

**A proof of the second part of Yudovich's theorem, the
existence and uniqueness of an associated flow, for
unbounded vorticity**

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1. PREFACE

Our goal is to prove a version of the second, “Lagrangian,” part of Yudovich’s Theorem as it appears in Theorem 5.1.1 p. 85 of [C3]. Our approach is to modify the argument in Chemin’s Section 5.2 for bounded vorticity. Initially, this document will make reference to that Section as needed rather than being self-contained, though over time I may make it self-contained—and possibly combine it with [Ky].

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2. A STATEMENT OF THE THEOREM

For completeness, we state Yudovich's theorem for unbounded vorticity in its entirety, though we must first describe how unbounded the vorticity is allowed to be. This is as in [Ky].

Let $\phi : [p_0, \infty) \rightarrow \mathbb{R}^+$ be a continuous function, where $p_0 > 1$. We define two functions, $\beta_{\epsilon, M, \phi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\beta_{M, \phi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, parameterized by ϵ in $(0, 1/p_0]$, $M > 0$, and ϕ :

$$\begin{aligned}\beta_{\epsilon, M, \phi}(x) &= M^\epsilon x^{1-\epsilon} \phi(1/\epsilon), \\ \beta_{M, \phi}(x) &= \inf \{ \beta_\epsilon(x) : \epsilon \in (0, 1/p_0] \}.\end{aligned}\tag{2.1}$$

For brevity, we usually write β_ϵ for $\beta_{\epsilon, M, \phi}$ and β for $\beta_{M, \phi}$, with the choices of M and ϕ being understood.

For all ϵ in $(0, 1/p_0]$, $\beta_\epsilon(x)$ is a monotonically increasing function continuous in x and in ϵ , with $\lim_{x \rightarrow 0^+} \beta_\epsilon(x) = 0$. It follows that β is a monotonically increasing continuous function and that $\lim_{x \rightarrow 0^+} \beta(x) = 0$. Also, $\beta(x) \leq \beta_\epsilon(x)$ for all ϵ in $(0, 1/p_0]$ and $x \in \mathbb{R}^+$.

Definition 2.1. A continuous function $\theta : [p_0, \infty) \rightarrow \mathbb{R}^+$ is called *admissible* if

$$\int_0^1 \frac{ds}{\beta_{M, \phi}(s)} = \infty,$$

where $\phi(p) = p\theta(p)$. This condition is independent of the choice of M .

We can now state a Yudovich's theorem for unbounded vorticity.

Theorem 2.1 (Yudovich's Theorem for Unbounded Vorticity). ***First part:** Let v_0 in E_m be a divergence-free vector field whose vorticity ω_0 is bounded in L^p -norm by $\theta(p)$ for some admissible function θ . Then there exists a unique weak solution v of (E). Moreover, v is in $C(\mathbb{R}; E_m) \cap L_{loc}^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))$. Also,*

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p} \text{ for all } 1 \leq p \leq \infty.\tag{2.2}$$

***Second part:** Moreover, the vector field has a flow. More precisely, there exists a unique mapping ψ , continuous from $\mathbb{R} \times \mathbb{R}^2$ to \mathbb{R}^2 , such that*

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x)) ds.$$

If $\Gamma_t : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$\int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta_{1, \phi}(r)} = t,$$

then $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the modulus of continuity of the flow at time t .

See [Ky] for the precise definition of a weak solution to the Euler equations and for a proof of the first part of the theorem. In the remainder

of this document, we prove the second part (what Chemin refers to as the “Lagrangian” part) of the theorem.

3. REPLACEMENT FOR LOG-LIPSCHITZIAN PROPERTY OF THE VELOCITY

The first step in Chemin’s proof of the second part of Yudovich’s theorem for bounded vorticity is a step he never states or proves—to show that the velocity field is log-Lipschitzian. This gives Chemin a function $\mu(r) = Cr(1 - \ln r)$ that satisfies the following five properties:

- (1) $|v(t, x) - v(t, x')| \leq \mu(|x - x'|)$ for all $t \geq 0$,
- (2) $\mu : [0, 1] \rightarrow [0, \infty)$,
- (3) $\mu(0) = 0$,
- (4) μ is nondecreasing (strictly increasing is not required), and
- (5) μ satisfies:

$$\int_0^1 \frac{da}{\mu(a)} = \infty. \quad (3.1)$$

(We can also express the first property as saying that the function μ is a bound on the modulus of continuity of $v(t)$.)

These properties of μ allow Chemin to apply the machinery of Osgood’s lemma to prove the existence and uniqueness of a continuous flow associated with the velocity field.

Our goal in this section is to establish that a function μ with all these properties also exists in the case of unbounded Yudovich vorticity—that is, such that the associated function β of Definition 2.1 is admissible—thus establishing the existence and uniqueness of a continuous flow. Our approach is to modify the proof in [Kc] that the vector field v is log-Lipschitzian when the initial vorticity is bounded.

Before proceeding, we make a comment on the second property. To apply Osgood’s lemma, we need μ to be defined on the domain $[0, \infty)$. We show, however, in Section 5 that if we can define μ on the domain $[0, 1]$, we can always extend it to the domain $[0, \infty)$ in such a way that it continues to obey the other four properties. Thus, we need only define μ on the domain $[0, 1]$.

From the Biot-Savart law, we have

$$\begin{aligned} I &:= |v(t, x) - v(t, x')| \\ &= \frac{1}{2\pi} \int \omega(t, y) \left[\frac{(x - y)^\perp}{|x - y|^2} - \frac{(x' - y)^\perp}{|x' - y|^2} \right] dy \\ &\leq \frac{1}{2\pi} \sqrt{(I_1)^2 + (I_2)^2}, \end{aligned}$$

where

$$I_2 := \int |\omega(t, y)| \left| \frac{x^1 - y^1}{|x - y|^2} - \frac{(x')^1 - y^1}{|x' - y|^2} \right| dy,$$

and where I_1 is defined similarly.

The technique we use to bound I_2 will clearly apply equally well to I_1 , so we will deal only with I_2 .

Let $a = |x - x'|/2$, and let R and λ be fixed positive real numbers. We will specify the values of R and λ later, but it will be true that $R \geq \lambda$. Assume also that $|x - x'| \leq 1$. Then we can split I_2 into three integrals:

$$I_2 = J + K + L,$$

where

$$J := \int_{B_{\lambda a}} |\omega(t, y)| f(y) dy, \quad K := \int_{B_R \setminus B_{\lambda a}} |\omega(t, y)| f(y) dy,$$

and

$$L := \int_{\mathbb{R}^2 \setminus B_R} |\omega(t, y)| f(y) dy,$$

with

$$f(y) := \left| \frac{x^1 - y^1}{|x - y|^2} - \frac{(x')^1 - y^1}{|x' - y|^2} \right|.$$

We bound J , K , and L differently.

Let $1 \leq p < \infty$ and $1 < q \leq \infty$ be such that $1/p + 1/q = 1$. Then

$$\begin{aligned} J &\leq \int_{B_{\lambda a}} |\omega(t, y)| \left[\frac{|x^1 - y^1|}{|x - y|^2} + \frac{|(x')^1 - y^1|}{|x' - y|^2} \right] dy \\ &\leq 2 \int_{B_{2\lambda a}} |\omega(t, y)| \frac{|y^1|}{|y|^2} dy \leq 2 \|\omega(t)\|_{L^p} \|1/|y|\|_{L^q(B_{2\lambda a})}. \end{aligned}$$

But,

$$\begin{aligned} \|1/|y|\|_{L^q(B_{2\lambda a})} &= \left(\int_{B_{2\lambda a}} \frac{1}{|y|^q} dy \right)^{1/q} = \left(2\pi \int_0^{2\lambda a} \frac{1}{r^q} r dr \right)^{1/q} \\ &= \left(2\pi \left[\frac{r^{2-q}}{2-q} \right]_0^{2\lambda a} \right)^{1/q} = \left(\frac{2\pi(2\lambda)^{2-q}}{2-q} \right)^{1/q} a^{2/q-1}. \end{aligned}$$

Thus,

$$J \leq C_1(q) \|\omega^0\|_{L^p} a^{2/q-1},$$

where

$$C_1(q) = 2 \left(\frac{2\pi(2\lambda)^{2-q}}{2-q} \right)^{1/q}.$$

To bound K , place the origin halfway between x and x' , with x' placed at $(a, 0)$ and x at $(-a, 0)$. Then

$$\begin{aligned} (x')^1 - y^1 &= a - r \cos \theta, & |x' - y|^2 &= a^2 + r^2 - 2ar \cos \theta, \\ x^1 - y^1 &= -a - r \cos \theta, & |x - y|^2 &= a^2 + r^2 + 2ar \cos \theta, \end{aligned}$$

so

$$\begin{aligned} f(y) = f(r, \theta) &= \left| \frac{-a - r \cos \theta}{a^2 + r^2 + 2ar \cos \theta} - \frac{a - r \cos \theta}{a^2 + r^2 - 2ar \cos \theta} \right| \\ &= \left| \frac{2a(a^2 + r^2 - 2r^2 \cos^2 \theta)}{a^4 + 2a^2r^2 + r^4 - 4a^2r^2 \cos^2 \theta} \right|, \end{aligned} \quad (3.2)$$

and

$$K \leq \|\omega(t)\|_{L^p} \left(\int_0^{2\pi} \int_{\lambda a}^R f(r, \theta)^q r \, dr \, d\theta \right)^{1/q}. \quad (3.3)$$

Now, the minimum value of r in the integrand of K is λa , so if we choose $\lambda \geq 1$, then $a \leq r$ so

$$|2a(a^2 + r^2 - 2r^2 \cos^2 \theta)| \leq |2a(r^2 + r^2 - 2r^2 \cos^2 \theta)| \leq 8ar^2.$$

If, further, we choose λ so that

$$a^4 + 2a^2r^2 + r^4 - 4a^2r^2 = a^4 - 2a^2r^2 + r^4 \geq \frac{1}{2}r^4, \quad (3.4)$$

then

$$a^4 + 2a^2r^2 + r^4 - 4a^2r^2 \cos^2 \theta \geq a^4 + 2a^2r^2 + r^4 - 4a^2r^2 \geq \frac{1}{2}r^4, \quad (3.5)$$

and it follows from (3.2) that

$$f(r, \theta) \leq \frac{8ar^2}{\frac{1}{2}r^4} = \frac{16a}{r^2}. \quad (3.6)$$

The condition (3.4) is equivalent to

$$\begin{aligned} 1 - 2 \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4 &\geq \frac{1}{2} \left(\frac{r}{a}\right)^4 \\ \implies \left(\frac{r}{a}\right)^4 - 4 \left(\frac{r}{a}\right)^2 + 2 &\geq 0, \end{aligned}$$

which, by the quadratic equation, will follow if $r/a \geq 2 + \sqrt{2}$, this in turn being insured by the choice

$$\lambda = 2 + \sqrt{2}.$$

We can then choose R to be any fixed real number greater than or equal to λ , so that $R \geq \lambda a$, a being no greater than 1. It will be convenient to choose

$$R = \lambda = 2 + \sqrt{2}.$$

Then from (3.3) and (3.6),

$$\begin{aligned}
K &\leq \|\omega(t)\|_{L^p} \left(2\pi \int_{\lambda a}^{\lambda} \left(\frac{16a}{r^2} \right)^q r dr \right)^{1/q} \\
&\leq 16 \|\omega^0\|_{L^p} (2\pi)^{1/q} a \left(\int_{\lambda a}^{\lambda} r^{1-2q} dr \right)^{1/q} \\
&= 16 \|\omega^0\|_{L^p} (2\pi)^{1/q} a \left(\left[\frac{r^{2-2q}}{2-2q} \right]_{\lambda a}^{\lambda} \right)^{1/q} \\
&= 16 \|\omega^0\|_{L^p} \left(\frac{\pi}{1-q} \right)^{1/q} a (\lambda^{2-2q} - (\lambda a)^{2-2q})^{1/q} \\
&= 16 \|\omega^0\|_{L^p} \left(\frac{\pi}{q-1} \right)^{1/q} \lambda^{2/q-2} a (a^{2-2q} - 1)^{1/q} \\
&= C_2(q) \|\omega^0\|_{L^p} a \left(\frac{a^{2-2q} - 1}{q-1} \right)^{1/q},
\end{aligned}$$

where

$$C_2(q) = 16\pi^{1/q} \lambda^{2/q-2}.$$

Todo: We don't need a fixed \bar{p} : Using p we would end up with $L \leq C \|\omega^0\|_{L^p} a$. This is a worse estimate, but the same as the bound that ultimately results on K .

As with J and K , we bound L using Hölder's inequality, but with a fixed choice of complementary exponents \bar{p} and \bar{q} , with $p_0 \leq \bar{p} < \infty$, where p_0 is as in Definition 2.1. This gives

$$L \leq \|\omega(t)\|_{L^{\bar{p}}(\mathbb{R}^2 \setminus B_R)} \|f\|_{L^{\bar{q}}(\mathbb{R}^2 \setminus B_R)} \leq 16a \|\omega^0\|_{L^{\bar{p}}} \left\| \frac{1}{r^2} \right\|_{L^{\bar{q}}(\mathbb{R}^2 \setminus B_R)},$$

where we used (3.6). But,

$$\begin{aligned}
\left\| \frac{1}{r^2} \right\|_{L^{\bar{q}}(\mathbb{R}^2 \setminus B_R)} &= \left(2\pi \int_{\lambda}^{\infty} \frac{1}{r^{2\bar{q}}} r dr \right)^{1/\bar{q}} \\
&= (2\pi)^{1/\bar{q}} \left(\left[\frac{r^{2-2\bar{q}}}{2-2\bar{q}} \right]_{\lambda}^{\infty} \right)^{1/\bar{q}} = \left(\frac{\pi}{\bar{q}-1} \right)^{1/\bar{q}} \lambda^{2/\bar{q}-2}.
\end{aligned}$$

Thus,

$$L \leq C_3 a,$$

where

$$C_3 = 16 \|\omega^0\|_{L^{\bar{p}}} \left(\frac{\pi}{\bar{q}-1} \right)^{1/\bar{q}} \lambda^{2/\bar{q}-2}.$$

From our bounds on J , K , and L , we have

$$I_2 \leq C_1(q) \|\omega^0\|_{L^p} a^{2/q-1} + C_2(q) \|\omega^0\|_{L^p} a \left(\frac{a^{2-2q} - 1}{q-1} \right)^{1/q} + C_3 a.$$

With the identical bound on I_1 , we can write

$$\begin{aligned} I &\leq \frac{1}{2\pi} \sqrt{(I_1)^2 + (I_2)^2} \\ &\leq \frac{1}{\pi} \left[C_1(q) \|\omega^0\|_{L^p} a^{2/q-1} + C_2(q) \|\omega^0\|_{L^p} a \left(\frac{a^{2-2q} - 1}{q-1} \right)^{1/q} + C_3 a \right]. \end{aligned} \quad (3.7)$$

We will reexpress this bound in terms of p rather than q , because it will allow us to recognize the connection with Yudovich's bounds on the L^p norms more easily.

Because the condition on admissibility in Definition 2.1 does not depend on the choice of p_0 , we can assume that $p_0 > 2$, in which case $C_1(q)$ and $C_2(q)$ are bounded for p in $[p_0, \infty)$ by constants that we will simply call C_1 and C_2 .

Then, using

$$\frac{1}{q} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}, \quad q-1 = \frac{1}{p-1},$$

we have

$$\begin{aligned} a \left(\frac{a^{2-2q} - 1}{q-1} \right)^{1/q} &= a \left(\frac{a^{-2/(p-1)} - 1}{1/(p-1)} \right)^{(p-1)/p} \\ &= a (p-1)^{(p-1)/p} \left(a^{-2/(p-1)} - 1 \right)^{(p-1)/p} \\ &\leq a (p-1) \left(a^{-2/(p-1)} - 1 \right)^{(p-1)/p} \\ &\leq a (p-1) \left(a^{-2/(p-1)} \right)^{(p-1)/p} \\ &= (p-1) a^{1-2/p} \leq p a^{1-2/p}. \end{aligned}$$

But

$$\frac{2}{q} - 1 = 1 - \frac{2}{p},$$

so from (3.7), we have

$$\begin{aligned} I &\leq \frac{1}{\pi} \left[C_1 \|\omega^0\|_{L^p} a^{1-2/p} + C_2 \|\omega^0\|_{L^p} p a^{1-2/p} + C_3 a \right] \\ &\leq C \|\omega^0\|_{L^p} p a^{1-2/p} \leq C a^{1-2/p} p \theta(p) = C a^{1-2/p} \phi(p), \end{aligned} \quad (3.8)$$

where the functions θ and ϕ are as in Definition 2.1, and where we used the fact that for a in $(0, 1/2]$, $a \leq a^{1-2/p} \leq p a^{1-2/p}$.

Since (3.8) holds for all p in $[p_0, \infty)$, it follows that

$$I \leq \inf \left\{ C a^{1-2/p} \phi(p) : p \in [p_0, \infty) \right\},$$

or,

$$|v(t, x) - v(t, x')| \leq \mu(|x - x'|), \quad (3.9)$$

where

$$\mu(r) := \inf \left\{ C \left(\frac{r}{2} \right)^{1-2/p} \phi(p) : p \in [p_0, \infty) \right\}. \quad (3.10)$$

The first three required properties of μ follow immediately from (3.10).

To show that μ is nondecreasing, first observe that $r \mapsto (r/2)^{1-2\epsilon}$ is an increasing function for all ϵ in $(0, 1/p_0]$, since, having assumed that $p_0 > 2$, it follows that $1 - 2\epsilon > 0$. Thus, if $r < s$ then

$$\begin{aligned} \mu(r) &= \inf \{ C(r/2)^{1-2\epsilon} \phi(1/\epsilon) : \epsilon \in (0, 1/p_0] \} \\ &\leq \inf \{ C(s/2)^{1-2\epsilon} \phi(1/\epsilon) : \epsilon \in (0, 1/p_0] \} = \mu(s). \end{aligned}$$

Finally, to verify (3.1), we express μ in terms of the function β of Definition 2.1:

$$\begin{aligned} \mu(r) &= \inf \{ C(r/2)^{1-2\epsilon} \phi(1/\epsilon) : \epsilon \in (0, 1/p_0] \} \\ &= (C/r) \inf \{ (r/2)^{2-2\epsilon} \phi(1/\epsilon) : \epsilon \in (0, 1/p_0] \} \\ &= (C/r) \inf \{ (r^2/4)^{1-\epsilon} \phi(1/\epsilon) : \epsilon \in (0, 1/p_0] \} \\ &= (C/r) \beta(r^2/4). \end{aligned} \quad (3.11)$$

Then, making the change of variables $u = r^2/4$,

$$\begin{aligned} \int_0^1 \frac{dr}{\mu(r)} &= \int_0^1 \frac{dr}{(C/r)\beta(r^2/4)} = C \int_0^1 \frac{r dr}{\beta(r^2/4)} \\ &= C \int_0^{1/4} \frac{du}{\beta(u)} > \infty, \end{aligned} \quad (3.12)$$

since β is an admissible function by assumption. Thus, μ satisfies the final of its required properties. As mentioned earlier, this is enough to establish, using the argument in Section 5.2 of Chemin, the existence and uniqueness of a continuous flow.

It is worth observing that the function μ is independent of time. Also, since $\mu(0) = 0$, the vector field $v(t)$ is uniformly continuous (see, for instance, exercise 17 p. 14 of [WZ]).

Finally, if the vorticity is bounded, then

$$\beta(r) := -C \|\omega^0\|_{L^{p_0} \cap L^\infty} r \ln r,$$

for $r \leq e^{-p_0}$ (see, for example, [Ksc]). This gives

$$\begin{aligned} \mu(r) &= (C/r)\beta(r^2/4) = (C/r) (-C \|\omega^0\|_{L^{p_0} \cap L^\infty} (r^2/4) \ln(r^2/4)) \\ &= -C \|\omega^0\|_{L^{p_0} \cap L^\infty} r (2 \ln r - \ln 4) \leq C \|\omega^0\|_{L^{p_0} \cap L^\infty} r (1 - \ln r), \end{aligned}$$

in accord with Chemin's result for bounded vorticity.

4. BOUNDING THE MODULUS OF CONTINUITY OF THE FLOW

The flow ψ that results from the argument in Section 3 can be written in the form

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x)) ds, \quad (4.1)$$

as in Theorem 5.2.1 p. 90 of Chemin. To bound the modulus of continuity of the flow, we want to determine how far apart two nearby points at time zero can become after time t . Toward this end, let x_1 and x_2 be two points in the plane and let $x_i(t) = \psi(t, x_i)$, $i = 1, 2$, be the corresponding trajectories of these points along the flow (So $x_i = x_i(0)$, which is just slightly confusing notation.) Then from (4.1),

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq |x_1 - x_2| + \int_0^t |v(s, \psi_1(s, x)) - v(s, \psi_2(s, x))| ds \\ &= |x_1 - x_2| + \int_0^t |v(s, x_1(s)) - v(s, x_2(s))| ds \\ &\leq |x_1 - x_2| + \int_0^t \mu(x_1(s) - x_2(s)) ds. \end{aligned}$$

Applying Osgood's lemma in the the form of Theorem 5.2.2 p. 91 of [C3] with $\rho(t) = |x_1(t) - x_2(t)|$, $a = |x_1 - x_2|$, and $\gamma(t) = 1$, we conclude that

$$\begin{aligned} - \int_{|x_1(t) - x_2(t)|}^1 \frac{dr}{\mu(r)} + \int_{|x_1 - x_2|}^1 \frac{dr}{\mu(r)} &= \int_{|x_1 - x_2|}^{|x_1(t) - x_2(t)|} \frac{dr}{\mu(r)} \\ &= \int_{|x_1 - x_2|/4}^{|x_1(t) - x_2(t)|/4} \frac{dr}{\beta(r)} \leq t, \end{aligned} \quad (4.2)$$

where in the last integration we changed variables as in (3.12).

Let $\Gamma_t : [0, \infty) \rightarrow [0, \infty)$ be defined by

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = \int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta(r)} = t.$$

It follows that $|x_1(t) - x_2(t)| \leq \Gamma_t(|x_1 - x_2|)$, so $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the modulus of continuity of the flow at time t . Arguing much as at the end of the proof of Theorem 2.2 of [K] and in Section 4 of [K], it follows that $|x_1(t) - x_2(t)| \leq \Gamma_t(|x_1 - x_2|) \rightarrow 0$ as $|x_1 - x_2| \rightarrow 0$, but that the bound on the convergence can be arbitrarily slow for initial vorticities that satisfy Yudovich bounds on the growth of their vorticity. It follows that the bound on the Hölder exponent can be arbitrarily small or even fail to exist.

Speculative comment: In fact, a bound may not even exist even for any positive time. To prove this, suppose that

$$\Gamma_t(x) = -\frac{t}{\ln|x_1 - x_2|} \quad (4.3)$$

for small x . Then for any γ in $(0, 1]$,

$$\begin{aligned} \lim_{x_2 \rightarrow x_1} \frac{|x_1(t) - x_2(t)|}{|x_1 - x_2|^\gamma} &= \lim_{x \rightarrow 0^+} \frac{-t/\ln x}{x^\gamma} = - \lim_{x \rightarrow 0^+} \frac{tx^{-\gamma}}{\ln x} \\ &= \lim_{x \rightarrow 0^+} t\gamma \frac{x^{\gamma-1}}{1/x} = \lim_{x \rightarrow 0^+} t\gamma x^\gamma = \infty, \end{aligned}$$

meaning that the velocity field $v(t)$ is not Hölder continuous at any time $t > 0$.

What makes this speculative is that it is not clear that (4.3) is a possible bound. We know that for any fixed t we can achieve such a bound, for we have great freedom in choosing how $\Gamma_t(x)$ behaves as a function of x . We have not shown how it can behave as a function of t , however.

5. EXTENDING THE DOMAIN OF μ TO $[0, \infty]$

If we wish to apply Osgood's lemma as is, we need to extend the domain of our function μ to all of $[0, \infty)$ while maintaining the validity of the other four properties described in Section 3. Only the first and fourth properties are of any concern, since the remaining two are not affected by any change to the value of μ on $[0, 1]$.

It is possible to make this extension because we know that the $v(t)$ is bounded, even for the case of unbounded vorticity. Thus,

$$|v(t, x) - v(t, x')| \leq C,$$

for some constant C . If $\mu(1) < C$, extend μ linearly so that $\mu(2) = C$, then let $\mu(x) = C$ for all $x \geq 2$. Otherwise, let $\mu(x) = \mu(1)$ for all $x \geq 1$. Note that both the first and fourth properties are maintained.

Now, the constant C increases with time because our bound on $\|v(t)\|_{L^\infty}$ increases with time (see the equation preceding Equation (5.2) p. 87 of [C3] for the case of bounded vorticity; the form is essentially the same for unbounded vorticity). So if we wish our bound to apply for only a fixed time interval $[0, T]$, we choose $C = C(T)$. This will not affect the inversion of (4.2) to obtain the function Γ_t , since for sufficiently small values of $|x_1 - x_2|$, both $|x_1 - x_2|$ and $|x_1(t) - x_2(t)|$ will be less than 1, and we did not change the value of $\mu(x)$ for $x < 1$.

Thus, we need only worry about the value of μ on the domain $[0, 1]$.

(It might be possible to arrive at the same conclusion by reexamining the argument for the existence of the flow on p. 92-93 of [C3], but I have not looked into this. Modifying Chemin's argument for the uniqueness of the flow, however, is easy, since given that $x_1(t)$ and $x_2(t)$ are two continuous solutions to (ODE) on p. 91 of Chemin, we know that the function ρ can be restricted to the domain $[0, 1]$ for some finite time, so the proof of Osgood's lemma still applies. Thus, $x_1(t) = x_2(t)$ for this finite time. But then there can be no greatest time over which they are equal, since we can always extend the equality at least some finite time further. If this approach

does work, it has the perhaps inconsequential advantage of not requiring the velocity field to be bounded.)

6. THE DETAILS OF THE REST OF CHEMIN'S PROOF

We now give the details of Chemin's proof that given a function μ with the five properties of Section 3, that a continuous flow associated with the velocity field v exists and is unique. We do this in just slightly less abstraction, however.

Theorem 6.1. *For all x_0 in \mathbb{R}^2 , there exists a unique continuous function $x : [0, \infty) \rightarrow \mathbb{R}^2$ —the trajectory of the point x_0 —such that*

$$x(t) = x_0 + \int_0^t v(s, x(s)) ds. \tag{6.1}$$

Proof. First we prove uniqueness. Suppose x and x' are two solutions to (6.1). Let

$$\rho(t) = |x(t) - x'(t)|.$$

Then

$$\begin{aligned} \rho(t) &= |x(t) - x'(t)| = \left| \int_0^t v(s, x(s)) - v(s, x'(s)) ds \right| \\ &\leq \int_0^t |v(s, x(s)) - v(s, x'(s))| ds \leq \int_0^t \mu(|x(s) - x'(s)|) ds \\ &= \int_0^t \mu(\rho(s)) ds. \end{aligned}$$

By Osgood's lemma it follows that ρ is identically zero on $[0, t]$. Since t is an arbitrary positive number, it follows that any solution that exists to (6.1) is unique on $[0, \infty)$.

To establish existence, we use a classical Picard scheme. We construct a series of trajectories, $\{x_k\}_{k=1}^\infty$, inductively by

$$x_1(t) = x_0 + \int_0^t v(s, x_0) ds$$

and

$$x_{k+1}(t) = x_0 + \int_0^t v(s, x_k) ds.$$

We restrict ourselves to the interval $[0, T]$, where $T > 0$ is arbitrary. Since we know that v is bounded over such an interval by some constant, M ,

$$|x_{k+1}(t)| \leq |x_0| + \int_0^t |v(s, x_k(s))| ds \leq |x_0| + MT,$$

so the sequence $\{x_k\}$ is bounded over the interval $[0, T]$.

The proof of the corresponding fact in Chemin (which he does not give) is more involved, since he is being more abstract.

We will now show that $\{x_k\}$ is a Cauchy sequence in the space of continuous functions from the interval $[0, T]$ to \mathbb{R}^2 . Let

$$\rho_{k+1,n}(t) = |x_{k+1+n}(t) - x_{k+1}(t)|.$$

Then

$$\begin{aligned} 0 \leq \rho_{k+1,n}(t) &= \left| \int_0^t v(s, x_{k+n}) - v(s, x_k) ds \right| \\ &\leq \int_0^t |v(s, x_{k+n}) - v(s, x_k)| ds \leq \int_0^t \mu(|x_{k+n}(s) - x_k(s)|) ds \\ &= \int_0^t \mu(\rho_{k,n}(s)) ds. \end{aligned}$$

Letting

$$\rho_k(t) = \sup_{n \geq 0} \rho_{k+1,n}(t),$$

we have

$$0 \leq \rho_{k+1}(t) \leq \sup_{n \geq 0} \int_0^t \mu(\rho_{k,n}(s)) ds \leq \int_0^t \sup_{n \geq 0} \mu(\rho_{k,n}(s)) ds,$$

the last inequality being by Fatou's lemma. Because μ is increasing, it follows that

$$\tilde{\rho}(t) := \limsup_{k \rightarrow \infty} \rho_k(t) \leq \int_0^t \mu(\tilde{\rho}(s)) ds.$$

By Osgood's lemma, it follows that $\tilde{\rho}$ is identically zero on the interval $[0, T]$, which means that the sequence $\{x_k\}$ is Cauchy in the complete space of continuous functions from the interval $[0, T]$ to \mathbb{R}^2 . Hence, $x(t) = \lim_{k \rightarrow \infty} x_k(t)$ exists and is continuous—and also must satisfy (6.1).

Since $T > 0$ was arbitrary, it follows that a unique continuous x satisfying (6.1) exists on the interval $[0, \infty)$. \square

The flow is now defined by

$$\psi(t, x_0) := x(t).$$

The continuity of ψ in time follow from Theorem 6.1, the continuity in space by the results of Section 4.

Fluid mechanics references appear first, then general references.

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