STREAM FUNCTIONS FOR DIVERGENCE-FREE VECTOR FIELDS
Constructive Version

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Abstract. In 1990, Von Wahl and, independently, Borchers and Sohr showed that a divergence-free vector field u in a 3D bounded domain that is tangential to the boundary can be written as the curl of a vector field vanishing on the boundary of the domain. We extend this result to higher dimension and to Lipschitz boundaries in a form suitable for integration in flat space, showing that u can be written as the divergence of an antisymmetric matrix field. We also demonstrate how obtaining a kernel for such a matrix field is dual to obtaining a Biot-Savart kernel for the domain.

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Constructive Version

This version includes a constructive, more geometric (and much longer) approach to obtaining the stream function in Section 7. It also includes Appendices A and B.

Blue italicized text in smaller fonts contains parenthetical comments or details of proofs not intended for publication.

1. Overview

Let u be a divergence-free vector field on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, that is tangential to the boundary. For a simply connected domain, it is well known that in two dimensions, $u = \nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)$ for a *stream function*, ψ, vanishing on the boundary. It is also well known that in three dimensions, we can write $u = \operatorname{curl} \psi$, where now the *vector potential* ψ is a divergence-free vector field tangential to the boundary. Perhaps somewhat less well-known is that ψ can also be chosen (non-uniquely) to vanish on the boundary, though sacrificing the divergence-free condition. This 3D form of the vector potential was developed in [9, 26], where it is studied in Sobolev, Hölder spaces, for $C^{1,1}$, C^∞ boundaries, respectively.

In higher dimension, we can no longer use a vector field as the potential; instead, we will use an antisymmetric matrix field A vanishing on the boundary, for which $u = \operatorname{div} A$, the divergence applied to A row-by-row. This was the manner it was utilized in [20], without, however, the key antisymmetric condition.

Our main result is Theorem 1.1.

Theorem 1.1. *Let H be the space of divergence-free vector fields on Ω that are tangential to the boundary and that have L^2 coefficients. Let H_c be the closed subspace of* curl*-free vector fields (see (3.1)) in H, let H_0 be its orthogonal complement in H, and let*

$$X_0 := \{A \in H_0^1(\Omega)^{d\times d} \colon A \text{ antisymmetric}\}.$$

Then $H_0 = \operatorname{div} X_0$, and there exists a bounded linear map $S \colon H_0 \to X_0$ with $\operatorname{div} Su = u$.

Specializing to $d = 2, 3$, we can write

$$H_0 = \begin{cases} \nabla^\perp H_0^1(\Omega), & d = 2, \\ \operatorname{curl}_3 H_0^1(\Omega)^3, & d = 3. \end{cases}$$

Because the term *matrix potential* is commonly used in the literature for other purposes, we will adopt the 2D terminology for all dimensions, calling A the *stream function* for u.

Closely connected to stream functions is the Hodge decomposition of L^2-vector fields on Ω. Indeed, one form of the Hodge decomposition in 3D is

$$H = H_c \oplus \operatorname{curl}(H \cap H^1(\Omega)^3).$$

That is, each element of $H_0 := H_c^\perp$ is the image of a classical, divergence-free vector potential tangential to the boundary. Moreover, for any $u \in H_0$, the boundary value problem

$$\begin{cases} \operatorname{curl} \psi = u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is (non-uniquely) solvable, and gives the 3D form of the stream function in Theorem 1.1.

In fact, solving the analog of (1.1) in any dimension in the more general setting of an oriented manifold with boundary was worked out by Schwarz in [22]. He shows that for such a manifold with $C^{1,1}$ boundary, given a 1-form α having L^2-regularity and vanishing normal component, the boundary value problem

$$\begin{cases} \delta\beta = \alpha & \text{on } M, \\ \beta|_{\partial M} = 0 & \text{on } \partial M \end{cases}$$

(δ is the codifferential) is solvable for a 2-form having H^1-regularity if and only if

$$\int_M \alpha \wedge *\lambda = 0 \text{ for all } \lambda \in \mathcal{H}_N^1(\Omega).$$

Here, $\mathcal{H}_N^1(\Omega)$ is the space of harmonic fields having vanishing normal component, the analog of H_c, and the integral condition on α defines the analog of H_0. (Appendix A has a more detailed account.)

Schwarz's result is not restricted to 1-forms, but holds for k-forms and also allows non-zero boundary values. It is restricted, however, to $C^{1,1}$ boundaries. For manifolds embedded in $\mathbb{R}^d$, this restriction is loosened in [21], which applies to boundaries even less regular than Lipschitz. The authors show that, given an $(\ell-1)$-form α for any $0 \le l \le d-1$, there exists an ℓ-form β having prescribed boundary value for which $\delta\beta = \alpha$. They assume, however, that the $(\ell-1)$-st Betti number vanishes. Since we need such a result for $\ell = 2$, this means that the first Betti number must vanish, which means that Ω must be simply connected, an assumption we wish to avoid.

We present our derivation of a stream function here, therefore, because it applies to non-simply connected domains having only a Lipschitz continuous boundary. Moreover, we obtain the stream function non-constructively, using simple functional analytic arguments, avoiding entirely the language of differential forms, making it more accessible and self-contained for our intended primary audience of analysts working in flat space.

Central to our approach is the fact that the divergence operator maps vector fields in $H^1_0(\Omega)^d$ *onto* $L^2_0(\Omega)$, the space of L^2 functions with mean zero. For arbitrary domains, this is a result of Bogovskiĭ [7, 8] (see Lemma 2.11, below). Bogovskiĭ produces an integral kernel for solving the problem $\operatorname{div} u = f$ in a star-shaped domain. This kernel and adaptations of it have been used in other approaches to Theorem 1.1 in 3D, such as [6] for star-shaped domains, but we use Bogovskiĭ's result as a "black box," for with it, we can easily obtain Theorem 1.1 except for the key antisymmetric condition on the stream function.

Nevertheless, the partially constructive 3D approach taken in [9] can be adapted, using aspects of the geometric approach taken in [23], to obtain the same result. We present this approach in Section 7. It relies, however, upon two lemmas that hold true for manifolds in $\mathbb{R}^d$ with smooth boundary, but whose proofs for Lipschitz boundaries do not, as far as the author can determine, appear in the literature. Hence, this approach is incomplete.

We also present in Appendix A an overview of the results as presented in [23] as regards the Hodge decomposition and what we are calling stream functions in the language of differential forms, making the connection with the "flat space" approach we have taken.

We assume that Ω is a bounded, connected, open subset of $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary, $\partial\Omega$. We define the L^2-based Sobolev spaces, $H^k(\Omega)$ and $H^k_0(\Omega)$, for nonnegative k in the usual way (the boundary is regular enough that all standard definitions are equivalent). Identifying L^2 with its own dual, we also define the dual spaces, $H^{-k}(\Omega) := H^k_0(\Omega)'$.

We will work with the classical function spaces, H and V, of incompressible fluid mechanics:

$$\begin{aligned} H &:= \{u \in L^2(\Omega)^d \colon \operatorname{div} u = 0,\, u \cdot \boldsymbol{n} = 0\}, \\ V &:= \{u \in H^1_0(\Omega)^d \colon \operatorname{div} u = 0\}. \end{aligned} \tag{1.2}$$

The divergence here is defined in terms of weak derivatives, and $u \cdot \boldsymbol{n}$ is defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$ in terms of a trace (see Lemma 2.2), $\boldsymbol{n}$ being the outward unit normal vector. Both H and V are Hilbert spaces with norms and inner products as subspaces of L^2 and H^1_0. By virtue of the Poincaré inequality, we can use

$$\begin{aligned} (f, g)_{H^1_0} &:= (\nabla f, \nabla g)_{L^2}, \quad \|f\|_{H^1_0} := \|\nabla f\|_{L^2}, \\ (u, v)_V &:= (\nabla u, \nabla v)_{L^2}, \quad \|u\|_V := \|\nabla u\|_{L^2}. \end{aligned}$$

With these very cursory definitions out of the way, we give in Section 2 some further necessary background material drawn mostly from [14, 16]. In Section 3, we prove our main result, Theorem 1.1, extending it to the space V in Section 4. In Section 5 we show how the classical 3D vector potentials can be obtained from the stream function of Theorem 1.1.

In Section 6 we demonstrate that the Biot-Savart law, which recovers a vector field in H_0 from its vorticity (curl), is, in a precise way, dual to the problem of obtaining a stream function from a velocity field in H_0. We show that if there is an integral kernel associated with one of these problems it is also the kernel associated with the other problem.

In Appendix A we present the Hodge-Morrey decomposition of L^2 differential forms corresponding to the space H by using the results of [23], and give a few results regarding differential forms that we need in the proof of Theorem 1.1. Finally, in Appendix B, we present, for comparison, an outline of the more classical characterization of the space H_0.

Throughout, we follow the convention that $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_H$.

We write (u, v) for the inner product in L^2 or H. We write v^i for the i-th coordinate of a vector v; A^i_j for the element in the i-th row, j-th column of a matrix A; A^i for the i-th row of A; A_j for the j-th column of A. We follow the convention that repeated indices are implicitly summed, even when both indices are superscripts or both are subscripts.

2. Background material

Here, we present a number of tools we will use in what follows. The results themselves are classical, but their form and proofs are based primarily upon Galdi's invaluable introductory chapters in [14] along with material from the equally invaluable [16]. Table 1 converts some of Galdi's notation to the notation we are using, which may be useful for the reader who wishes to examine our explicit references to Galdi's text.

Table 1. Some notation in Galdi's [14]

Galdi	Our notation
$\mathcal{D}(\Omega)$	$\mathcal{V} = V \cap C_0^\infty(\Omega)$: *divergence-free* test functions
H_2	the space H defined in (1.2)
H^1 or H_2^1	$H \cap H^1(\Omega)$, with H as defined in in (1.2)
D^m	$\dot{H}^m(\Omega)$, the homogeneous Sobolev space
D_0^m	$\dot{H}_0^m(\Omega)$, the homogeneous Sobolev space
	(for us, Ω is bounded, so $\dot{H}_0^m(\Omega) = H_0^m(\Omega)$)

Definition 2.1. *As in* [25], *we define the space*

$$E(\Omega) := \{u \in L^2(\Omega)^d \colon \operatorname{div} u \in L^2(\Omega)\},$$

endowed with the norm, $\|u\| + \|\operatorname{div} u\|$. *We also define the space,*

$$\widetilde{E}(\Omega) := \{u \in L^2(\Omega)^3 \colon \operatorname{curl} u \in L^2(\Omega)\},$$

endowed with the norm, $\|u\| + \|\operatorname{curl} u\|$. *We use* $\widetilde{E}(\Omega)$ *only in 3D.*

We frequently integrate by parts using Lemma 2.2 (see Theorem 2.5 and (2.17) of [16]):

Lemma 2.2. *There exists a normal trace operator from* $E(\Omega)$ *to* $H^{-1/2}(\partial\Omega)$ *that continuously extends* $u \mapsto u \cdot \boldsymbol{n}|_{\partial\Omega}$ *from* $C(\overline{\Omega})$ *to* $E(\Omega)$. *We will simply write* $u \cdot \boldsymbol{n}$ *rather than naming this trace operator. For all* $u \in E(\Omega)$, $\varphi \in H^1(\Omega)$,

$$(u, \nabla\varphi) = -(\operatorname{div} u, \varphi) + \int_{\partial\Omega} (u \cdot \boldsymbol{n})\varphi,$$

where we have written $(u \cdot \boldsymbol{n}, \varphi)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$ *in the form of a boundary integral.*

In 3D, we also have the following (see Theorem 2.11 of [16]):

Lemma 2.3. *In 3D, there exists a tangential trace operator from* $\widetilde{E}(\Omega)$ *to* $H^{-1/2}(\partial\Omega)$ *that continuously extends* $u \mapsto u \times \boldsymbol{n}|_{\partial\Omega}$ *from* $C(\overline{\Omega})$ *to* $\widetilde{E}(\Omega)$. *We will simply write* $u \times \boldsymbol{n}$ *rather than naming this operator. For all* $u \in \widetilde{E}(\Omega)$, $\varphi \in H^1(\Omega)$,

$$(\operatorname{curl} u, \varphi) = (u, \operatorname{curl} \varphi) + \int_{\partial\Omega} (u \times \boldsymbol{n}) \cdot \varphi.$$

Poincaré's inequality holds not just for V, but for the larger space $H \cap H^1(\Omega)^d$:

Lemma 2.4. *There exists a constant* $C = C(\Omega)$ *such that for all* $u \in H \cap H^1(\Omega)^d$,

$$\|u\| \le C \|\nabla u\|.$$

Proof. For any $u \in H$,

$$\int_\Omega u^j = \int_\Omega u \cdot \nabla x^j = -\int_\Omega \operatorname{div} u \, x^j + \int_{\partial\Omega} (u \cdot \boldsymbol{n}) \, x^j = 0.$$

Hence, u has mean value zero, so Poincaré's inequality holds in the form stated. □

The well-posedness of solutions to the (stationary) Stokes problem is a classical, deep result, that lies at the heart of much of what we do. We will rely heavily upon the following version of it:

Proposition 2.5. *For any $f \in H^{-1}(\Omega)^d$, the (stationary) Stokes problem,*

$$\begin{cases} -\Delta v + \nabla q = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

has a unique (up to an additive constant for q) weak solution, $(v, q) \in H^1(\Omega)^d \times L^2(\mathbb{R})$. Moreover,

$$\|v\|_{H^1} + \|q\| \le C \|f\|_{H^{-1}}.$$

Proof. See, for instance, Proposition 4.2 of [3] or Theorem IV.1.1 of [14]. □

The well-posedness of the Stokes problem quickly yields a proof of the version of de Rham's lemma in Proposition 2.6. (This makes de Rham's lemma appear quite simple, yet de Rham's lemma is generally used in the proof of the well-posedness of the Stokes problem, as it is in the proof in [14] that we referenced. This perceived simplicity, then, is merely a consequence of the presentation, and hardly a self-contained proof.)

Proposition 2.6 (de Rham's Lemma)**.** *A vector field $f \in H^{-1}(\Omega)$ is the gradient of an L^2 function if and only if*

$$(f, v) = 0 \text{ for all } v \in V.$$

Proof. The forward direction is immediate. For the converse, given $f \in H^{-1}(\Omega)^d$, let $(v, q) \in V \times L^2(\Omega)$ be the solution to (2.1) given by Proposition 2.5. Since $v \in V$, we then have

$$0 = (f, v) = (-\Delta v, v) + (\nabla q, v) = \|\nabla v\|^2.$$

Hence, $v = 0$, so $f = \nabla q$. □

Key tools for us will be the decomposition of vector fields in $H_0^1(\Omega)$ given in Proposition 2.7 and the surjectivity of the divergence operator in Lemma 2.11. These results employ the space

$$L_0^2(\Omega) := \{f \in L^2(\Omega) \colon \int_\Omega f = 0\}.$$

Proposition 2.7. *The orthogonal decomposition, $H_0^1(\Omega)^d = V \oplus V^\perp$, holds with*

$$V^\perp = \{z \in H_0^1(\Omega)^d \colon \Delta z = \nabla q \text{ for some } q \in L^2(\Omega)\} \tag{2.2}$$

and $\|P_{V^\perp}\varphi\| \le C \|\operatorname{div} \varphi\|$. Moreover, the orthogonal projection $P_V \colon H_0^1(\Omega)^d \to V$ given by $\varphi = P_V \varphi + z$, where $(z, q) \in H_0^1(\Omega)^d \times L^2(\Omega)$ is a weak solution to

$$\begin{cases} -\Delta z + \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div} z = \operatorname{div} \varphi & \text{in } \Omega, \\ z = 0 & \text{on } \Omega. \end{cases} \tag{2.3}$$

Proof. This decomposition is given in Corollary 2.3 p. 23 of [16] (also see Lemma 2.2 of [18]). We give a proof here for completeness.

Starting with $\varphi \in H_0^1(\Omega)^d$, set $g = \operatorname{div} \varphi \in L^2(\Omega)$ and solve (non-uniquely), $\operatorname{div} w = g$ for $w \in H_0^1(\Omega)$. That we can solve this is a matter we will return to in Section 3; specifically, see Lemma 2.11. We have, $\|w\|_{H^1(\Omega)} \le C \|g\|$, as shown, for instance, in Exercise III.3.8 of [14].

Next let $f = \Delta w \in H^{-1}(\Omega)^d$, and let (v, q) be the unique solution to (2.1). Set $z = v + w$ and observe that $-\Delta z + \nabla q = f - \Delta w = 0$, $\operatorname{div} z = g = \operatorname{div} \varphi$, and $z = 0$ on $\partial\Omega$. Hence, (z, q) is a solution to (2.3), and we see that $P_V \varphi = \varphi - z$. Moreover,

$$(P_V \varphi, z)_V = (\nabla P_V \varphi, \nabla z) = -(\Delta z, P_V \varphi)_{H^{-1}, H_0^1} = -(\nabla q, P_V \varphi)_{H^{-1}, H_0^1} = 0.$$

Hence, we see that $z \in V^\perp$, so $V^\perp$ contains the set on the righthand side of (2.2).

It remains to show that $V^\perp$ contains *only* the set on the righthand side of (2.2). To see this, suppose that $z \in V^\perp$. Let $u \in V$ be arbitrary. Then

$$(u, z)_V = (\nabla u, \nabla z) = (\Delta z, u)_{H^{-1}, H_0^1} = 0.$$

Thus, $\Delta z = \nabla q$ for some $q \in L^2(\Omega)$ by Proposition 2.6. The bound $\|P_{V^\perp} \varphi\| \le C \|\operatorname{div} \varphi\|$ follows, for instance, from the Stokes problem bound in Exercise IV.1.1 of [14]. □

We could have directly used the solution to (2.3) to obtain the decomposition of $H_0^1(\Omega)^d$, but we wished to reduce the problem to the classical Stokes problem and (non-unique) inversion of the divergence operator.

Remark 2.8. *Going a little beyond (2.2), there is a bijection between $\nabla L^2(\Omega)$ and $V^\perp$ that comes from solving, for a given $q \in L^2(\Omega)$, the elliptic problem, $\Delta z^i = \partial_i q$ in Ω, $z^i = 0$ on $\partial\Omega$ for each i. We never, however, make use of this bijection.*

Remark 2.9. *Corollary 2.3 p. 23 of [16] gives the decomposition in Proposition 2.7, also using, as we did, a solution to the Stokes problem to obtain it. Interestingly, Amrouche and Girault in [3] invert this approach, using the decomposition to prove the existence of a solution to the Stokes problem. They then go on to give a proof of the decomposition that does not require knowledge of the existence of a solution to the Stokes problem (though it uses, and proves, that any solution satisfies certain estimates). All this is done in $W^{k,p}$ spaces for $p \in (1, \infty)$ and goes far beyond our purposes here.*

Note that the solution of (2.1) can be rephrased as follows:

Proposition 2.10.

$$H^{-1}(\Omega)^d = \Delta V \oplus \nabla L^2(\Omega) = \Delta V \oplus \Delta V^\perp = \Delta H_0^1(\Omega)^d.$$

Proof. Let $f \in H^{-1}(\Omega)^d$ and let (v, q) solve (2.1). This gives $H^{-1}(\Omega)^d = \Delta V + \nabla L^2(\Omega)$, and the uniqueness of the solution shows that the decomposition is a direct sum. Then (2.2) shows that $\Delta V^\perp = \nabla L^2(\Omega) = \nabla L^2(\Omega)$, hence also $H^{-1}(\Omega)^d = \Delta V \oplus \Delta V^\perp = \Delta(V + V^\perp) = \Delta H_0^1(\Omega)^d$, where we invoked Proposition 2.7. □

Lemma 2.11. *[Bogovskiĭ [7, 8]] For any $f \in L_0^2(\Omega)$ there exists $v \in H_0^1(\Omega)^d$ for which $\operatorname{div} v = f$. We can choose the (non-unique) solutions in such a way as to define a bounded linear operator $R\colon L_0^2(\Omega) \to H_0^1(\Omega)^d$ for which $\|\nabla Rf\| \le C \|f\|$. Moreover, we can assume that R maps into the space $V^\perp$.*

Proof. For the proof of all but the last sentence, see Bogovskiĭ [7, 8] or Theorem 2.4 of [9]. Then, for any $f \in L_0^2(\Omega)$, $\operatorname{div}(P_{V^\perp} Rf) = \operatorname{div} Rf = f$ and

$$\|\nabla(P_{V^\perp} Rf)\| = \|P_{V^\perp} Rf\|_{H_0^1(\Omega)^d} \le \|Rf\|_{H_0^1(\Omega)^d} = \|\nabla Rf\|.$$

So because $P_{V^\perp}$ is a continuous linear operator, we can replace R by $P_{V^\perp} R$. □

In fact, Bogovskiĭ in [7, 8] showed that the divergence is surjective for an arbitrary domain in $\mathbb{R}^d$. See, for instance, the historical comments on pages 208-209 of [2].

The difficult part of proving Lemma 2.11 is obtaining the surjectivity of the divergence as a map from $H_0^1(\Omega)^d$ to $L_0^2(\Omega)$: once that is obtained (or even just that the range of div is closed), the bounded linear (partial) inverse map R follows from basic functional analysis, by arguing much as we do in the proof of Theorem 1.1 in Section 3. (And see Remark 3.8.)

Moreover, since $P_{V^\perp}$ does not change the divergence of a vector field, the constant in the inequality in Lemma 2.11 is at least as small as the constant in Proposition 2.7. (This is a little misleading, however, as Lemma 2.11 is generally used to prove the estimates on the Stokes problem that lead to the inequality in Proposition 2.7.)

From R of Lemma 2.11, we define a matrix-valued operator, which we continue to call R, by applying R on each component of any vector in $L_0^2(\Omega)^d$:

$$R\colon L_0^2(\Omega)^d \to H_0^1(\Omega)^{d\times d}, \quad (Ru)^i := Ru^i. \tag{2.4}$$

We have been somewhat formal in our proof of Proposition 2.7, as we never gave a definition of a weak solution to (2.3) or even to the special case in (2.1). For this purpose, we unwind the definitions and results in [14][1], leading to the following:

Definition 2.12. *The pair $(z, q) \in H_0^1(\Omega) \times L^2(\Omega)$ is a weak solution to (2.3) if $z = v + w$, where $v, w \in H_0^1(\Omega)$, $\operatorname{div} w = \operatorname{div} \varphi$, and*

$$\begin{aligned} &(\nabla v, \nabla \psi) = \langle f, \psi\rangle \text{ for all } \psi \in \mathcal{V}, \\ &(\nabla v, \nabla \alpha) = \langle f, \alpha\rangle + (q, \operatorname{div} \alpha) \text{ for all } \alpha \in C_0^\infty(\Omega), \end{aligned}$$

where $f = \Delta w$ and $\mathcal{V} := V \cap C_0^\infty(\Omega)$ (this is what Galdi, very confusingly, calls $\mathcal{D}(\Omega)$). Also, $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now, since

$$(\nabla v, \nabla \psi) = (\nabla z, \nabla \psi) - (\nabla w, \nabla \psi) = (\nabla z, \nabla \psi) + (\Delta w, \psi) = (\nabla z, \nabla \psi) + (f, \psi)$$

and, similarly,

$$(\nabla v, \nabla \alpha) = (\nabla z, \nabla \alpha) + (f, \alpha),$$

we see that

$$\begin{aligned} &(\nabla z, \nabla \psi) = 0 \text{ for all } \psi \in V, \\ &(\nabla z, \nabla \alpha) = (q, \operatorname{div} \alpha) \text{ for all } \alpha \in C_0^\infty(\Omega). \end{aligned} \tag{2.5}$$

In the first equality we used the density of $\mathcal{V}$ in V. Although f is eliminated in (2.5), q still appears, and q ultimately derives from f. Hence, we cannot use these identities together to define a weak solution.

We have the following simple proposition:

Proposition 2.13. *If $u \in V$ then for all $\varphi \in H_0^1(\Omega)$,*

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi).$$

If $u \in V^\perp$ then for all $\varphi \in H_0^1(\Omega)$,

$$(\Delta u, \varphi) = (\Delta u, P_{V^\perp} \varphi).$$

[1]See Definition IV.1.1, Remark IV.1.1, (IV.1.3), Lemma IV.1.1 in [14], and note the sign change, since Galdi solves $\Delta v - \nabla q = f$.

Proof. Let $\varphi \in H_0^1(\Omega)$, which we can write as $\varphi = P_V \varphi + z$, as in Proposition 2.7. For $u \in V$,

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi) + (\Delta u, z) = (\Delta u, P_V \varphi) - (\nabla u, \nabla z) = (\Delta u, P_V \varphi)$$

by $(2.5)_1$.

For $u \in V^\perp$, we know by Proposition 2.7 that $\Delta u = \nabla q$ for some $q \in L^2(\Omega)$. Hence,

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi) + (\Delta u, z) = (\nabla q, P_V \varphi) + (\Delta u, P_{V^\perp} \varphi) = (\Delta u, P_{V^\perp} \varphi). \qquad \square$$

A simple and immediate consequence of Proposition 2.13 is the following:

Corollary 2.14. *Let $v \in \Delta V$. Then for all $\varphi \in H_0^1(\Omega)$,*

$$(v, \varphi) = (v, P_V \varphi).$$

Moreover,

$$(v, \psi) = 0 \text{ for all } \psi \in V \iff (v, \varphi) = 0 \text{ for all } \varphi \in H_0^1(\Omega).$$

Remark 2.15. *We might interpret Corollary 2.14 as saying that ΔV is a near proxy for V' as a distribution space, touching upon the subject of the next section.*

3. Proof of main result

In this section we prove our main result, Theorem 1.1. We present first some imporant existing results then establish a series of lemmas and propositions we will use in the (short) body of the proof of Theorem 1.1, with which we close the section.

Define the subspace

$$H_c := \{u \in H \colon \operatorname{curl} u = 0\}$$

of H. Here, we use the curl operator on $\mathbb{R}^d$ in the form,

$$\operatorname{curl} u := \nabla u - (\nabla u)^T. \tag{3.1}$$

That is, $\operatorname{curl} u$ is twice the antisymmetric gradient, the $d \times d$ matrix-valued function with $(\operatorname{curl} u)^i_j = \partial_j u^i - \partial_i u^j$. This form of the curl is convenient for integrating by parts (applying the divergence theorem) in flat space. In 2D, we can define $\operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$, the scalar curl, and in 3D we can define it as a vector in the usual way, denoting it curl_3 for clarity.

We have the following simple lemma:

Lemma 3.1. $H_c \subseteq \{v \in H \colon \Delta v = 0\}$.

Proof. Let $v \in H_c$, meaning that $\operatorname{div} v = 0$ and $\operatorname{curl} v = 0$. Then

$$\Delta v = \operatorname{div} \nabla v = \operatorname{div}(\nabla v - (\nabla v)^T) + \operatorname{div}(\nabla v)^T = \operatorname{div} \operatorname{curl} v = 0,$$

since $(\operatorname{div}(\nabla v)^T)^i = \partial_j \partial_i v^j = \partial_i \operatorname{div} v = 0$. $\square$

H_c is clearly closed, so we can define

$$H_0 := H_c^\perp,$$

the orthogonal complement of H_c in H. Hence, $H = H_0 \oplus H_c$.

Remark 3.2. *H_c is finite-dimensional for a large class of domains for which $\partial\Omega$ has a finite number of components. For smooth boundaries, this follows, for instance, from the discussion in Section 4.1 of [15]. For special classes of 3D Lipschitz domains, Helmholtz domains of [5], H_c (and H_0) can be characterized by making "cuts" in Ω that leave the remaining domain simply connected. This idea goes back to Helmholtz; see the historical comments in [11].*

This is the definition of H_0 that we will use to prove Theorem 1.1, as stated precisely in Theorem 1.1, below. We can view Theorem 1.1 as giving a direct characterization of H_0, but there is another direct characterization most often employed in 2 and 3 dimensions in terms of the vanishing of internal fluxes. We outline that perhaps somewhat more geometrical characterization of H_0 in Appendix B.

In [20] (Corollary 7.5), the simple tool in Lemma 3.3 was used to investigate conditions under which solutions to the Navier-Stokes equation for incompressible fluids converge to a solution to the Euler equations (the so-called *vanishing viscosity limit*).

Lemma 3.3. *For any $u \in H$ there exists (a non-unique) $A \in H_0^1(\Omega)^{d\times d}$ such that $u = \operatorname{div} A$; that is, such that $u^i = \partial_j A^i_j$.*

The idea of the proof is that a simple integration by parts as in the proof of Lemma 2.4 shows that each component of any $v \in H$ lies in $L^2_0(\Omega)$. But by Lemma 2.11, div maps $H^1_0(\Omega)^d$ onto $L^2_0(\Omega)$, so we can obtain each row of A independently. The proof of Lemma 3.3 is therefore quite simple, but it relies on the powerful and deep result in Lemma 2.11.

Left open in [20] was whether it could be assured that A in Lemma 3.3 is antisymmetric. In fact, such antisymmetry can be obtained, and was obtained in 3D by Borchers and Sohr in Theorem 2.1, Corollary 2.2 of [9], whose lowest regularity result can be stated as follows:

Lemma 3.4. *Assume that $d = 3$ and $\partial\Omega$ is $C^{1,1}$. For any $u \in H_0$ there exists $v \in H^1_0(\Omega)^3$ such that $u = \operatorname{curl}_3 v$ and $\Delta \operatorname{div} v = 0$. Moreover, one can choose the solutions in such a way as to define a bounded linear operator $S\colon H_0 \to H^1_0(\Omega)^3$ with $\|\nabla S u\| \le C\,\|u\|$.*

To see that Lemma 3.4 provides a 3D form of an extension of Lemma 3.3 to antisymmetric matrices, note that any 3×3 antisymmetric matrix can be written in the form,

$$A = \begin{pmatrix} 0 & \psi^3 & -\psi^2 \\ -\psi^3 & 0 & \psi^1 \\ \psi^2 & -\psi^1 & 0 \end{pmatrix}. \tag{3.2}$$

We can define a bijection Q from a vector in $\mathbb{R}^3$ to an antisymmetric $d\times d$ matrix, by setting $Q(\psi) = Q(\psi^1, \psi^2, \psi^3)$ to be the matrix in (3.2), and we can write that $\operatorname{div} Q\psi = \operatorname{curl}_3 \psi$. The claim in Theorem 1.1, then, is the natural extension of Lemma 3.4 to $d \ge 2$.

The simple argument in Proposition 3.5 shows that $\operatorname{div} X_0$ is at least dense in H_0:

Proposition 3.5. $H_0 = \overline{\operatorname{div} X_0}$.

Proof. First, we show that $\operatorname{div} X_0$ is a subspace of H. To see this, observe that if $u \in \operatorname{div} X_0$ then $u^i = \operatorname{div} A^i = \partial_j A^i_j$. Hence, $\operatorname{div} u = \partial_{ij} A^i_j = -\partial_{ij} A^j_i = -\partial_{ji} A^i_j = -\partial_{ij} A^i_j = -\operatorname{div} u$, so $\operatorname{div} u = 0$. (That $\operatorname{div} u = \operatorname{div}\operatorname{div} A = 0$ is a reflection of $\delta^2 = 0$ when A is expressed as a 2-form as in Appendix A.)

Moreover, since A^i_j is constant along the boundary, ∇A^i_j is normal to the boundary, so we can write, $\nabla A^i_j = \alpha^i_j \boldsymbol{n}$, where

$$\alpha^i_j = \frac{\partial A^i_j}{\partial \boldsymbol{n}} = -\frac{\partial A^j_i}{\partial \boldsymbol{n}} = -\alpha^j_i.$$

Then,

$$\partial_j A^i_j = \nabla A^i_j \cdot \mathbf{e}^j = \alpha^i_j \boldsymbol{n} \cdot \mathbf{e}^j = \alpha^i_j n^j$$

so, using that $\alpha^i_j = -\alpha^j_i$,

$$u \cdot \boldsymbol{n} = \operatorname{div} A \cdot \boldsymbol{n} = \operatorname{div} A^i n^i = \partial_j A^i_j n^i = \alpha^i_j n^j n^i = -\alpha^j_i n^j n^i = -\alpha^i_j n^j n^i = -u \cdot \boldsymbol{n},$$

so $u \cdot \boldsymbol{n} = 0$. We conclude that $\operatorname{div} X_0 \subseteq H$.

Here is the proof that $\operatorname{div} X_0 \subseteq H$ specifically in three dimensions, which gives maybe a little extra insight. We have,

$$A = \begin{pmatrix} 0 & f & g \\ -f & 0 & h \\ -g & -h & 0 \end{pmatrix},$$

so $\psi = Q^{-1} A = (h, -g, f)$, and

$$u = \operatorname{curl}_3 \psi = (\partial_2 f + \partial_3 g, -\partial_1 f + \partial_3 h, -\partial_1 g - \partial_2 h).$$

We then automatically have

$$0 = \operatorname{div} u = \partial_{12} f + \partial_{13} g - \partial_{21} f + \partial_{23} h - \partial_{31} g - \partial_{32} h,$$

as required. Also,

$$0 = u \cdot \boldsymbol{n} = \operatorname{curl}_3 \psi \cdot \boldsymbol{n} = (\partial_2 f + \partial_3 g) n^1 + (-\partial_1 f + \partial_3 h) n^2 + (-\partial_1 g - \partial_2 h) n^3.$$

But A, and so f, g, and h, are constant along each boundary component. This means that ∇f is parallel to $\boldsymbol{n}$, so that

$$\partial_j f = \nabla f \cdot e^j = \frac{\partial f}{\partial \boldsymbol{n}} n^j$$

and similarly,

$$\partial_j g = \frac{\partial g}{\partial \boldsymbol{n}} n^j, \quad \partial_j h = \frac{\partial h}{\partial \boldsymbol{n}} n^j.$$

Thus,

$$\begin{aligned} &(\partial_2 f + \partial_3 g) n^1 + (-\partial_1 f + \partial_3 h) n^2 + (-\partial_1 g - \partial_2 h) n^3 \\ &\quad = (\frac{\partial f}{\partial \boldsymbol{n}} n^2 + \frac{\partial g}{\partial \boldsymbol{n}} n^3) n^1 + (-\frac{\partial f}{\partial \boldsymbol{n}} n^1 + \frac{\partial h}{\partial \boldsymbol{n}} n^3) n^2 + (-\frac{\partial g}{\partial \boldsymbol{n}} n^1 - \frac{\partial h}{\partial \boldsymbol{n}} n^2) n^3 = 0. \end{aligned}$$

We now show that $(\operatorname{div} X_0)^\perp = H_c$. Let $A \in X_0$ and $v \in H$ be arbitrary. Then $u := \operatorname{div} A$ is an arbitrary element of $\operatorname{div} X_0$. Applying Lemma 2.2 and using $A = 0$ on $\partial\Omega$,

$$\begin{aligned} (u, v) &= (\operatorname{div} A, v) = -(A, \nabla v) = -(A, \nabla v - (\nabla v)^T) - (A, (\nabla v)^T) \\ &= -(A, \operatorname{curl} v) - (A^T, \nabla v) = -(A, \operatorname{curl} v) + (A, \nabla v). \end{aligned}$$

Hence, $(A, \nabla v) = (1/2)(A, \operatorname{curl} v)$, and because both A and $\operatorname{curl} v$ are antisymmetric,

$$(u, v) = -(A, \nabla v) = -\frac{1}{2}(A, \operatorname{curl} v) = -\sum_{i<j} A^i_j (\operatorname{curl} v)^i_j.$$

We can choose the components A^i_j independently for $i < j$, and $H^1_0(\Omega)$ is dense in $L^2(\Omega)$, so we conclude that $(u, v) = 0$ for all $u \in \operatorname{div} X_0$ if and only if $\operatorname{curl} v = 0$; that is, if and only if $v \in H_c$. It then follows that $(\operatorname{div} X_0)^\perp = H_c$ so that, in fact, $\overline{\operatorname{div} X_0} = ((\operatorname{div} X_0)^\perp)^\perp = H_c^\perp = H_0$. □

As we see in the proof of Proposition 3.5, the antisymmetry of $A \in X_0$ insures that $\operatorname{div} A \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. This need not be true without a symmetry assumption, but if $\operatorname{div} A$ happens to be in H so does A^T, as we see in Lemma 3.6.

Lemma 3.6. *Let $A \in H^1_0(\Omega)^{d \times d}$, with no symmetry assumption, but with $\operatorname{div} A = u \in H$. Then $\operatorname{div} A^T$ is also in H.*

Proof. We have, $0 = \operatorname{div} u = \operatorname{div} \operatorname{div} A = \partial_i \partial_j A^i_j = \partial_i \partial_j A^j_i = \operatorname{div} \operatorname{div} A^T$. Decomposing A into its symmetric and antisymmetric parts, $A_S = (1/2)(A + A^T)$ and $A_A = (1/2)(A - A^T) \in X_0$, it follows that $\operatorname{div} \operatorname{div} A_S = 0$ and, from Proposition 3.5, that $\operatorname{div} A_A \in H_0$. Hence, $\operatorname{div} A^T = \operatorname{div} A - 2 \operatorname{div} A_A \in H$. □

The operator R of (2.4) allows us to easily establish that $\operatorname{div} X_0$ actually yields all of H_0:

Proposition 3.7. $H_0 = \operatorname{div} X_0$.

Proof. We have, $\operatorname{div} X_0 = \operatorname{div}(R \operatorname{div} X_0) = \operatorname{div} Y$, where $Y = R \operatorname{div} X_0$. It follows from Proposition 3.5 that $\operatorname{div} Y$ is dense in H_0. If we can show that it is closed, then we are done.

Let (u_n) be a sequence in $\operatorname{div} Y$ converging to u in H_0. Then $u_n = \operatorname{div} B_n$ with $B_n = Ru_n$ in Y, and we have from Lemma 2.11 that $\|\nabla B_n\| \le C \|u_n\|$. Since (u_n) converges, it is Cauchy and hence (B_n) is Cauchy and so converges to some $B \in Y$ with $u = \operatorname{div} B$. This shows that $H_0 = \operatorname{div} Y = \operatorname{div} X_0$. □

It remains only to obtain the bounded linear map S of Theorem 1.1. Examining the proof of Proposition 3.7, we see that $B_n = Ru_n$ in Y has some D_n in X_0 for which $R \operatorname{div} D_n = B_n$, but the convergence of (B_n) does not mean the convergence of (D_n). To surmount this difficulty, and obtain S, we restrict the domain of div to a subspace:

Proof of Theorem 1.1. Observe that $\operatorname{div} A = \operatorname{div} B$ for $A, B \in X_0$ if and only if $B = A+E$ for some E in $V^d \cap X_0$, a closed subspace of X_0. Letting $Y_0 = (V^d \cap X_0)^\perp$, the orthogonal complement of $V^d \cap X_0$ in X_0 *as a Hilbert space*, $\operatorname{div}\colon Y_0 \to H_0$ is a continuous bijection. It follows from a corollary of the open mapping theorem (see, for instance, Corollary 2.7 of [10]) that the inverse map, $S := \operatorname{div}|_{Y_0}^{-1}$, is also continuous. But this means that, $\|Su\|_{X_0} = \|Su\|_{Y_0} \le C \|u\|_{H_0}$, giving us the bounded linear map of Theorem 1.1. □

The Baire category theorem appears through the proof of the corollary to the open mapping theorem we applied. Hence, the constant we obtain in $\|\nabla Su\| \le C \|\operatorname{div} u\|$ is not effectively computable, although we can see that C is no smaller than the constant in Lemma 2.11.

Remark 3.8. *Although the adjoints to the two forms of* div *appearing in Lemma 2.11 and Theorem 1.1 never appear explicitly, they are, in a sense, hiding in the proofs. It can be shown that the adjoint of* $\operatorname{div}\colon X_0 \to H_0$ *is* $-(1/2)\operatorname{curl}$, *whose null space is* H_c. *Since* div *is a closed map,* $\operatorname{div} X_0$ *is closed if and only if it equals* $H_c^\perp =: H_0$. *Similarly, it can be shown that the adjoint of* $\operatorname{div}\colon H_0^1(\Omega)^d \to L_0^2(\Omega)$ *is* $-\nabla$, *whose null space is trivial. Hence,* $\operatorname{div} H_0^1(\Omega)^d$ *is closed if only if it equals all of* $L_0^2(\mathbb{R}^d)$. *Proving that the range of either version of* div *is closed is the hard part of each proof, but we were able to leverage the powerful result in Lemma 2.11 to obtain the hard part for Theorem 1.1 with minimal effort.*

We avoided characterizing the space $Y_0 = (V^d \cap X_0)^\perp$ *explicitly, but given that the adjoint of* $\operatorname{div}\colon X_0 \to H_0$ *is* $-(1/2)\operatorname{curl}$, *one can show that* $Y_0 = \{z \in X_0 \colon \Delta z = \operatorname{curl} q \text{ for some } q \in L_0^2(\Omega)^d\}$, *in analogy with Proposition 2.7. In 3D, this is* $Y_0 = \{z \in H_0^1(\Omega)^3 \colon \Delta z = \operatorname{curl}_3 q, q \in L_0^2(\Omega)^d\}$, *which yields* $\Delta \operatorname{div} Su = 0$, *as in Lemma 3.4.*

4. Higher regularity

Bogovskiĭ in [7, 8] showed more than what we stated in Lemma 2.11 (see Theorem 2.4 of [9]):

Lemma 4.1. *[Bogovskiĭ [7, 8]] Let* $p \in (1, \infty)$ *and* $m \ge 0$ *be an integer. Define* $H_{0,0}^{m,p}(\Omega)$ *to be the functions in* $H_0^{m,p}(\Omega)$ *having mean zero. There exists a bounded linear operator* $R = R_{m,p}\colon H_{0,0}^{m,p}(\Omega) \to H_0^{m+1,p}(\Omega)^d$ *satisfying* $\operatorname{div} Rf = f$ *with* $\|\nabla^{m+1} Rf\|_{L^p(\Omega)} \le C \|\nabla^m f\|_{L^p(\Omega)}$.

Restricting ourselves to $p = 2$, we define, as in (2.4), a matrix-valued operator $R_m = R_{m,2}$:

$$R_m \colon H_0^m(\Omega)^d \to H_0^{m+1}(\Omega)^{d\times d}, \quad (R_m u)^i := R_m u^i.$$

We will use Lemma 4.1 to study the stream function for an element of V.

Theorem 4.2. *The map* S *of Theorem 1.1 also maps* $V \cap H_0$ *continuously onto* $Y_0 \cap H_0^2(\Omega)^{d\times d}$, *where* $Y_0 = (V^d \cap X_0)^\perp$.

Proof. The space $Y_0^2 := Y_0 \cap H_0^2(\Omega)^{d\times d}$ is dense in Y_0 and $\operatorname{div}\colon Y_0 \to H_0$ is a continuous surjection, so $\operatorname{div} Y_0^2$ is dense in H_0. Moreover, $\operatorname{div} Y_0^2 \subseteq V \cap H_0$, so $\operatorname{div} Y_0^2$ is dense in $V \cap H_0$.

Then, arguing as in the proof of Proposition 3.7, $\operatorname{div} Y_0^2 = \operatorname{div}(R_1 \operatorname{div} Y_0^2)$ is closed in $V \cap H_0$ and hence $\operatorname{div} Y_0^2 = V \cap H_0$. Because $\operatorname{div}|_{Y_0}$ is injective it also holds that $\operatorname{div}|_{Y_0^2}$ is injective. Finally, arguing as in the proof of Theorem 1.1, the inverse map, $\operatorname{div}|_{Y_0^2}^{-1}$, is continuous. But this is the same map S as in Theorem 1.1, restricted to $V \cap H_0$. □

Remark 4.3. *Using R_m, one can extend Theorem 4.2 to $S\colon H_0 \cap H_0^m(\Omega)^d \to Y_0 \cap H_0^{m+1}(\Omega)^{d\times d}$, though its utility is likely limited for $m \geq 2$. Similarly, one can employ Lemma 4.1 to develop L^p bounds in analog with Theorem 1.1.*

5. 3D vector potentials

In 2D, the stream function of Theorem 1.1 is unique in that no other $A \in X_0$ satisfies $\operatorname{div} A = u$ for a given $u \in H_0$. This is not, however, true in any higher dimension. Let us take a closer look at 3D. There, for $u \in H_0$, our "stream function" is to satisfy

$$\begin{cases} \operatorname{curl}_3 \psi = u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

We have, however, complete freedom to choose the divergence. Hence, if p is any scalar field for which $\nabla p = 0$ on $\partial\Omega$ (for instance, any $p \in H_0^2(\Omega)$) then we also have

$$\begin{cases} \operatorname{curl}_3(\psi + \nabla p) = u & \text{in } \Omega, \\ \psi + \nabla p = 0 & \text{on } \partial\Omega, \end{cases}$$

so $\psi + \nabla p$ is also a stream function.

This kind of argument also leads to the perhaps more familiar formulation of a 3D stream function in Proposition 5.1.

We can use Theorem 1.1 to obtain the more classical versions of 3D stream functions or vector potentials of Propositions 5.1 and 5.2 (cf., Theorems 3.5 and 3.6 Chapter I of [16] or Theorem 3.12 and 3.17 of [1]).

Proposition 5.1. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in H$ for which $\operatorname{curl}_3 \overline{\psi} = u$. The vector potential is unique up to the addition of an arbitrary element in H_c; or, equivalently, the vector potential is unique if we require it to lie in H_0. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in H \cap H^1(\Omega)^3$.*

Proof. First, we show existence. Let ψ be the 3D stream function given by Theorem 1.1 and let p be the unique (up to an additive constant) solution to the Neumann problem,

$$\begin{cases} \Delta p = -\operatorname{div} \psi & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

If $\partial\Omega$ is Lipschitz, we can only conclude that $p \in H^1(\Omega)$ so $\nabla p \in L^2(\Omega)^3$, but if $\partial\Omega$ is $C^{1,1}$ then $p \in H^2(\Omega)$ so $\nabla p \in H^1(\Omega)^3$. Letting $\overline{\psi} = \psi + \nabla p$, we see that

$$\begin{cases} \operatorname{curl}_3 \overline{\psi} = u & \text{in } \Omega, \\ \operatorname{div} \overline{\psi} = 0 & \text{in } \Omega, \\ \overline{\psi} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Hence, $\overline{\psi} \in H$ with $\operatorname{curl}_3 \overline{\psi} = u$, as required, with $\overline{\psi} \in H \cap H^1(\Omega)^3$ if $\partial\Omega$ is $C^{1,1}$.

Adding any element of H_c to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in H and is curl-free; that is, it lies in H_c. This proves the uniqueness statement. □

The need for a more regular boundary in Proposition 5.1 arose from the need to obtain a classical solution to an elliptic problem, an issue we avoided in the proof of Theorem 1.1.

Define the space,

$$\widetilde{H} := \{\psi \in L^2(\Omega)^3 \colon \operatorname{div}\psi = 0, \operatorname{curl}\psi \in L^2(\Omega)^3, \psi \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

with the norm $\|\psi\|_{\widetilde{H}} := \|\psi\| + \|\operatorname{curl}\psi\|$. That $\psi \times \boldsymbol{n}$ makes sense in terms of a trace is shown in Theorem 2.11 of [16]. Also let

$$\widetilde{H}_c := \{\psi \in \widetilde{H} \colon \operatorname{curl}\psi = 0\}.$$

Proposition 5.2. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in \widetilde{H}$ for which $\operatorname{curl}_3 \overline{\psi} = u$. The vector potential is unique up to the addition of an arbitrary element in $\widetilde{H}_c$. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in \widetilde{H} \cap H^1(\Omega)^3$.*

Proof. The proof is the same as that of Proposition 5.1, but using the boundary condition $p = 0$ on $\partial\Omega$ in (5.1), noting that then $\nabla p \times \boldsymbol{n} = 0$. As in (5.2), this gives $\operatorname{curl}_3 \overline{\psi} = u$ and $\operatorname{div}\overline{\psi} = 0$ but with $\overline{\psi} \times \boldsymbol{n} = \psi \times \boldsymbol{n} + \nabla p \times \boldsymbol{n} = 0$ on $\partial\Omega$. Adding any element of $\widetilde{H}_c$ to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in $\widetilde{H}$ and is curl-free; that is, it lies in $\widetilde{H}_c$. This proves the uniqueness statement. □

Suppose that $\Omega \subseteq \mathbb{R}^3$ has a finite number of boundary components $\Gamma_0, \cdots, \Gamma_N$. Then the vector potential $\overline{\psi}$ of Proposition 5.2 is unique if one imposes the condition $\int_{\Gamma_i} \overline{\psi} \cdot \boldsymbol{n} = 0$ for all i. This is shown in Theorem 3.6 Chapter I of [16] and 3.17 of [1]. The idea, in essence, is to use the boundary condition $p = c_i$ on Γ_i instead of $p = 0$ on $\partial\Omega$ in (5.1), and show that, fixing $c_0 = 0$, there exists a unique choice of the c_i such that $\int_{\Gamma_i} \nabla p \cdot \boldsymbol{n} = -\int_{\Gamma_i} \psi \cdot \boldsymbol{n}$ for all i. See, for instance, the argument on pages 49-50 of [16].

Remark 5.3. *The boundary condition $\psi \times \boldsymbol{n} = 0$ in the definition of $\widetilde{H}$ corresponds to $A\boldsymbol{n} = 0$ via the bijection given by (3.2). This suggests that Proposition 5.2 has a natural higher-dimensional formulation. Indeed for smooth boundaries it does, as follows from Theorem 3.1.1 of [23], in which $\overline{\psi}$ becomes a co-closed 2-form.*

6. A Biot-Savart kernel?

The Biot-Savart law is the classical method for obtaining a vector field in, say $H_0 \cap H^1(\Omega)^d$, having a given vorticity in $L^2(\Omega)$. But the existence of an integral representation for this law, that is, of a Biot-Savart kernel, for a bounded domain is a largely open question: the existence for all of $\mathbb{R}^d$ and for a bounded domain in $\mathbb{R}^2$ is quite classical, but only recently, in [13], has a kernel for a 3D bounded domain been obtained, and that was for domains with smooth boundary. In dimensions higher than 3 a kernel has not been obtained even for smooth domains. (Also, see the introductory comments in [13].)

We can show, however, the conditional result in Theorem 6.1: a Biot-Savart kernel exists if and only if a kernel for the stream function exists, and there is a duality between them.

Theorem 6.1. *We say that $K \in L^1(\Omega^2)^d$ is a kernel for the Biot-Savart law on Ω if for all antisymmetric $B \in C(\overline{\Omega})^{d \times d}$,*

$$u^i(x) = \int_\Omega K^j(x, y) B^i_j(y)\, dy \tag{6.1}$$

lies in H_0 with $\operatorname{curl} u = B$. We say that $T \in L^1(\Omega^2)^d$ is a kernel for the stream function on Ω if for all $v \in H_0 \cap C^\infty(\overline{\Omega})^d$,

$$A^i_j(y) = \int_\Omega T_j(x, y) v^i(x)\, dx - \int_\Omega T_i(x, y) v^j(x)\, dx \tag{6.2}$$

lies in X_0 *with* $\operatorname{div} A = v$. *A kernel* K *exists if and only if a kernel* T *exists, and in such a case, we can set* $K = T$.

Proof. Assume that T exists. Let $v \in H_0 \cap C^\infty(\overline{\Omega})^d$ and let A be as given in (6.2). Let $u \in H_0 \cap C^\infty(\overline{\Omega})^d$ with $\operatorname{curl} u = B$. Then, applying Fubini's theorem,

$$\begin{aligned}
(2u, v) &= 2(u, \operatorname{div} A) = -2(\nabla u, A) = -(\nabla u, A) - ((\nabla u)^T, A^T) \\
&= -(\nabla u, A) + ((\nabla u)^T, A) = -(\operatorname{curl} u, A) = -(B, A) \\
&= \int_\Omega \int_\Omega B^i_j(y) \left[T_i(x, y) v^j(x)\, dx - \int_\Omega T_j(x, y) v^i(x)\, dx \right] dy \\
&= \int_\Omega \int_\Omega B^i_j(y) T_i(x, y) v^j(x)\, dx - \int_\Omega \int_\Omega B^i_j(y) T_j(x, y) v^i(x)\, dx\, dy \\
&= \int_\Omega \int_\Omega B^i_j(y) T_i(x, y) v^j(x)\, dx - \int_\Omega \int_\Omega B^j_i(y) T_i(x, y) v^j(x)\, dx\, dy \\
&= 2 \int_\Omega \int_\Omega B^i_j(y) T_i(x, y) v^j(x)\, dx\, dy = (2w, v),
\end{aligned}$$

where

$$w(x) = \int_\Omega T_i(x, y) B^i_j(y)\, dy.$$

Since $H_0 \cap C^\infty(\overline{\Omega})^d$ is dense in H_0 it follows that we must have $u = w$. Examining (6.1), then, we see that we can set $K = T$.

To show that the existence of K implies the existence of T, we reverse the order of the integrations by parts. □

7. A constructive approach

In this section, we give a more constructive, somewhat geometric proof of Theorem 1.1. We will need, however, a key fact, described in Remark 7.2, not firmly established in the literature concerning smooth manifolds embedded in $\mathbb{R}^d$ having Lipschitz boundaries. Hence, this approach should be considered incomplete (though it would be complete for smooth boundaries).

Our starting point is Proposition 3.5, which we can use to obtain important information about any element of H_0:

Corollary 7.1. *Let* $u \in H_0$ *and let* C *be any generator of* $H_{d-2}(\Omega^C, \partial\Omega^C; \mathbb{R})$. *If* Σ *is any* $(d-1)$*-cycle in* Ω *for which* $\partial\Sigma = C$ *then*

$$\int_\Sigma u \cdot \boldsymbol{n} = 0.$$

Proof. Let (ψ_n) be a sequence in X_0 with $u_n := \operatorname{div} \psi_n \to u$ in H, the existence of such a sequence being assured by Proposition 3.5. Then, using that for a $(d-1)$-form, $*\delta = **d* = (-1)^d d*$ and applying Stokes's theorem,

$$\int_\Sigma u_n \cdot \boldsymbol{n} = \int_\Sigma *\xi u_n = \int_\Sigma *\xi \operatorname{div} \psi_n = \int_\Sigma *\delta\theta\psi_n = (-1)^d \int_\Sigma d * \theta\psi_n = (-1)^d \int_C *\theta\psi_n = 0.$$

In the first step, we used Lemma A.10. In the last step, we used that ψ_n, and so $\theta\psi_n$ and $*\theta\psi_n$, vanish on the boundary. (See Remark 7.2.)

We will apply the analog of Lemma 2.2 for Σ, where now $E(\Omega) = E(\Sigma)$. Since u_n and u are divergence-free, observe that $u_n \to u$ in $E(\Sigma)$ so $u_n \cdot \boldsymbol{n} \to u \cdot \boldsymbol{n}$ in $H^{-\frac{1}{2}}(C)$, from which the result follows. □

Remark 7.2 (Difficulty in the proof of Corollary 7.1)**.** *Being a Lipschitz domain in $\mathbb{R}^d$, Ω is also a topological manifold, and so singular homology makes sense for it. However, for integration, we need to have some degree of smoothness to the chains or cycles over which we are integrating. In particular, while the generator C of $H_{d-2}(\Omega^C, \partial\Omega^C; \mathbb{R})$ will have Lipschitz regularity, we have no inherent regularity at all of the $(d-1)$-cycle Σ for which $\partial\Sigma = C$. Lipschitz regularity of C is sufficient, but the lack of regularity of Σ is an obstacle.*

Alternate proof of Theorem 1.1. In 2D, any $A \in H_0$ would be of the form

$$A = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix}, \tag{7.1}$$

so then $\operatorname{div} A = (-\partial_2\psi, \partial_1\psi) = \nabla^\perp\psi = u$. Hence, ψ is the classical 2D stream function for $u \in H_0$ (so also A is unique, since we require it to vanish on the boundary.)

In 3D, making the bijection in (3.2), we see that

$$u = (\partial_2\psi^3 - \partial_3\psi^2, \partial_1\psi^3 - \partial_3\psi^1, \partial_1\psi^2 - \partial_2\psi^1) = \operatorname{curl}_3 \psi.$$

Hence if the general result for $d \geq 3$ holds, we obtain the 3D result.

So assume now that $d \geq 3$. We will establish the expression for H_0 following as closely as possible the 3D argument in [9].

Assume that $u \in H_0$ and let $\mathcal{E}_0 u$ be u extended by zero to all of $\mathbb{R}^d$. Then $\mathcal{E}_0 u \in L^2(\mathbb{R}^d)$ and $\operatorname{div} \mathcal{E}_0 u = 0$ still holds in the sense of distributions, so $\mathcal{E}_0 u \in H(\mathbb{R}^d)$. Let G be the fundamental solution to the Laplacian in $\mathbb{R}^d$, so $\Delta G * f = f$, and define the antisymmetric matrix-valued function ψ by

$$\psi^i_j := \partial_j G * \mathcal{E}_0 u^i - \partial_i G * \mathcal{E}_0 u^j.$$

(Formally, $\psi = G * (\operatorname{curl} \mathcal{E}_0 u)$.) Then

$$(\operatorname{div} \psi)^i = \partial_{jj} G * \mathcal{E}_0 u^i - \partial_i G * \partial_j \mathcal{E}_0 u^j = \Delta G * \mathcal{E}_0 u^i - \partial_i G * (\operatorname{div} \mathcal{E}_0 u) = \mathcal{E}_0 u^i.$$

These calculations are as convolutions of $\mathcal{E}_0 u \in \mathcal{E}'(\mathbb{R}^d)$, the space of compactly supported distributions, with derivatives of $G \in \mathcal{D}'(\mathbb{R}^d)$, the space of distributions. Or the convolution defining ψ can be viewed as the convolution of the L^1_{loc}-function $\partial_i G$ with the compactly supported $\mathcal{E}_0 u^j$, and the expression for $\operatorname{div} \psi$ can be verified by a standard limiting argument. Hence, $\mathcal{E}_0 u = \operatorname{div} \psi$: this is a form of the Biot-Savart law (see, for example, Chapter 1 of [12]).

Moreover, for any k,

$$\partial_k \psi^i_j := \partial_k \partial_j G * \mathcal{E}_0 u^i - \partial_k \partial_i G * \mathcal{E}_0 u^j.$$

This calculation holds as the convolution of an element in $\mathcal{E}'(\mathbb{R}^d)$ with an element of $\mathcal{D}'(\mathbb{R}^d)$, but in that form, $\partial_k \partial_j G *$ is not a Calderon-Zygmund operator. A more careful, but standard, argument (see, for instance, Proposition 6.1 of [4]) would give that

$$\partial_k \partial_j G * \mathcal{E}_0 u^i(x) = \frac{\delta_{jk}}{d} \mathcal{E}_0 u^i(x) + \text{p. v.} \int \partial_k \partial_j G(x-y)\, \mathcal{E}_0 u^i(y)\, dy.$$

The principal value integrals are Calderon-Zygmund operators applied to $\mathcal{E}_0 u^i$, so each term on the right-hand side lies in $L^2(\mathbb{R}^d)$. Hence, $\partial_k \psi^i_j \in L^2(\mathbb{R}^d)$ so $\psi \in H^1(\mathbb{R}^d)^{d \times d}$.

Nonetheless, ψ does not satisfy the boundary condition, $\psi = 0$ on $\partial\Omega$. To correct for this, let us first consider the 3D approach taken in [9], using the bijection Q given by (3.2).

In the language of the 3D curl, we have $\operatorname{curl}_3 Q^{-1}\psi = \mathcal{E}_0 u$ on $\mathbb{R}^3$. In particular, $\operatorname{curl}_3 Q^{-1}\psi = 0$ on $U := \mathbb{R}^3 \setminus \overline{\Omega}$. Let γ be any simple closed curve that is a generator of $H_1(\Omega^C, \partial\Omega^C; \mathbb{R})$

that generate $H_{d-1}(\Omega, \partial\Omega; \mathbb{R})$, the $(d-1)$-dimensional real homology class of Ω relative to its boundary and let Σ be a smooth surface in Ω whose boundary is γ. Then by Stokes's theorem, and using that $Q^{-1}\psi \in H^1(\mathbb{R}^3)$ so its trace on $\partial\Omega$ is well-defined,

$$\int_\gamma Q^{-1}\psi \cdot ds = \int_{\Sigma_i} \operatorname{curl}_3 Q^{-1}\psi \cdot \boldsymbol{n} = \int_{\Sigma_i} u \cdot \boldsymbol{n} = 0, \tag{7.2}$$

the last equality following from Corollary 7.1 since $u \in H_0$. It follows that $\psi = \nabla p$ on U (by the classical, 3D version of Lemma A.9) for some $p \in H^2(U)$ (since $\nabla p = \psi \in H^1(\mathbb{R}^2)^{d\times d}$). Extend p to lie in $H^2(\mathbb{R}^d)$ using Theorem 5' p. 181 of [24] (and a cutoff function inside Ω). Let $N = Q\nabla p$. Then $A := \psi - N \in H^1_0(\Omega)^{d\times d}$ and is antisymmetric, and $\operatorname{div} A = \operatorname{div}(\psi - N) = \operatorname{div}\psi = u$.

In higher dimension, the argument is similar, though now we need to use the language of differential forms. In Appendix A, we define a bijection θ that maps ψ to a $(d-2)$-form on Ω, and a bijection ξ that maps vector fields on Ω to $d-1$ forms on Ω with the property that $d\theta = \xi \operatorname{div}$. Then $d\theta\psi = \xi(\mathcal{E}_0 u)$, which vanishes on $U := \mathbb{R}^d \setminus \overline{\Omega}$; that is, $\theta\psi$ is closed on U.

We now show that, in fact, $\theta\psi$ is exact on U. Let C be any generator of $H_{d-2}(\Omega^C, \partial\Omega^C; \mathbb{R})$ and let Σ be a $(d-1)$-cycle whose boundary is C. Now, although ψ does not vanish on $\partial\Omega$, we still have $\operatorname{div}\psi = u$ on Ω, and so can integrate just as in the proof of Corollary 7.1, though now we do so in reverse order:

$$\int_C *\theta\psi = (-1)^d \int_\Sigma d * \theta\psi = \int_\Sigma *\delta \operatorname{div}\psi = \int_\Sigma *\xi \operatorname{div}\psi = \int_\Sigma \xi u = \int_\Sigma u \cdot \boldsymbol{n} = 0. \tag{7.3}$$

The vanishing of the final integral follows from Corollary 7.1 since $u \in H_0$.

It follows from Lemma A.9 that $*\theta\psi$ is exact on U. Thus, $*\theta\psi = dp$ for some 0-form $p \in H^2(U)$. We then extend p to $H^2(\mathbb{R}^d)$, and set $A = \psi - \theta^{-1}dp$, where we used that $**$ is the identity when applied to a $(d-2)$-form.

Although A is not unique, we have constructed it in an unambiguous way (that depended only upon our choice of extension operator from $H^2(U)$ to $H^2(\mathbb{R}^d)$). Hence, the operator $S\colon H_0 \to X_0$, $Su = A$, is well-defined, and $\|Su\|_{X_0} \le C\|u\|_H$. $\square$

Remark 7.3. *Rather than extending p into Ω using an extension operator, as we did in the proof of Theorem 1.1, which requires only (in fact, less than) Lipschitz regularity of the boundary, the authors of [9], working specifically in 3D, solve a biharmonic equation on Ω to obtain the equivalent of what we have called N in the proof of Theorem 1.1. This requires a $C^{1,1}$ boundary to know that $N \in H^1(\Omega)$, but gives that $\operatorname{div} v$ in Lemma 3.4 is harmonic on Ω. Assuming that $u \in H_0$ vanishes to order m on the boundary, they use a solution of a higher-order polyharmonic equation with higher-regularity boundaries, to obtain higher regularity of A. We will consider, in Theorem 7.5, only the one additional derivative of regularity gained by assuming that $u \in V$ (but without adding additional regularity on the boundary), as velocity fields vanishing to higher order on the boundary are not common in fluid mechanics applications.*

Remark 7.4. *In the proof of Proposition 5.1, we used the stream function of Theorem 1.1 to obtain the classical stream function. In light of Remark A.4 and the way we integrated in (7.3), we could have reversed this, obtaining the stream function of Theorem 1.1 from that of Proposition 5.1.*

Theorem 7.5 gives the regularity of the stream function that results if we assume that u is in V.

Theorem 7.5. *Define the space,*

$$X_0^2 := X_0 \cap H^2(\Omega)^{d\times d} \text{ with the } H^2(\Omega)^{d\times d}\text{-norm.}$$

The operator S defined in Theorem 1.1 maps $V \cap H_0$ continuously into X_0^2.

Proof. We follow the proof of Theorem 1.1, letting $u \in V \cap H_0$. Because $u \in H_0^1(\Omega)^d$, $\mathcal{E}_0 u \in H^1(\mathbb{R}^d)^d$. Hence,

$$\nabla \psi_j^i := \partial_j G * \nabla \mathcal{E}_0 u^i - \partial_i G * \nabla \mathcal{E}_0 u^j,$$

where we have convolutions of an L^1_{loc} function with a compactly supported L^∞ function; thus, we can treat the convolutions in either of the two ways we treated them in the proof of Theorem 1.1. It follows as in the remainder of that proof that $A \in H^2(\Omega)$ and the operator S is continuous from $V \cap H_0$ into X_0^2. □

Finally, we have the following simple but useful bound in Lemma 7.6, a generalization of sorts of Corollary 3.2 of [19]:

Lemma 7.6. *Assume that $\partial\Omega$ is C^k for some $k \in (1, \infty]$. Let X be any function space embedded in H that contains $C^k(\Omega)$. For any $u \in X$,*

$$\|u\|_X \le \|P_{H_0} u\|_X + C(X) \|u\|_H .$$

Proof. Since H_c is finite dimensional, it has some orthonormal basis $w_1, \dots, w_N$, and one can show by elliptic regularity theory that, in fact, each $w_j \in C^k(\overline{\Omega})$. Hence, $H_c u = \sum_j (u, w_j) w_j$ with

$$\sum_j (u, w_j)^2 = \|H_c u\|^2 \le \|u\|^2 .$$

It follows that

$$\begin{aligned} \|u\|_X &\le \|P_{H_0} u\|_X + \|P_{H_c} u\|_X \le \|P_{H_0} u\|_X + \sum_{j=1}^n |(u, w_j)| \, \|w_j\|_X \\ &\le \|P_{H_0} u\|_X + C(X) \|u\|_H . \end{aligned}$$

□

Appendix A. Differential forms point of view

We have been treating Ω as an open subset of $\mathbb{R}^d$. We wish now to also treat it as an oriented manifold with boundary: more specifically, as a ∂-manifold, as given in Definition 1.2 of [23]. We write $\mathcal{A}^k(H^j(\Omega))$ for the space of k-forms on Ω having coefficients in $H^j(\Omega)$. We identify a vector field $v \in H^j(\Omega)^d \cong T\Omega(\Omega)$ with a 1-form in $\mathcal{A}^1(H^j(\Omega))$ the usual way by the bijection,

$$\xi(v^1, \dots, v^d) = v^1 dx_1 + \dots + v^d dx_d.$$

Defining

$$X := \{\text{antisymmetric } A \in H^1(\Omega)^{d \times d}\},$$

we define the bijection,

$$\begin{aligned} &\theta \colon X \to \mathcal{A}^2(H^1(\Omega)), \\ &\theta A = (-1)^d \sum_{j > i} A^i_j dx_i \wedge dx_j . \end{aligned}$$

Here, δ is the codifferential operator, defined by

$$\begin{aligned} &\delta \colon \mathcal{A}^k(H^j(\Omega)) \to \mathcal{A}^{k-1}(H^j(\Omega)), \\ &\delta \omega = (-1)^{d(k+1)+1} * d(*\omega), \end{aligned}$$

where $*$ is the Hodge dual operator.

We will use the notation,

$$dx_{I(i)} = dx_1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge \cdots dx^d,$$

and similarly, $dx_{I(i,j)}$, $i \neq j$, is the wedge product of $dx_1 \wedge \cdots \wedge dx^d$ with dx^i and dx^j excluded.

Since we are working in flat space, $*\colon \mathcal{A}^2(H^j(\Omega)) \to \mathcal{A}^{d-2}(H^j(\Omega))$ can be defined by requiring that

$$*(dx_{i_1} \wedge dx_{i_k}) = (-1)^n dx_{j_1} \wedge dx_{j_{d-k}},$$

where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{d-k}$, $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{d-k}\} = \{1, \dots, d\}$ and n is the sign of the permutation, $(i_1, \dots, i_k, j_1, \dots, j_{d-k})$. It follows, in particular, that for $i < j$,

$$*(dx_i \wedge dx_j) = (-1)^{i+j} dx_{I(i,j)}.$$

Similarly, we can define $*\colon \mathcal{A}^1(H^j(\Omega)) \to \mathcal{A}^{d-1}(H^j(\Omega))$ by requiring that

$$*dx_j = (-1)^{j-1} dx_{I(j)}.$$

Observe that for $j > i$,

$$\begin{aligned} *dx_{I(i)} &= (-1)^{i-1} dx_i, \\ d\, dx_{I(i)} &= (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_d, \\ dx_{I(i,j)} &= (-1)^{i-1} dx_{I(j)} - (-1)^{j-1} dx_{I(i)}, \\ dx^i \wedge dx_{I(i,j)} &= (-1)^{i-1}\, dx_{I(j)}, \\ dx^j \wedge dx_{I(i,j)} &= (-1)^j\, dx_{I(i)}. \end{aligned} \tag{A.1}$$

For instance, the wedge product in the fourth identity involves $i-1$ transpositions while that in the fifth identity involves $j-2$ transpositions and $(-1)^{j-2} = (-1)^j$.

Lemma A.1. *For all $A \in X$,*

$$\delta(\theta A) = \xi \operatorname{div} A \text{ for all } A \in X. \tag{A.2}$$

Proof. Let $A \in X$. We see, then, that

$$\begin{aligned} \delta(\theta A) &= (-1)^{d(2+1)+1}(-1)^d * d \sum_{j>i} A^i_j * (dx_i \wedge dx_j) = - * d \sum_{j>i} (-1)^{i+j} A^i_j dx_{I(i,j)} \\ &= - * \sum_{j>i} (-1)^{i+j} \left[(-1)^{i-1} \partial_i A^i_j dx_{I(j)} + (-1)^j \partial_j A^i_j dx_{I(i)} \right] \\ &= - * \sum_{j>i} \left[(-1)^{j-1} \partial_i A^i_j dx_{I(j)} + (-1)^i \partial_j A^i_j dx_{I(i)} \right] \\ &= - * \left[\sum_{j<i} (-1)^{i-1} \partial_j A^j_i dx_{I(i)} + \sum_{j>i} (-1)^i \partial_j A^i_j dx_{I(i)}, \right] \end{aligned}$$

where we used (A.1).

But, again using (A.1), $*dx_{I(i)} = (-1)^{i-1} dx_i$, so

$$\delta(\theta A) = - \left[\sum_{j<i} \partial_j A^j_i dx_i - \sum_{j>i} \partial_j A^i_j dx_i \right].$$

On the other hand,

$$\xi \operatorname{div} A = \xi \sum_i \sum_j \partial_j A^i_j \mathbf{e}_i = \sum_i \sum_j \partial_j A^i_j\, dx_i = \sum_{j<i} \partial_j A^i_j\, dx_i + \sum_{j>i} \partial_j A^i_j\, dx_i$$

$$= -\sum_{j<i} \partial_j A_i^j \, dx_i + \sum_{j>i} \partial_j A_j^i \, dx_i,$$

since A is antisymmetric as a $d \times d$ matrix. This gives (A.2). □

We have, then, the bijections,

$$\xi \colon L^2(\Omega)^d \to \mathcal{A}^1(L^2(\Omega)), \quad \theta \colon X \to \mathcal{A}^2(H^1(\Omega)),$$

so that the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\ \theta\ } & \mathcal{A}^2(H^1(\Omega)) \\ \text{curl} \uparrow \downarrow \text{div} & & d \uparrow \downarrow \delta \\ L^2(\Omega)^d & \xrightarrow{\ \xi\ } & \mathcal{A}^1(L^2(\Omega)) \\ \nabla \uparrow \downarrow \text{div} & & d \uparrow \downarrow \delta \\ H^{-1}(\Omega) & & \mathcal{A}^0(H^{-1}(\Omega)), \end{array}$$

is commutative[2]; that is, so that (A.2) holds.

Theorem 2.4.2 of [23] gives the Hodge-Morrey decomposition of forms in L^2, which for k-forms is

$$\mathcal{A}^k(L^2(\Omega)) = \mathcal{E}^k(\Omega) \oplus \mathcal{C}^k(\Omega) \oplus \mathcal{H}^k(\Omega),$$

where

$$\mathcal{E}^k(\Omega) := \{d\alpha \colon \alpha \in \mathcal{A}^{k-1}(\Omega),\, \boldsymbol{t}\alpha = 0\},$$
$$\mathcal{C}^k(\Omega) := \{\delta\beta \colon \beta \in \mathcal{A}^{k+1}(\Omega),\, \boldsymbol{n}\beta = 0\},$$
$$\mathcal{H}^k(\Omega) := \{\lambda \in \mathcal{A}^k(L^2(\Omega)) \colon d\lambda = 0,\, \delta\lambda = 0\},$$

where $\boldsymbol{t}$, $\boldsymbol{n}$ give the tangential and normal components of a form on the boundary. It is important to note that such components vanishing do not (necessarily) directly transfer to what happens to the corresponding vector field or matrix under the mappings θ and ξ we have defined. Rather, for a k-form ω, $\boldsymbol{t}\omega$ is defined by its action on vector fields by

$$\boldsymbol{t}\omega(X_1, \dots, X_k) := \omega(X_1^{||}, \dots, X_k^{||}),$$

where $\omega(X_1^{||}, \dots, X_k^{||})$ are the components of the vector fields $X_1, \dots, X_k$ parallel to (tangent to) the boundary. Then

$$\boldsymbol{n}\omega(X_1, \dots, X_k) := \omega(X_1, \dots, X_k) - \boldsymbol{t}\omega(X_1, \dots, X_k).$$

The Hodge-Morrey decomposition is a full decomposition of k-forms in L^2; we are interested in the subspace of those 1-forms in L^2 corresponding to divergence-free vector fields tangential to the boundary. That is, we wish to calculate

$$\xi\left(\xi^{-1}(\mathcal{A}^1(L^2(\Omega))) \cap H\right).$$

Since in our correspondence, div of an L^2 vector field corresponds to δ of a 1-form, we should first determine the subspaces of the Hodge-Morrey decomposition whose codifferential vanishes, and whose normal components–when translated to vector fields—vanish:

$$\mathcal{E}^1_{\sigma,\boldsymbol{n}}(\Omega) := \{d\alpha \colon \alpha \in \mathcal{A}^0(\Omega),\, \boldsymbol{t}\alpha = 0,\, \delta d\alpha = 0,\, (\xi^{-1} d\alpha) \cdot \boldsymbol{n} = 0\},$$
$$\mathcal{C}^1_{\sigma,\boldsymbol{n}}(\Omega) := \{\delta\beta \colon \beta \in \mathcal{A}^2(\Omega),\, \boldsymbol{n}\beta = 0,\, \delta^2\beta = 0,\, (\xi^{-1}\delta\beta) \cdot \boldsymbol{n} = 0\},$$
$$\mathcal{H}^1_{\sigma,\boldsymbol{n}}(\Omega) := \{\lambda \in \mathcal{A}^1(L^2(\Omega)) \colon d\lambda = 0,\, \delta\lambda = 0,\, (\xi^{-1}\lambda) \cdot \boldsymbol{n} = 0\}.$$

[2]The solid lines indicate the maps that commute; the dashed lines indicate maps in the reverse direction that are not the inverses of those in the solid lines

Lemma A.2. $\mathcal{C}^1_{\sigma,\boldsymbol{n}}(\Omega) = \mathcal{C}^1(\Omega)$, $\mathcal{E}^1_{\sigma,\boldsymbol{n}}(\Omega) = \{0\}$.

Proof. Let $d\alpha \in \mathcal{E}^1_{\sigma,\boldsymbol{n}}(\Omega)$, and let $u = \xi^{-1} d\alpha$. Now,

$$d\alpha = \sum_{i=1}^d \partial_i \alpha dx_i = \xi \nabla \alpha,$$

where we are treating α interchangeably as a 0-form and as a scalar-valued function. Then, using (A.1),

$$\delta d\alpha = (-1)^{2d+1} * d * d\alpha = -\sum_{i=1}^d *d*(\partial_i \alpha dx_i).$$

But,

$$\begin{aligned} *d*(\partial_i \alpha dx_i) &= *d(\partial_i \alpha * dx_i) = *d(\partial_i \alpha (-1)^{i-1} dx_{I(i)}) = (-1)^{i-1} * d(\partial_i \alpha dx_{I(i)}) \\ &= (-1)^{i-1} * (-1)^{i-1} \partial_{ii} \alpha \, dx_1 \wedge \cdots \wedge dx_d = \partial_{ii} \alpha \, *(dx_1 \wedge \cdots \wedge dx_d) = \partial_{ii} \alpha. \end{aligned}$$

Hence,

$$\delta d\alpha = -\sum_{i=1}^d \partial_{ii} \alpha = -\Delta \alpha = 0.$$

This agrees with $\operatorname{div} u = \operatorname{div} \nabla \alpha = \Delta \alpha = 0$.

Since also we require that $u \cdot \boldsymbol{n} = (\xi^{-1} d\alpha) \cdot \boldsymbol{n} = 0$, we have $\Delta h = 0$ with $\nabla h \cdot \boldsymbol{n} = 0$, and we conclude that h is constant on Ω. But then $u = \nabla h \equiv 0$, and we find that $\mathcal{E}^1_{\sigma,\boldsymbol{n}}(\Omega) = \{0\}$.

Now let $\delta\beta \in \mathcal{C}^1_{\sigma,\boldsymbol{n}}(\Omega)$. Then Then $\delta^2 \beta = 0$ automatically and hence poses no additional restriction. So let β be any form in $\mathcal{A}^2(\Omega)$ for which $\boldsymbol{n}\beta = 0$. Then by Proposition 1.2.6 of [23], $\boldsymbol{n}(\delta\beta) = \delta(\boldsymbol{n}\beta) = 0$. But $\delta\beta$ is a 1-form, so $\beta = \sum_i v^i dx_i$ for some $v^i \in L^2(\Omega)$. So let $\boldsymbol{n}$ be the unit normal vector field and extend it, via the collar theorem, into Ω. Then

$$\boldsymbol{n}(\delta\beta) = (\delta\beta)(\boldsymbol{n}) - \omega(\boldsymbol{n}^{||}) = (\delta\beta)(\boldsymbol{n}) = \sum_i v^i n^i = \xi^{-1}(\delta\beta) \cdot \boldsymbol{n}.$$

That is, $\xi^{-1}(\delta\beta) \cdot \boldsymbol{n} = 0$ also poses no additional restriction, and we see that $\mathcal{C}^1_{\sigma,\boldsymbol{n}}(\Omega) = \mathcal{C}^1(\Omega)$. □

We have the immediate corollary:

Corollary A.3. $H_0 = \xi^{-1}(\mathcal{C}^1(\Omega))$.

Remark A.4. *Corollary A.3 can be viewed as the differenential forms analog of Proposition 5.1 in any dimension:*

$$\xi(H_0) = \delta\{\beta \in \mathcal{A}^2(\Omega) \colon \boldsymbol{n}\beta = 0\}.$$

Here, β is the stream function whose normal component vanishes on the boundary and δ is playing the role of the curl operator.

Remark A.5. *Similar reasoning shows that also $H_0 = (*\xi)^{-1} \mathcal{E}^{d-1}(\Omega)$.*

In fact, using the tools developed in Chapter 3 of [23], we can obtain the differential form equivalent of Theorem 1.1 in fairly short order:

Theorem A.6. *Let M be a ∂-manifold with C^∞ boundary. Define*

$$\widetilde{H}_0 := \{\alpha \in \mathcal{A}^1(\Omega) \colon \boldsymbol{n}\alpha = 0, \int_M \alpha \wedge *\lambda = 0 \text{ for all } \lambda \in \mathcal{H}^1_N(\Omega)\},$$

where

$$\mathcal{H}^1_N(\Omega) := \{\lambda \in \mathcal{A}^1(L^2(\Omega)) \colon d\lambda = 0,\, \delta\lambda = 0,\, \boldsymbol{n}\lambda = 0\}.$$

Then

$$\widetilde{H}_0 = \delta\{\beta \in \mathcal{A}^2(\Omega) \colon \beta|_{\partial M} = 0\}.$$

Proof. Given $\alpha \in \widetilde{H}_0$, it is always possible, by Corollary 3.3.4 of [23], to solve the boundary value problem,

$$\begin{cases} \delta\beta = \alpha & \text{on } M, \\ \beta|_{\partial M} = 0 & \text{on } \partial M. \end{cases}$$

□

Remark A.7. *In Theorem A.6, we are using $\widetilde{H}_0$ as a convenient proxy for H_0 as given by Remark A.4. Better would be to see it as equivalent to the homology-based version of H_0 given in Section 7. Exploring these issues would take us too far afield, however.*

Remark A.8. *We could also have proved Theorem A.6 by first establishing that the form of H_0 as given by Remark A.4 holds. Then, letting $\beta \in H_0$, we could "correct" its boundary value by subtracting from it the solution to*

$$\begin{cases} \delta\gamma = 0 & \textit{on } M, \\ \gamma|_{\partial M} = \beta & \textit{on } \partial M, \end{cases}$$

which we also can solve by applying Corollary 3.3.4 of [23]. This much less direct approach is in sympathy with the "corrector" argument we made in the proof of Theorem 1.1 (involving (7.3)). In fact, we could have used Corollary 3.3.4 of [23] in the proof of Theorem 1.1 to correct the boundary value without resorting to knowledge of the domain exterior to Ω. We wished, however, to obtain a result for Lipschitz boundaries and to, as much as possible, keep the argument in the language of "flat space."

The following two lemmas were used in Section 7. Lemma A.9 gives a convenient test for the exactness of a closed k-form on a manifold with boundary. It follows from Lemma 3.2.1 with Theorem 3.2.3 of [23]), along with a remark following the statement of Corollary 3.2.4 of [23]. Lemma A.10 relates integration of a $d-1$ form and a classical integral of a vector field.

Lemma A.9. *A closed k-form α, $0 \le k \le d$, on a manifold with boundary is exact if and only if*

$$\int_C \alpha = 0$$

for any k-cycle C in the manifold. It suffices to only consider k-cycles that are generators of $H_k(M, \partial M; \mathbb{R})$.

Lemma A.10. *Let Σ be a $(d-1)$-cycle (or more generally a $(d-1)$-chain), in $\mathbb{R}^d$, and also write Σ for the corresponding subset of $\mathbb{R}^d$. Then for any divergence-free vector field on $\mathbb{R}^d$,*

$$\int_\Sigma u \cdot \boldsymbol{n} = \int_\Sigma u \cdot \boldsymbol{dS} = \int_\Sigma *\xi u.$$

The first two integrals are different ways to write the classical "surface" integral, while the last integral is the integration of a $(d-1)$-form.

Proof. This is a standard calculation. See, for instance, the example on page 169-170 in [17], which is worked out explicitly for $d = 3$. □

Appendix B. A characterization H_0 in 2D and 3D

In this section we outline the characterization of H_0 that is more commonly used in 2D and 3D. The characterization applies to all dimensions $d \geq 2$, but the topological issues for $d \geq 4$ become more complex. Since our purpose is to be motivational, we will content ourselves with being a little imprecise about some of our arguments.

Let $\Gamma_1, \dots, \Gamma_{N+1}$, be the $N+1$ components of $\partial\Omega$ with Γ_{N+1} the boundary of the unbounded component of Ω^C. Let $\Sigma_1, \dots, \Sigma_N$ be pairwise disjoint Lipschitz regular $(d-1)$-submanifolds of Ω that generate $H_{d-1}(\Omega, \partial\Omega; \mathbb{R})$, the $(d-1)$-dimensional real homology class of Ω relative to its boundary. For a vector field $v \in H$, the internal flux across Σ_i is the value of

$$\int_{\Sigma_i} v \cdot \boldsymbol{n}.$$

Because v is divergence-free and tangential to the boundary, it is easy to see that the internal fluxes do not depend upon the specific choices of the Σ_i.

In 2D, each Σ_i is a curve from one boundary component to another, its boundary being two points, one on one boundary component the other on another boundary component. In 3D, each Σ_i is a surface whose boundary is a curve lying in a single component of the boundary. In 4D, each Σ_i is a 3-manifold, whose boundary is a 2-manifold that lies in one component of the boundary. The deepest fact about homology that we will use is the following:

Lemma B.1. *$\{\partial\Sigma_1, \dots, \partial\Sigma_N\}$ is a complete set of generators for $H_{d-2}(\partial\Omega^C; \mathbb{R})$, an homology group on the boundary of Ω. Because of this, it is also a complete set of generators for $H_{d-2}(\Omega^C, \partial\Omega^C; \mathbb{R})$.*

Proposition B.2 gives a direct characterization of H_0. We prove it using ideas from Appendix I of [25].

Proposition B.2. *$H_0 = \{v \in H \colon$ all internal fluxes are zero$\}$.*

Proof. Let $\widetilde{H}_0 = \{v \in H \colon \text{all internal fluxes are zero}\}$. Let $\dot{\Omega}$ be the simply connected open subset of Ω having a Lipschitz boundary that is produced by cutting along (that is, removing) each Σ_i. (We know that $\dot{\Omega}$ is simply connected, for otherwise we would obtain an additional generator for $H_{d-1}(\Omega, \partial\Omega; \mathbb{R})$.) Let $h \in H_c$. Then on $\dot{\Omega}$, h is curl-free (closed when viewed as a 1-form) and so is exact; hence, $h = \nabla p$ for some $p \in H^1(\dot{\Omega})$. Of necessity, the jump $[p]_i$ across each Σ_i is constant along Σ_i. (Or, we can view p as multi-valued on Ω with $v = \nabla p$.)

Now let $v \in H$ be arbitrary. Then

$$\begin{aligned}(h, v) &= \int_\Omega h \cdot v = \int_{\dot{\Omega}} \nabla p \cdot v = -\int_{\dot{\Omega}} p \operatorname{div} v + \int_{\partial\dot{\Omega}} p(v \cdot \boldsymbol{n}) \\ &= \int_{\partial\Omega} p(v \cdot \boldsymbol{n}) + \sum_i \int_{\Sigma_i} p(v \cdot \boldsymbol{n}) = \sum_i [p]_i \int_{\Sigma_i} v \cdot \boldsymbol{n}.\end{aligned}$$

This will vanish if and only if $v \in \widetilde{H}_0$. We conclude that $H_0 := H_c^\perp = \widetilde{H}_0$. □

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