_1, 0) = -yv^1$$

uniformly over y and time, since ψ' is uniformly continuous on $[0, T] \times \overline{\Omega}$. Here, we used that $v = \nabla^\perp \psi = (\partial_2 \psi, J\partial_1 \psi)$ in coordinates by Lemma 2.2, so $\partial_2 \psi = v^1$.

Again invoking uniform continuity (of J, and ϕ) to obtain limits, we see that

$$\int_{\Gamma_\delta^k} \varphi_\delta'' \psi\, \phi \to -\int_0^\ell v^1(x_1, 0)\phi(x_1, 0)J(x_1, 0)\, dx_1 \int_0^1 y\varphi''(y)\, dy = -\int_{\Sigma_k} (v \cdot \boldsymbol{\tau})\phi,$$

since

$$\int_0^1 y\varphi''(y)\, dy = \big[y\varphi'(y)\big]_0^1 - \int_0^1 \varphi'(y)\, dy = [0 - 0] - (-1) = 1.$$

From these limits, we see that (B.4) follows. □

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