Vortex Sheets

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`http://www.ma.utexas.edu/users/kelliher/FluidMechanics/FluidMechanics.html`

Preface

This document was written in preparation for a series of four lectures in the Fluid Mechanics Seminar at the University of Texas Austin, Spring 2003. This was a "learning" seminar with detailed lectures on each topic. The topic of my lectures is the result of Delort in [2] on vortex sheets. My lectures followed a series of four lectures on vortex patches by Roman Shvydkoy; the two sets of lectures are logically independent but complementary.

There are a number of published versions of Delort's original result. (In Section 1, I briefly discuss them.) I follow Chemin's mathematical narrative of Delort's result as it appears in Chapter 6 of [1], the flow and details of his argument, but a lot of the understanding of what is really going on, and some of the math, comes from Chapter 11 of Majda and Bertozzi's [7], especially Section 11.4, and also some results from Chapters 8 through 10.

Nonetheless, the reader should view this document as an expanded rewrite of Chapter 6 of [1]. And view it with **CAUTION**, since I have undoubtedly got a few things wrong.

The setting throughout is that of a perfect incompressible fluid (zero viscosity) in two-dimensional Euclidean space.

CONTENTS

1. INTRODUCTION

We adopt Chemin's definition of a *vortex sheet* as a measure on the plane derived from the length of a compact C^1 curve. We fix such a curve in the plane, and define the Borel measure, ω, for any open subset, Ω, of the plane by setting $\omega(\Omega)$ to be the total length of those portions of the curve that lie in Ω. This truly results in a Borel measure since it is clearly countably additive on open sets, and hence on the σ-algebra of Borel sets.

When we talk about a vortex sheet, it is really with the understanding that it is the initial vorticity of a fluid, or even sometimes a solution given that initial vorticity. The vortex sheet problem is to find out whatever we can about the existence, uniqueness, regularity, or any other useful property of weak solutions to the Euler equations, given an initial vorticity that is a vortex sheet.

A vortex patch has a jump discontinuity in the vorticity across a curve (which is the boundary of some domain), but the velocity remains continuous. A vortex sheet exhibits a discontinuity in the velocity itself across a curve.

In contrast to the approach to the vortex patch problem, we don't want to be dealing directly with the curve itself. So we deal with a slightly more general initial vorticity as described in Theorem 1.1, the proof of which is the main goal of this document:

Theorem 1.1 (Delort). *Let m be a real number, and let v_0 be a divergence-free vector field belonging to the space E_m for $m \in \mathbb{R}$ (E_m is defined in Section A). Suppose that ω_0, its initial vorticity, is a finite measure with a positive singular part. Then there exists a weak solution, (v, p), to the Euler equations, belonging to the space $L^\infty_{loc}(\mathbb{R}; E_m) \times L^\infty_{loc}(\mathbb{R}; \mathcal{F}^{-1}(L^2 + L^\infty))$.*

Furthermore, for each time t, the vorticity ω_t is a bounded measure with positive singular part, whose total mass is bounded in time (almost everywhere) by the total mass of the initial vorticity.[1]

The measure ω_0 is a *signed*, or *real*, measure, a subclass of complex measure (it does not take complex values, but can take negative values, and hence is not a positive measure). In Section B we present some results from measure theory that we will need in our proof of Theorem 1.1.

Theorem 1.1 makes no claim regarding uniqueness, and in fact no uniqueness result is currently known. And neither existence nor uniqueness is known in the case where the singular part of the vorticity can have mixed sign. (See [7] p. 432.)

Majda and Bertozzi in [7] refer to Theorem 1.1 as the "existence of weak solutions with vortex-sheet initial data of distinguished sign." So Chemin's definition of a vortex sheet is made to roughly match the known results (as

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[1] Chemin doesn't make the restriction "almost anywhere," but I don't know how to avoid it.

well as being physically motivated), while Majda and Bertozzi restrict the domain of their definition in the statement of the result.

TODO: Include Chemin's argument on the bottom of p. 109 on why this is more general than the vortex sheet problem. This is actually a very technical argument when one takes into account that it makes use of Proposition 2.2.6, which I haven't even worked through yet.

There are a number of published proofs of Theorem 1.1, none of which I have done more than peruse. The first was that of Delort in [2]. Chemin's account in [1] is based upon [5].

Majda in [6] produces a variant of the Delort's proof that serves as the basis for Majda and Bertozzi's approach in Section 11.4 of [7].

Finally, Evans and Müller in [4] give an approach that involves Hardy spaces in a manner I do not understand. I believe, at this point, though, that if I do pursue this topic further, the next thing I will do is study this paper.

2. OUTLINE OF APPROACH

To prove Theorem 1.1 we follow the usual procedure of regularizing the initial velocity, v_0, using an approximation of the identity, ρ_n (a *mollifier* in the terminology of [7]), to obtain a sequence of functions,

$$v_{0,n} = \rho_n * v_0, \tag{2.1}$$

representing the initial velocity and converging to v_0 in all L^p-norms, $1 \leq p < \infty$, in which v_0 lies. The function ρ is assumed to be smooth, compactly supported, nonnegative, and have norms $\|\rho\|_{L^1} = 1$ and $\|\rho\|_{L^\infty} = 1$. We also assume, for simplicity in dealing with radial-energy decompositions, that ρ is radially symmetric.

By Theorem C.1, there exists a unique global strong solution, v_n, of the Euler equations with initial velocity $v_{0,n}$, which is continuous in the time variable, smooth in the space variables, and belongs to $L^\infty_{loc}(\mathbb{R}; E_m)$. The solutions $\{v_n\}$ (or rather some subsequence of them) converge weakly (in the sense of test functions) to a vector field v, as we show[2] in Section 3. We do not know, a priori, that v is a solution in any sense to the Euler equations—that is what we must prove.

We also establish in Section 3 properties of the limiting vorticity that will be critical in proving in Section 7 that v is a solution to the Euler equations.

To make sense of the term "weak solution" in Theorem 1.1, we need to give a weak formulation of the Euler equations appropriate for the study of vortex sheets. This we do in Section 4. We show that if $\left\{v_n^1 v_n^2\right\}$ converges weakly to $v^1 v^2$, then v *will* be a weak solution to the Euler equations with initial velocity v_0. So we have reduced the proof of the main part of Theorem 1.1 to proving that $\left\{v_n^1 v_n^2\right\}$ converges weakly to $v^1 v^2$.

Toward establishing weak convergence of $\left\{v_n^1 v_n^2\right\}$ to $v^1 v^2$, we show in Section 5 that for any test function g,

$$\left\langle v_n^1 v_n^2, g \right\rangle = \int_{\mathbb{R}^5} G(t, x, y)\, d\mu_n(t, x, y), \tag{2.2}$$

where

$$d\mu_n(t, x, y) = \omega_n(t, x) \omega_n(t, y)\, dt\, dx\, dy, \tag{2.3}$$

and where G is defined in Section 5. In Section 6 we show that G is a bounded function vanishing at infinity, and continuous on the subset $(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2) \setminus \mathbf{D}$, where $\mathbf{D} = \mathbb{R} \times \operatorname{diag}(\mathbb{R}^2 \times \mathbb{R}^2)$. This fact is what Majda and Bertozzi call in [7] p. 442, "Delort's (1991) key new observation." It appears in all versions of the proof mentioned in Section 1.

We use these properties of G in Section 7 to show that $\left\{v_n^1 v_n^2\right\}$ weakly converges to $v^1 v^2$, thereby completing the proof of Theorem 1.1.

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[2] A confusing aspect of Chapter 6 of [1] is that Chemin doesn't prove this weak convergence until near the end of the chapter, but starts referring almost at the beginning to a vector field v without ever saying what it is.

3. WEAK CONVERGENCE OF $\{v_n\}$

In this section we establish the weak convergence of v_n to a weak solution v. We also establish some properties of the associated vorticities ω_n and ω that will be important in Section 7.

Our approach can be summarized as follows:

1. Prove that the sequence $\{\omega_n\}$, where $\omega_n = \omega(v_n)$, is bounded in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ (see Lemma 3.2). We employ an energy estimate to accomplish this. Weak convergence of $\{\omega_n\}$ to some ω (Theorem 3.3) follows by the Banach-Alaoglu theorem.
2. Prove that a bounded measure lying in $H^{-1}(\mathbb{R}^2)$ is continuous (see Lemma 3.4). (A measure is continuous if the measure of a single point is always zero).
3. Decompose ω_0 into a positive and negative absolutely continuous part and a singular part, show that each part flows along the vorticity lines, prove that ω can be taken to be a finite measure, and show that the absolutely continuous parts weakly approach functions in L^1 (see Theorem 3.6).
4. Prove that $|\omega_n|$ converges weakly to a finite positive measure ω^p that is continuous (see Theorem 3.6). This is the critical fact needed in Section 7.
5. Show that corresponding to $\omega_n \rightharpoonup \omega$ there is a sequence of velocities $v_n \rightharpoonup v$, where v is in $L^\infty_{loc}(\mathbb{R}; E_m)$ (see Theorem 3.7).

We will have to deal with the two different kinds of weak convergence, similar yet subtly different:

1. Convergence in the sense of integration with respect to a test function. When the test functions are smooth compactly supported functions on some subset Ω of $\mathbb{R}^n$, our notation will be
$$f_n \rightharpoonup f,$$
where $\{f_n\}$ is a sequence of functions or distributions on Ω and f is its weak limit in the sense that for all φ in $C_0^\infty(\Omega)$,
$$\lim_{n\to\infty} \int_\Omega f_n\varphi = \int_\Omega f\varphi,$$
where the measure is Lebesgue.
2. When we say that a sequence of bounded measures $\{\mu_n\}$ on a space X converges weakly to another measure μ, which we write as
$$\mu_n \rightharpoonup \mu,$$
we mean convergence in the weak topology on the bounded measures. This means that the limit of any bounded linear functional (member of the dual space, X^*) applied to a sequence is equal to the bounded linear functional applied to the weak limit of that sequence. That

is, for any linear functional λ in X^*,

$$\lim_{n\to\infty} \lambda(\mu_n) = \lambda(\mu).$$

Without exception, all our weakly convergent measures will lie in the space $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$. By Theorem C.3 and the fact that L^1 is the dual of L^∞_{loc}, the dual space to $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ is isomorphic to $L^1(\mathbb{R}; H^1(\mathbb{R}^2))$, with an element φ in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ corresponding to the element $\lambda = (f \mapsto \langle \varphi, f\rangle)$.

So $\mu_n \rightharpoonup \mu$ means that for all φ in $L^1(\mathbb{R}; H^1(\mathbb{R}^2))$,

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^2} \varphi \, d\mu_n = \int_{[0,T]\times\mathbb{R}^2} \varphi \, d\mu.$$

The function φ, then, serves as a test function, just as it did in our first definition of weak convergence. $L^1(\mathbb{R}; H^1(\mathbb{R}^2))$ contains $C_0([0,T]\times\mathbb{R}^2)$, the space of compactly supported continuous functions, so we also have convergence in the same sense as in Section 4.

On p. 54-59 of [11] is an informative discussion of weak convergence of Radon measures (which includes our measures). Of particular interest is the proof of Theorem 2, which is less abstract (and more general) than the proof of the weak convergence of a bounded sequence of bounded measures that we will give, which relies upon the Banach-Alaoglu theorem (Theorem C.9).

Lemma 3.1. *For any real s, $\partial_k : H^s(\mathbb{R}^2) \to H^{s-1}(\mathbb{R}^2)$, and for any $u \in H^s(\mathbb{R}^2)$, $\|\partial_k u\|_{H^{s-1}(\mathbb{R}^2)} \le \|u\|_{H^s(\mathbb{R}^2)}$.*

Proof. We have

$$\begin{aligned}
\|\partial_k u\|^2_{H^{s-1}(\mathbb{R}^2)} &= \int_{\mathbb{R}^d} (1+|\xi|^2)^{s-1} \left|\widehat{\partial_k u}\right|^2 d\xi \\
&= \int_{\mathbb{R}^d} (1+|\xi|^2)^{s-1} |\xi^k|^2 \, |\widehat{u}|^2 \, d\xi \\
&\le \int_{\mathbb{R}^d} (1+|\xi|^2)^{s-1} |\xi|^2 \, |\widehat{u}|^2 \, d\xi \\
&= \int_{\mathbb{R}^d} (1+|\xi|^2)^{s} \frac{|\xi|^2}{1+|\xi|^2} \, |\widehat{u}|^2 \, d\xi \\
&\le \int_{\mathbb{R}^d} (1+|\xi|^2)^{s} \, |\widehat{u}|^2 \, d\xi = \|u\|^2_{H^s(\mathbb{R}^2)}.
\end{aligned}$$

□

Lemma 3.2. *The sequence $\{\omega_n\}$ is bounded in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$.*

Proof. Let

$$v_0 = \sigma + w$$

be a radial-energy decomposition of v_0, where σ is a stationary solution and w is in $L^2(\mathbb{R}^2)$. Then

$$v_{n,0} = \rho_n * v_0 = \rho_n * \sigma + \rho_n * w. \tag{3.1}$$

The vorticity $\omega(\rho_n * \sigma) = \rho_n * \omega(\sigma)$ is compactly supported and radially symmetric since both ρ_n and $\omega(\sigma)$ have these same properties. So there exists some stationary distribution σ' whose vorticity is the same as that of $\rho_n * \sigma$'s (because we can integrate the vorticity as in Equation (A.1)).

Because ρ is compactly supported, $\rho_n * \sigma$ decays as σ does—like $C/|x|$. Also, $\operatorname{div}(\rho_n * \sigma) = \rho_n * \operatorname{div} \sigma = 0$ and $\operatorname{div} \sigma' = 0$, so $\rho_n * \sigma$ and σ' are two smooth divergence-free vector fields with identical vorticity that both decay like $C/|x|$. By Theorem A.3, $\rho_n * \sigma = \sigma'$, so $\rho_n * \sigma$ is a stationary solution.

Since also $\rho_n * w$ is in L^2, we conclude that Equation (3.1) is a radial-energy decomposition of $v_n(0)$.

By Theorem B.5 and the second comment following it,

$$\|\rho_n * \omega_0\|_{L^1} \le \|\rho_n\|_{L^1} \|\omega_0\|_{\mathcal{M}} \le \|\omega_0\|_{\mathcal{M}},$$

so the total vorticity of $\omega_{n,0}$ is bounded, and $\omega_{n,0}$ belongs to some E_{m_n} where m_n need not equal m. By Theorem C.1, $v_n(t)$ is in E_{m_n} for all time. Thus, we can use the same stationary solution $\sigma_n = \rho_n * \sigma$ for all time, writing the radial-energy decomposition of v_n as

$$v_n(t) = \sigma_n + w_n(t) = \rho_n * \sigma + w_n(t).$$

Using Lemma 3.1,

$$\begin{aligned}
\|\omega_n(t)\|_{H^{-1}} &= \|\omega(\rho_n * \sigma) + \omega(w_n(t))\|_{H^{-1}} \\
&\le \|\rho_n * \omega(\sigma)\|_{H^{-1}} + \|\partial_1 w_n(t)\|_{H^{-1}} + \|\partial_2 w_n(t)\|_{H^{-1}} \\
&\le \|\rho_n * h\|_{H^{-1}} + 2\|w_n(t)\|_{H^0},
\end{aligned}$$

where h, the vorticity of σ, is compactly supported.

Now,

$$\begin{aligned}
\|\rho_n * h\|_{H^{-1}} &= \int_{\mathbb{R}^2} \frac{\left|\widehat{\rho_n * h}(\xi)\right|^2}{1 + |\xi|^2}\, d\xi = \int_{\mathbb{R}^2} \frac{\left|\widehat{\rho_n(\xi)}\right|^2 \left|\widehat{h}(\xi)\right|^2}{1 + |\xi|^2}\, d\xi \\
&\le \|\widehat{\rho_n}\|_{L^\infty}^2 \int_{\mathbb{R}^2} \frac{\left|\widehat{h}(\xi)\right|^2}{1 + |\xi|^2}\, d\xi \le \|\rho_n\|_{L^1}^2 \int_{\mathbb{R}^2} \frac{\left|\widehat{h}(\xi)\right|^2}{1 + |\xi|^2}\, d\xi \\
&\le \int_{\mathbb{R}^2} \frac{\left|\widehat{h}(\xi)\right|^2}{1 + |\xi|^2}\, d\xi = \|\sigma\|_{H^{-1}(\mathbb{R}^2)},
\end{aligned}$$

which is finite because h is compactly supported, so $\widehat{h}(\xi)$ decays like $C/|\xi|$. Also,

$$\|w_n(t)\|_{H^0} = \|w_n(t)\|_{L^2} \le \|v_n(t) - \sigma_n\|_{L^2}.$$

Therefore,

$$\|\omega_n(t)\|_{H^{-1}} \le C + 2\,\|v_n(t) - \sigma_n\|_{L^2}.$$

But by Theorem C.1 we have the following energy estimate for initial velocity in $E_m \cap C^r$ (the initial velocity for the regularized solution is in $E_m \cap C^\infty$):

$$\|v_n(t) - \sigma_n\|_{L^2} \le \|v_n(0) - \sigma_n\|_{L^2}\, e^{t\|\nabla\sigma_n\|_{L^\infty}}.$$

Thus,

$$\begin{aligned}
\|\omega_n(t)\|_{H^{-1}} &\le C + 2\,\|v_n(0) - \sigma_n\|_{L^2}\, e^{t\|\nabla\sigma_n\|_{L^\infty}} \\
&= C + 2\,\|\rho_n * v_0 - \rho_n * \sigma\|_{L^2}\, e^{t\|\nabla(\rho_n * \sigma)\|_{L^\infty}} \\
&\le C + 2\,\|\rho_n\|_{L^1}^2\, \|v_0 - \sigma\|_{L^2}\, e^{t\|\rho_n * \nabla\sigma\|_{L^\infty}} \\
&\le C + 2\,\|v_0 - \sigma\|_{L^2}\, e^{t\|\rho_n\|_{L^1}\|\nabla\sigma\|_{L^\infty}} \\
&= C + 2\,\|v_0 - \sigma\|_{L^2}\, e^{t\|\nabla\sigma\|_{L^\infty}}.
\end{aligned}$$

Thus, $\|\omega_n(t)\|_{H^{-1}}$ is uniformly bounded in n and in time (over any finite time interval), and the lemma is proved. ☐

Theorem 3.3. *$\{\omega_n\} \rightharpoonup \omega$ for some ω in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ (after extracting a subsequence).*

Proof. $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ is a topological vector space and by Lemma 3.2, the sequence $\{\omega_n\}$ lies within a ball in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$. But this ball is compact in the weak-* topology by the Banach-Alaoglu theorem (see Theorem C.9), so some subsequence of $\{\omega_n\}$ converges in the weak-* topology to some ω in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$. Identifying the dual spaces , we conclude that this subsequence converges in the weak topology. Without loss of generality, we assume that the subsequence is the sequence $\{\omega_n\}$ itself. ☐

Does this argument make sense?

Comment: When we extract a subsequence of $\{\omega_n\}$ reindex it and identify it with itself, we also reindex the associated sequences, $\{v_{0,n}\}$, and $\{v_n\}$. At this point, these are the only associated sequences, but later there will be more; when reindexing, we always mean to include all the sequences. Another way of looking at this is that we are only reindexing the one sequence $\{\rho_n\}$, since all the other sequences are defined in terms of $\{\rho_n\}$.

Lemma 3.4. *If the measure μ belongs to $H^{-1}(\mathbb{R}^2)$ it is continuous.*

Proof. Let h be a smooth, compactly supported, nonnegative function with h(0) = 1, and define for any $\lambda > 0$,

$$h_{\lambda,x}(y) = h\left(\frac{x-y}{\lambda}\right).$$

If follows by a simple change of variables that $\|h_{\lambda,x}\|_{L^2} = \lambda \|h\|_{L^2}$.

For any smooth, compactly supported function θ, we have, using the change of variables $y = x - \lambda z$, which has Jacobian λ^2 (since we are in two dimensions),

$$\partial_i h_{\lambda,x}(y) = \partial_i \left(h\left(\frac{x-y}{\lambda}\right)\right) = \frac{1}{\lambda}(\partial_i h)\left(\frac{x-y}{\lambda}\right) = \frac{1}{\lambda}(\partial_i h)(z).$$

It follows that

$$\begin{aligned}\langle \partial_i h_{\lambda,x}, \theta\rangle &= \int \partial_i h_{\lambda,x}(y)\theta(y)\,dy = \int \frac{1}{\lambda}(\partial_i h)(z)\theta(x-\lambda z)\,\lambda^2 dz\\ &= \lambda \int \partial_i h(z)\theta(x-\lambda z)\,dz\\ &\le \lambda \|\partial_i h\|_{L^1} \|\theta\|_{L^\infty}.\end{aligned}$$

Thus, as λ approaches 0, $h_{\lambda,x}$ approaches 0 strongly in L^2 and its derivatives approach 0 weakly, so $h_{\lambda,x}$ must approach 0 weakly in H^1. Since H^{-1} is dual to H^1 (Theorem C.3), μ acting on H^1 by the inner product is a linear functional on H^1, so

$$\lim_{\lambda\to 0} \langle \mu, h_{\lambda,x}\rangle = 0$$

by the definition of weak convergence of $h_{\lambda,x}$ to zero.

As for pointwise convergence, for all y in $\mathbb{R}^2$,

$$\lim_{\lambda\to 0} h_{\lambda,x} = \mathbf{1}_{\{x\}}(y).$$

Because $|h_{\lambda,x}| \le \|h\|_{L^\infty}$ and μ is a finite measure, $h_{\lambda,x}$ is dominated by $\|h\|_{L^\infty} \in L^1(\mu)$. Thus, we can apply Lebesgue's dominated convergence theorem to conclude that

$$\begin{aligned}0 = \lim_{\lambda\to 0}\langle \mu, h_{\lambda,x}\rangle &= \lim_{\lambda\to 0}\int_{\mathbb{R}^2} h_{\lambda,x}\,d\mu = \int_{\mathbb{R}^2} \lim_{\lambda\to 0} h_{\lambda,x}\,d\mu\\ &= \int_{\mathbb{R}^2} \mathbf{1}_{\{x\}}\,d\mu = \mu(\{x\}).\end{aligned}$$

This completes the proof of the lemma. ☐

Corollary 3.5. *The sequence* $\{\omega_n\}$ *and* ω *are continuous measures.*

Proof. This follows immediately from Theorem 3.3 and Lemma 3.4. (Of course, ω_n is smooth, so continuity follows for it anyway.) ☐

Theorem 3.6. *After extracting an appropriate subsequence of* $\{\omega_n\}$ *and identifying it with the sequence itself, the following hold:*

1. *The limit measure* ω *is a finite measure lying in* $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$.
2. *There exists a positive continuous measure* ω^p *such that* $|\omega_n|$ *converges weakly to* ω^p.
3. *The measure* ω_t *is bounded with positive singular part, and its total mass is bounded almost everywhere in Lebesgue measure on time* t *by the total mass of the initial vorticity.*

Proof. **(1)**: The hypotheses of Theorem 1.1 allow us to decompose ω_0 as

$$\omega_0 = f_0^+ - f_0^- + \omega_0^s,$$

where

$$f_0^\pm \in L^1, \quad f_0^\pm \geq 0, \quad \text{and} \quad \omega_0^s \geq 0.$$

By Lemma B.2,

$$|\omega_0| = \omega_0 = f_0^+ + f_0^- + \omega_0^s.$$

Because vorticity is preserved along flow lines, and by Equation (2.1),

$$\omega_n(t, x) = f_n^+(t, x) - f_n^-(t, x) + \omega_n^s(t, x), \tag{3.2}$$

where

$$\begin{aligned} f_n^\pm(t, x) &= (\rho_n * f_0^\pm)(\psi_n^{-1}(t, x)), \\ \omega_n^s(t, x) &= (\rho_n * \omega_0^s)(\psi_n^{-1}(t, x)), \end{aligned} \tag{3.3}$$

and ψ_n is the flow corresponding to v_n.

Because ω_0 is a finite measure, each of $f^\pm$ and ω_0^s are finite. By Young's convolution inequality,

$$\left\| f_n^\pm \right\|_{L^1} \leq \|\rho_n\|_{L^1} \left\| f_0^\pm \right\|_{L^1} \leq \left\| f_0^\pm \right\|_{L^1}. \tag{3.4}$$

By Theorem B.5 and the second comment following it,

$$\|\omega_n^s\|_{L^1} \leq \|\rho_n\|_{L^1} \|\omega_0^s\|_{\mathcal{M}} = \|\omega_0^s\|_{\mathcal{M}}. \tag{3.5}$$

Therefore,

$$\|\omega_n\|_{\mathcal{M}} = \|\omega_n\|_{L^1} \leq \left\| f_0^+ \right\|_{L^1} + \left\| f_0^- \right\|_{L^1} + \|\omega_0^s\|_{\mathcal{M}},$$

meaning that $\{\omega_n\}$ is a bounded in the measure norm. By Corollary C.10, a subsequence of ω_n converges to some finite measure ω' in the weak-* topology on the space $\mathcal{M}$ of all bounded Borel[3] measures on $[0, T] \times \mathbb{R}^2$ and so, as before, in the weak topology. Since the bounded linear functionals on $\mathcal{M}$ are the space of all continuous functions vanishing at infinity, which include continuous compactly supported functions, it is also true that we have weak convergence in our other sense as well.

If that argument made sense before.

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[3] We restrict our spaces to Borel measures so that single point sets $\{x\}$, which are the countable intersections of open balls or radius $1/n$ about x, are measurable sets, and so it makes sense to speak of continuous measures. Each ω_n is Borel because it is smooth, and so absolutely continuous with respect to Lebesgue measure, which is Borel.

Identify the subsequence with the sequence $\{\omega_n\}$ itself. Then we have that $\omega_n \rightharpoonup \omega'$. But by Theorem 3.3, we can again extract a subsequence that converges to an ω'' that lies in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ and is still convergent in the weak-* topology on the space $\mathcal{M}$ of all bounded Borel measures on $[0, T] \times \mathbb{R}^2$. Identifying the subsequence with the sequence once more, and identifying ω with ω'', we have a sequence ω_n converging to a finite measure lying in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$.

We can argue similarly for f_n^+ (using the subsequence we just identified with ω_n as our starting point) to insure that f_n^+ converges weakly to a bounded measure f^+. Taking subsequences and arguing as before two more times, we finally arrive at sequences, $\{\omega_n\}$, $\{f_n^+\}$, $\{f_n^-\}$, and $\{\omega_n^s\}$ that converge individually to bounded measures ω, f^+, f^-, and ω^s, respectively, and are such that

$$\omega = f^+ - f^- + \omega^s.$$

(2): We need to establish that $f^\pm$ is absolutely continuous with respect to Lebesgue measure so we can view $f^\pm$ as being in $L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))$. It is sufficient to show that for all open sets E on $[0, T] \times \mathbb{R}^2$ that given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$m(E) < \delta \implies f^\pm(E) < \epsilon,$$

where m is Lebesgue measure.

Suppose first we can show that for any bounded integrable φ,

$$\lim_{n\to\infty} \int \varphi(t, x) f_n^\pm(t, x)\, dt\, dx = \lim_{n\to\infty} \int \varphi(t, x) f_0^\pm(\psi_n^{-1}(t, x))\, dt\, dx. \tag{3.6}$$

It follows by Lemma B.6 that[4]

$$\begin{aligned} f^\pm(E) &\le \liminf_{n\to\infty} f_n^\pm(E) = \liminf_{n\to\infty} \int \mathbf{1}_E(t, x) f_n^\pm(t, x)\, dt\, dx \\ &= \liminf_{n\to\infty} \int \mathbf{1}_E(t, x) f_0^\pm(\psi_n^{-1}(t, x))\, dt\, dx \\ &= \liminf_{n\to\infty} \int \mathbf{1}_E(t, \psi_n(t, x)) f_0^\pm(x)\, dt\, dx = f_0^\pm(\psi_n(E)). \end{aligned}$$

Because f_0 is absolutely continuous and because $m(\psi_n(E)) = m(E)$ by the preservation of Lebesgue measure under the flow, there is some delta for which we can we can insure that the right-hand side of the above inequality is smaller than ϵ, thus demonstrating the absolute continuity of $f^\pm$. Once we prove Equation (3.6), that is.

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[4] We are using f and f_n both as measures, applying them to a measurable set, and as functions, which is a bit confusing.

We have

$$\begin{aligned}\int \varphi(t,x) f_n^{\pm}(t,x)\,dt\,dx &= \int \varphi(t,x) f_0^{\pm}(\psi_n^{-1}(t,x))\,dt\,dx \\ &\quad + \int \varphi(t,x)(\rho_n * f_0^{\pm} - f_0^{\pm})(\psi_n^{-1}(t,x))\,dt\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\left| \int \varphi(t,x) f_n^{\pm}(t,x)\,dt\,dx - \int \varphi(t,x) f_0^{\pm}(\psi_n^{-1}(t,x))\,dt\,dx \right| \\ &\qquad \le \|\varphi\|_{L^\infty([0,T]\times\mathbb{R}^2)} \left\| \rho_n * f_0^{\pm} - f_0^{\pm} \right\|_{L^1(\mathbb{R}^2)},\end{aligned}$$

from which Equation (3.6) follows.

By Lemma B.6, for any compact measurable E in $[0,T]\times\mathbb{R}^2$,

$$\omega^s(E) \ge \limsup_{n\to\infty} \omega_n^s(E).$$

But by hypothesis of Theorem 1.1, ω_0^s is positive[5], hence ω_n^s is positive by Equation (3.3)[6], so ω^s is positive.

Now we can define a sequence $\{\omega_n^+\}$ of positive measures by

$$\omega_n^+ = f_n^+ + f_n^- + \omega_n^s.$$

Then $|\omega_n| \le \omega_n^+$ by Theorem B.1. Each of the components of ω_n^+ converge weakly, hence ω_n^+ itself converges weakly to a measure ω^+, and we can write

$$\omega^+ = f^+ + f^- + \omega^s.$$

After again extracting a subsequence, $|\omega_n|$ weakly converges to some bounded positive measure ω^p, and by Lemma B.7, $\omega^p \le \omega^+$.

Being locally integrable, $f^{\pm}$ are continuous measures, and ω is continuous by Corollary 3.5. Hence, for any point x in $\mathbb{R}\times\mathbb{R}^2$,

$$\begin{aligned}0 = \omega(\{x\}) &= f^+(\{x\}) - f^-(\{x\}) + \omega^s(\{x\}) \\ &= 0 + 0 + \omega^s(\{x\}),\end{aligned}$$

so $\omega^s(\{x\}) = 0$, meaning ω^s is continuous. But then

$$\omega^+(\{x\}) = f^+(\{x\}) + f^-(\{x\}) + \omega^s(\{x\}) = 0,$$

so ω^+ is continuous.

But $\omega^p \le \omega^+$, meaning that $\omega^p(E) \le \omega^+(E)$ for every measurable set E. Specifically, $\omega^p(\{x\}) \le \omega^+(\{x\}) = 0$, so ω^p is continuous. Thus, we have a positive continuous measure ω^p such that $|\omega_n|$ converges weakly to ω^p.

(3):All that remains to be shown is that the total mass of ω_t is bounded in time. Applying Theorem B.1 just as in our demonstration that $|\omega_n| \le \omega_n^+$,

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[5] This is the only place in the proof where the positivity of the singular part is used.

[6] This is why we added the restriction that ρ_n be nonnegative. Also notice that we are using the regularity of the measures.

we conclude that $|\omega| \le \omega^+$. Since these are both positive measures,

$$\|\omega\|_{\mathcal{M}} = |\omega|\,([0,T] \times \mathbb{R}^2) \le \omega^+([0,T] \times \mathbb{R}^2) = \left\|\omega^+\right\|_{\mathcal{M}},$$

where we fix $T > 0$. More generally, if we let I be a subinterval of $[0,T]$, then

$$|\omega|\,(I \times \mathbb{R}^2) \le \omega^+(I \times \mathbb{R}^2).$$

By the preservation of vorticity along flow lines,

$$\begin{aligned}\omega_n^+(I \times \mathbb{R}^2) &= m(I)(\rho_n * (f_0^+ + f_0^- + \omega_0^s))(\mathbb{R}^2) \\ &= m(I)(\rho_n * |\omega_0|)(\mathbb{R}^2) \le m(I)\,\|\rho_n\|_{L^1}\,\|\omega_0\|_{\mathcal{M}} \\ &\le m(I)\,\|\omega_0\|_{\mathcal{M}}.\end{aligned}$$

By Lemma B.7,

$$\omega^+(I \times \mathbb{R}^2) \le m(I)\,\|\omega_0\|_{\mathcal{M}},$$

so

$$|\omega|\,(I \times \mathbb{R}^2) \le m(I)\,\|\omega_0\|_{\mathcal{M}}. \tag{3.7}$$

Writing the first measure as an integral, we have

$$|\omega|\,(I \times \mathbb{R}^2) = \int_I |\omega_t|\,(\mathbb{R}^2)\,dt = \int_I \|\omega_t\|_{\mathcal{M}}\,dt \le m(I)\,\|\omega_0\|_{\mathcal{M}},$$

or,

$$\frac{1}{m(I)} \int_I \|\omega_t\|_{\mathcal{M}}\,dt \le \|\omega_0\|_{\mathcal{M}}.$$

Taking intervals I nicely shrinking to a point and applying the Lebesgue differentiation theorem we conclude that

$$\|\omega_t\|_{\mathcal{M}} \le \|\omega_0\|_{\mathcal{M}}$$

almost everywhere in Lebesgue measure on time. $\square$

Comment: The positivity of the singular part of the initial vorticity was needed to separate a positive from a negative part of the limiting vorticity simultaneously with separating the absolutely continuous from the singular part. This allowed us to demonstrate continuity of the measure ω^+. Also, if we weren't able to make such a separation, we wouldn't have been able to apply Theorem B.1 to conclude that $|\omega|$ is also continuous. We won't really appreciate why this was so important, though, until we see why continuity of ω^+ is so important in the proof of Theorem 7.1.

Comment: By the reasoning that led to Equation (3.7), without the added complication of introducing an interval in time, it would be tempting to conclude directly that $\|\omega_t\|_{\mathcal{M}} \le \|\omega_0\|_{\mathcal{M}}$ for all time t. However, ω_t doesn't exist as a limit of its own at each time; rather, we take the limiting measure ω on $[0,T] \times \mathbb{R}^2$ and then ω_t is the slice of ω at time t. I see no way to avoid this and obtain what Chemin claims, that the bound on total vorticity is for all time and not just almost everywhere in time.

Theorem 3.7. *$v_n \rightharpoonup v$ for some v in $L^\infty_{loc}(\mathbb{R}; E_m)$. Moreover, v satisfies the same energy estimate as for smooth initial data; namely,*

$$\|v(t) - \sigma\|_{L^2} \leq \|v_0 - \sigma\|_{L^2} e^{t\|\nabla\sigma\|_{L^\infty}},$$

where σ is a stationary solution in E_m.

Proof. From the proof of Lemma 3.2, the sequence $v_n - \sigma_n$ is bounded in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ by

$$\|v_0 - \sigma\|_{L^2} e^{T\|\nabla\sigma\|_{L^\infty}}.$$

Therefore, using Theorem C.9 as we have already done a number of times before, there is some subsequence of v_n, which we identify with itself (as we do all of the corresponding subsequences, of course), such that $v_n - \sigma_n$ converges weakly to some vector w in $L^\infty([0,T]; L^2(\mathbb{R}^2))$, and w satisfies the same bound in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ as the sequence $v_n - \sigma_n$.

But $\sigma_n \rightharpoonup \sigma$ by Theorem C.7[7], so

$$v_n - \sigma \rightharpoonup w \quad \text{or} \quad v_n \rightharpoonup v := \sigma + w,$$

meaning that v is in $L^\infty_{loc}(\mathbb{R}; E_m)$, and that $v_n - \sigma$ satisfies the energy estimate claimed. ☐

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[7] In fact, by Lemma A.6, $\sigma_n \to \sigma$ in $L^2(\mathbb{R}^2)$.

4. WEAK FORMULATION OF THE EULER EQUATIONS

Let us first review the weak formulation of the Euler equations for vortex patches, following [7] p. 309-311, so we can see why this weak solution fails to apply to vortex sheets. In a vortex patch, our initial vorticity ω_0 is in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, but has no assumed regularity.

Let $v(t,x)$ be a smooth two-dimensional velocity field for an incompressible fluid, and assume that its corresponding vorticity $\omega(t,x)$, which will also be smooth, vanishes at infinity. Then ω is a smooth solution to the two-dimensional Euler equations, which we can write as

$$\frac{D\omega}{Dt} = 0,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla.$$

The transport equation (Equation 1.16 p. 5 of [7]) for an incompressible fluid tells us then that

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi\omega \, dx = \int_{\mathbb{R}^2} \frac{D}{Dt}(\varphi\omega) \, dx = \int_{\mathbb{R}^2} \frac{D\varphi}{Dt} \omega \, dx,$$

where φ is any smooth, compactly supported function.

Integrating with respect to time gives

$$\int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x) \, dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega(0,x) \, dx = \int_0^T \int_{\mathbb{R}^2} \frac{D\varphi}{Dt} \omega \, dx.$$

This last equation applies to a broader class of both test functions (we only require one derivative) and vorticity (any vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ will do).

This formulation will not serve for vortex sheets, however, because the vorticity is not even a function and so not measurable with respect to Lebesgue measure. Another way of looking at this is that ω has a singular part, so we cannot write it in the form $\omega = h dx$ where h is the Radon-Nikodym derivative with respect to Lebesgue measure—if we could, we could substitute it into the formula above and have a weak formulation for vortex sheets.

Since the singular part of the vorticity seems to be such a stumbling block, we choose instead to formulate the equations in terms of the velocity which, while it is discontinuous, is at least locally Lebesgue measurable. So consider the Euler equations in their strong form:

$$(E) \quad \begin{cases} \frac{Dv}{Dt} = \partial_t v + v \cdot \nabla v = -\nabla p \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0, \end{cases} \tag{4.1}$$

Whatever our weak formulation may look like, it cannot include any derivatives of the velocity, for our velocities will not be continuous let alone differentiable. The usual way to get rid of derivatives in going to a weak formulation, and the way we will follow, is to integrate by parts, thereby moving the derivatives to the test function. A little preparation of (E) is what's needed, noting that for smooth velocities,

$$\begin{aligned} v \cdot \nabla v &:= \sum_j v^j \partial_j v = \sum_j v^j \sum_i \partial_j v^i \widehat{e}_i \\ &= \sum_i \left(\sum_j v^j \partial_j v^i \right) \widehat{e}_i = \sum_i (v \cdot (\nabla v^i)) \widehat{e}_i \qquad (4.2) \\ &= \sum_i \operatorname{div}(v^i v) e_i = \sum_i \left(\sum_j \partial_j (v^i v^j) \right) \widehat{e}_i \\ &=: \operatorname{div} v \otimes v. \end{aligned}$$

Here we used the fact that if $\operatorname{div} v = 0$, then $v \cdot \nabla a = \operatorname{div}(av)$ for any continuously differentiable function a to conclude that $v \cdot (\nabla v^i) = \operatorname{div}(v^i v)$.)

Thus the system (E) is equivalent for smooth functions to the following system of equations:

$$(E) \quad \begin{cases} \partial_t v + \operatorname{div} v \otimes v = -\nabla p \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0, \end{cases} . \qquad (4.3)$$

We have not removed the derivatives in $v \cdot \nabla v$, but they are now positioned to be removed by integrations by parts. Arguing much as we did before with the transport formula, substituting Equation (4.2), we obtain (as in p. 361-362 of [7]),

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi(T, x) \cdot v(T, x)\, dx - \int_{\mathbb{R}^2} \varphi(0, x) \cdot v_0\, dx \\ &\qquad = \int_0^T \int_{\mathbb{R}^2} \partial_t \varphi \cdot v + \nabla \varphi \cdot (v \otimes v)\, dx\, dt, \end{aligned}$$

where now the test function φ is to be any smooth compactly supported divergence-free vector field. Also, $\nabla \varphi$, the differential of φ, is a 2×2 matrix, as is $v \otimes v$, and we define $\nabla \varphi \cdot (v \otimes v)$ to be the sum of the product of the corresponding elements of each matrix. (The disappearance of the pressure in this equation was by the divergence theorem and the fact that φ has divergence zero.)

Chemin shows in Theorem 1.3.1(ii) p. 13 of [1] that the system (E) is equivalent to the system of equations,

$$\partial_t v + \frac{1}{2} \begin{pmatrix} \partial_1 & 2\partial_2 \\ -\partial_2 & 2\partial_1 \end{pmatrix} \begin{pmatrix} (v^1)^2 - (v^2)^2 \\ v^1 v^2 \end{pmatrix} = -\nabla q,$$

for some vector field q in $L^\infty_{loc}(\mathbb{R}; \mathcal{S}'(\mathbb{R}^2)) \cap \mathcal{F}_x^{-1}(L^\infty_{loc}(\mathbb{R}; L^2 + L^\infty))$ with, I suppose, the same interpretation in terms of integration against a test function (so the vector field q disappears and plays no direct role in our analysis).

Equivalently, we can let our test function $\varphi = \nabla^{\perp}\eta$, where η is a scalar function in $C_0^\infty([0,T] \times \mathbb{R}^2)$ (this follows from Theorem C.4). In effect, then, η becomes our (now scalar) test function. After integrating by parts, we obtain Equation (11.95) p. 435 of [7]:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} (\partial_2\eta)v^1 - (\partial_1\eta)v^2 \, dx \, dt \\ = \int_0^T \int_{\mathbb{R}^2} (\partial_1\partial_2\eta)((v^2)^2 - (v^1)^2) + (\partial_2^2\eta - \partial_1^2\eta)(v^1v^2) \, dx \, dt. \quad (4.4)\end{aligned}$$

This, finally, is our weak formulation of the Euler equations for a vortex sheet.

As we showed in Section 3, there exists a vector field v to which a sequence $\{v_n\}$ of approximate smooth solutions weakly converges. This means that

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^2} (\partial_2\eta)v_n^1 - (\partial_1\eta)v_n^2 \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} (\partial_2\eta)v^1 - (\partial_1\eta)v^2 \, dx \, dt.$$

Suppose that we can also show that

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^2} (\partial_1\partial_2\eta)((v_n^2)^2 - (v_n^1)^2) + (\partial_2^2\eta - \partial_1^2\eta)(v_n^1v_n^2) \, dx \, dt \\ = \int_0^T \int_{\mathbb{R}^2} (\partial_1\partial_2\eta)((v^2)^2 - (v^1)^2) + (\partial_2^2\eta - \partial_1^2\eta)(v^1v^2) \, dx \, dt.\end{aligned}$$

Then since each v_n satisfies Equation (4.4), v will satisfy it as well.

DiPerna and Majda show in [3] that rotation by a constant angle of 45 degrees transforms $(v^2)^2 - (v^1)^2$ into v^1v^2. Also, since η is in $C_0^\infty([0,T] \times \mathbb{R}^2)$, we can replace $\partial_1\partial_2\eta$ and $\partial_2^2\eta - \partial_1^2\eta$ by an arbitrary test function in $C_0^\infty([0,T] \times \mathbb{R}^2)$, reducing the proof of Theorem 1.1 to proving the following:

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^2} \varphi v_n^1 v_n^2 \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} \varphi v^1 v^2 \, dx \, dt \quad (4.5)$$

for all φ in $C_0^\infty([0,T] \times \mathbb{R}^2)$. In other words, we are reduced to showing that $v_n^1 v_n^2 \rightharpoonup v^1 v^2$.

Assuming that it is valid to apply the Biot-Savart law (Theorem C.5) to give v_n in terms of ω_n, then in combination with Fubini's theorem, we have Equation (2.2) and Equation (2.3), where

$$G(t,x,y) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{z^2 - y^2}{|z-y|^2} \frac{z^1 - x^1}{|z-x|^2} g(t,z)\, dz. \tag{5.1}$$

To apply the Biot-Savart law, it is sufficient to know that ω_n lies in some L^p space for $1 \le p < 2$. In fact, ω_n lies in all L^p spaces, $1 \le p \le \infty$—that is, $\omega_n \in L^1 \cap L^\infty$. Since all L^p norms of the vorticity are preserved, $\omega_n = \omega(v_n) \in L^1 \cap L^\infty$ if and only if $\omega(v_{0,n}) \in L^1 \cap L^\infty$. But,

$$\omega(v_{0,n}) = \omega(\rho_n * v_0) = \rho_n * \omega(v_0) = \rho_n * \omega_0.$$

By assumption, ω_0 is in L^1, being a finite measure. Thus,

$$\|\rho_n * \omega_0\|_{L^1} \le \|\rho_n\|_{L^1} \|\omega_0\|_{\mathcal{M}} < \infty$$

and

$$\|\rho_n * \omega_0\|_{L^\infty} \le \|\rho_n\|_{L^\infty} \|\omega_0\|_{\mathcal{M}} < \infty,$$

by Theorem B.5 and the first comment following it.

Our ultimate goal is to show weak convergence of $v_n^1 v_n^2$ to $v^1 v^2$. This is equivalent to showing that

$$\lim_{n\to\infty} \langle v_n^1 v_n^2, g\rangle = \langle v^1 v^2, g\rangle$$

or, in other words, bringing the limit under the integral sign in Equation (2.2). To do this we need some control on the behavior of the function G, the topic of the following section, and on μ_n, which we describe now.

TODO: I am pretty sure the rest of this section is not needed. Chemin argues this, but it seems redundant.

Define the norm of a bounded measure ν on Ω by

$$\|\nu\| := |\nu|(\Omega). \tag{5.2}$$

Setting $\Omega = [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$, because of the preservation of the vorticity along flow lines,

$$\begin{aligned}
\|\mu_n\| &= \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\omega_n(t,x)|\, |\omega_n(t,y)|\; dt\, dx\, dy \\
&= \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\omega_n(0,x)|\, |\omega_n(0,y)|\; dt\, dx\, dy \\
&= T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\omega_n(0,x)|\, |\omega_n(0,y)|\; dx\, dy \\
&= T \|\omega_n(0,\cdot)\|_{L^1}^2 = T \|\rho_n * \omega_0\|_{L^1}^2 \\
&\le T \|\rho_n\|_{L^1}^2 \|\omega_0\|_{L^1}^2 \le T \|\omega_0\|_{L^1}^2,
\end{aligned}$$

so the sequence $\{\mu_n\}$ is bounded.

Since $\{\mu_n\}$ is bounded, it lives in a ball (the unit ball, if we normalize the measures) in the space of all bounded Borel measures, $\mathcal{M}$, on $\mathbb{R}^7$. But $\mathcal{M}$

is a Banach space (its dual is the Banach space of all continuous functions vanishing at infinity), and the unit ball is compact in the weak$-^*$ topology by the Banach-Alaoglu theorem (see Theorem C.9), so some subsequence of $\{\mu_n\}$ converges weakly to a bounded measure μ. Without loss of generality, we assume that the subsequence is the sequence $\{\mu_n\}$ itself.

6. PROPERTIES OF THE FUNCTION G

The properties of G we will need are described in the following theorem:

Theorem 6.1. *Let* $E = (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2) \setminus \mathbf{D}$, *where* $\mathbf{D} = \mathbb{R} \ \times \operatorname{diag}(\mathbb{R}^2 \times \mathbb{R}^2)$. *Then the function G defined by Equation (5.1), restricted to E, is bounded, vanishes at infinity, and is continuous.*

Proof. G is well-defined on E, since off the diagonal of $\mathbb{R}^2 \times \mathbb{R}^2$, the singularities at $\{x\} \times \mathbb{R}^2$ and $\mathbb{R}^2 \times \{y\}$ are removable (since the function $|z|^{-1}$ is locally integrable in $\mathbb{R}^2$).

Vanishing at Infinity. To prove that G vanishes at infinity, assume first that t does not lie in the projection of $\operatorname{supp}(g)$ on the time axis. Then g is identically zero in the integrand so $G(t, x, y) = 0$. In the time axis, then, G is compactly supported and hence vanishes at infinity.

As for the space variables, we can see that for values of (x, y) sufficiently far from the projection of the support of g on $\mathbb{R}^2 \times \mathbb{R}^2$, that

$$|G(t, x, y)| \leq \frac{C}{\max\{|x|, |y|\}},$$

so G vanishes at infinity in the space variables as well.

Continuity. I suppose we could argue continuity of G in various ways. Chemin just says "The function $|z|^{-1}$ is locally integrable on $\mathbb{R}^2$; therefore Lebesgue's continuity theorem implies the continuity of G off the diagonal." Unfortunately, I don't know to which of the many theorems attributed to Lebesgue, "Lebesgue's continuity theorem" refers. Majda and Bertozzi in [7] p. 445-446 go to enormous pain with a direct calculation of the difference between $G(t, x + h, y)$ and $G(t, x, y)$ to argue continuity. I, however, shall argue as follows.

Let $p = (t, x, y)$, $h = (h_t, h_x, h_y)$, and write $f(t, x, y, z) = f(p, z)$ for the integrand in Equation (5.1), so

$$G(t, x, y) = \int_{\mathbb{R}^2} f(t, x, y, z)\, dz \quad \text{or} \quad G(p) = \int_{\mathbb{R}^2} f(p, z)\, dz,$$

where

$$f(t, x, y, z) = -\frac{1}{4\pi^2} \frac{z^2 - y^2}{|z - y|^2} \frac{z^1 - x^1}{|z - x|^2} g(t, z).$$

Let K be the projection of the support of g onto $\mathbb{R}^2$.

If x and y both lie outside K, then f is bounded on K since the zeros in the denominator are avoided. Thus, $|f(p + h, z)| \leq C\mathbf{1}_K$, which is in L^1. Then since $\lim_{h \to 0} f(p + h, z) = f(p, z)$, by Lebesgue's dominated convergence theorem, $\lim_{h \to 0} G(p + h) = G(p)$ (regardless of how h approaches zero), so G is continuous outside of K.

For x or y inside K (with $x \neq y$, since if $x = y$ then p is not in E), the situation changes only slightly. Because $|z|^{-1}$ is locally integrable (maybe this is Chemin's argument), we can choose a ball B around (x, y) such that once $(x + h_x, y + h_y)$ lies within $\frac{1}{2}B$, the contribution to both $G(p)$ and $G(p + h)$ of their respective integrands within B is less than $\epsilon/2$ for any given $\epsilon > 0$, thereby contributing no more than ϵ to $|G(p + h) - G(p)|$, while outside B, $|f(p + h, z)| \leq C\mathbf{1}_K$.

That is, for $(x + h_x, y + h_y)$ in $\frac{1}{2}B$,

$$\begin{aligned}|G(p+h) - G(p)| \leq &\int_B |f(p+h, z) - f(p, z)| \, dz \\ &+ \left| \int_{\mathbb{R}^2 \setminus B} f(p+h, z) - f(p, z) \, dz \right| .\end{aligned} \tag{6.1}$$

The first term in Equation (6.1) is no larger than

$$\begin{aligned}\int_B |f(p+h, z) - f(p, z)| \, dz &\leq \int_B |f(p+h, z)| \, dz + \int_B |f(p, z)| \, dz \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

As for the second term in Equation (6.1), we argue as we did for (x, y) both outside K, since $\left|f(p + h, z)|_{\mathbb{R}^2 \setminus B}\right| \leq C\mathbf{1}_{K \setminus B}$, which is in L^1, and $f(p + h, z)|_{\mathbb{R}^2 \setminus B}$ approaches $f(p, z)|_{\mathbb{R}^2 \setminus B}$ pointwise. Applying Lebesgue's dominated convergence theorem, we conclude that in the limit as h approaches 0, the second term in Equation (6.1) is 0. Thus, $\lim_{h \to 0} |G(p + h) - G(p)| < \epsilon$. But ϵ is arbitrary, so, in fact, $\lim_{h \to 0} G(p + h) = G(p)$.

Boundedness. Finally, we prove boundedness, which is more difficult.

Let

$$\widetilde{G}(t, y, w) = \int_{\mathbb{R}^2} \frac{z^2}{|z|^2} \frac{z^1 + w^1}{|z + w|^2} g(t, y + z) \, dz.$$

It follows by the change of variables $z' = z - y$ that

$$G(t, x, y) = -\frac{1}{4\pi^2} \widetilde{G}(t, y, y - x).$$

Since G and hence $\widetilde{G}$ vanish at infinity, $|\widetilde{G}|$ is less than say, 1, outside of some compact subset of E. So if we can show that $\widetilde{G}$ is bounded in *every* compact set K, then it is bounded in all of E.

Define

$$r(t, y, z) := g(t, y + z) - g(t, y),$$

so,

$$g(t, y + z) = g(t, y) + r(t, y, z) \quad \text{and} \quad r(t, y, 0) = 0. \tag{6.2}$$

Since g is compactly supported and smooth, its derivatives are bounded, meaning that r is smooth and has derivatives bounded by some constant C.

Thus,

$$\begin{aligned}|r(t,y,z)| &= |r(t,y,z)-r(t,y,0)| \\ &\le \int_0^z \left|\frac{\partial}{\partial w} r(t,y,w)\right| dz \le C\,|z|,\end{aligned} \tag{6.3}$$

a fact we will use a little later.

Let B be a ball containing the sum of the projection of K onto the *first* copy of $\mathbb{R}^2$ in E and of the projection onto $\mathbb{R}^2$ of the support of g. Let θ be a smooth compactly supported radial function, equal to 1 in a neighborhood of B, and taking values in $[0,1]$. Our definition of B insures that for any $(t,y,w) \in K$, $g(t,y+z) = 0$ whenever $z \notin B$—that is, whenever $\theta(z) \neq 1$. Thus,

$$\widetilde{G}(t,y,w) = \int \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2} g(t,y+z)\theta(z)\,dz.$$

Applying Equation (6.2), it follows that

$$\widetilde{G}(t,y,w) = \Theta(w)g(t,y) + \int_{\mathbb{R}^2} \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2}\theta(z)r(t,y,z)\,dz, \tag{6.4}$$

where

$$\Theta(w) = \int_{\mathbb{R}^2} \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2}\theta(z)\,dz. \tag{6.5}$$

Thus,

$$|\widetilde{G}(t,y,w) - g(t,y)\Theta(w)| \le \int \frac{|z^2|}{|z|^2}\frac{|z^1+w^1|}{|z+w|^2}\,|r(t,y,z)|\,\theta(z)\,dz.$$

But

$$\frac{|z^2|}{|z|^2} \le \frac{1}{|z|} \qquad \text{and} \qquad \frac{|z^1+w^1|}{|z+w|^2} \le \frac{1}{|z+w|}, \tag{6.6}$$

so, with Equation (6.3),

$$\begin{aligned}|\widetilde{G}(t,y,w) - g(t,y)\Theta(w)| &\le \int \frac{1}{|z|}\frac{1}{|z+w|}\,|r(t,y,z)|\,\theta(z)\,dz \\ &\le C\int \frac{1}{|z|}\frac{1}{|z+w|}\,|z|\,\theta(z)\,dz \\ &\le C\int_{B'} \frac{1}{|z+w|}\,dz \\ &= C\int_0^{2\pi}\int_0^{r'} \frac{1}{r} r\,dr\,d\theta = 2C\pi r' = C,\end{aligned} \tag{6.7}$$

where B' is a ball centered at $-w$ and containing B, and r' is the radius of B' (which is no larger than twice the radius of B, since $-w \in B$).

Since g is bounded, if Θ is bounded on the projection of K onto the last copy of $\mathbb{R}^2$, then by Equation (6.7) $\widetilde{G}$, and hence G, are bounded on K. Thus, all that remains of the proof of this theorem is to show that Θ is bounded on the projection of K, which we now proceed to do.

Let w be on the unit circle in the plane and let σ be any real number in the interval $(0,1]$. The reason we choose σ in this range is that we wish to prove the boundedness of $\Theta(w)$ for w in the unit disk minus the origin, the origin ($w=0$) corresponding to the projection of the exceptional set $\mathbf{D}$ onto the last copy of $\mathbb{R}^2$.

Make the change of variables $z' = z/\sigma$ in Equation (6.5)—so $z = \sigma z'$, which has Jacobian σ^2 (remember z is an ordered pair)—to give

$$\begin{aligned}\Theta(\sigma w) &= \int \frac{z^2}{|z|^2}\frac{z^1+\sigma w^1}{|z+\sigma w|^2}\theta(\sigma z)\,dz\\ &= \int \frac{\sigma z^2}{|\sigma z|^2}\frac{\sigma z^1+\sigma w^1}{|\sigma z+\sigma w|^2}\theta(\sigma z)\,\sigma^2 dz\\ &= \int \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2}\theta(\sigma z)\,dz.\end{aligned}$$

Now, the function,

$$\frac{z^1z^2}{|z|^4},$$

integrates to zero over any radially symmetric measurable set centered at the origin, so

$$\begin{aligned}&\Theta(\sigma w)-\Theta(w)\\ &\quad= \int_{\mathbb{R}^2} \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2}(\theta(\sigma z)-\theta(z))\,dz\\ &\quad= \int_{|z|=c}^{|z|=2} \frac{z^2}{|z|^2}\frac{z^1+w^1}{|z+w|^2}(\theta(\sigma z)-\theta(z))\,dz\\ &\qquad+\int_2^\infty \frac{z^2}{|z|^2}\left(\frac{z^1+w^1}{|z+w|^2}-\frac{z^1}{|z|^2}\right)(\theta(\sigma z)-\theta(z))\,dz.\end{aligned}$$

Here we can choose c to be the radius of B, for $\theta(\sigma z)-\theta(z)$ is zero on B.

Because $|\theta(\sigma z)-\theta(z)|\le 2$, we have

$$\begin{aligned}|\Theta(\sigma w)-\Theta(w)| \le 2\int_{|z|=c}^{|z|=2} \frac{|z^2|}{|z|^2}\frac{|z^1+w^1|}{|z+w|^2}\,dz\\ +2\int_{|z|=2}^{|z|=\infty} \frac{|z^2|}{|z|^2}\left|\frac{z^1+w^1}{|z+w|^2}-\frac{z^1}{|z|^2}\right|\,dz.\end{aligned}\tag{6.8}$$

To bound the first integral in Equation (6.8) we use Equation (6.6) to conclude that

$$\int_{|z|=c}^{|z|=2} \frac{|z^2|}{|z|^2}\frac{|z^1+w^1|}{|z+w|^2}\,dz \le \int_{|z|=c}^{|z|=2} \frac{1}{|z|}\frac{1}{|z+w|}\,dz \le C_1 < \infty,$$

the integral being finite because $|z+w|^{-1}$ is locally integrable, and where C_1 is independent of w.

The second integral in Equation (6.8) is bounded with the help of Lemma 6.2, below, giving

$$\begin{aligned}\int_{|z|=2}^{|z|=\infty} \frac{\left|z^2\right|}{|z|^2}\left|\frac{z^1+w^1}{|z+w|^2}-\frac{z^1}{|z|^2}\right| dz &\leq 40\int_2^\infty \frac{1}{|z|}\frac{1}{|z|^2}\,dz \\ &= 40\int_2^\infty \frac{1}{|z|^3}\,dz \leq C_2 < \infty.\end{aligned}$$

Thus, Equation (6.8) becomes,

$$|\Theta(\sigma w)-\Theta(w)| \leq 2C_1 + 2C_2.$$

This implies that Θ is bounded on the unit disk minus the origin since it never differs by more than a constant from some value it attains on the unit circle, and on the unit circle it is bounded, again because the function $|z|^{-1}$ is locally integrable. But the above argument can obviously be extended to any disk in $\mathbb{R}^2$ minus the origin, and hence to a disk containing the projection of K onto the last copy of $\mathbb{R}^2$, which completes the proof of Theorem 6.1 except for the proof of Lemma 6.2. ☐

Lemma 6.2. *For all w in on the unit circle in the plane and for all z such that $|z| \geq 2$,*

$$\left|\frac{z^1+w^1}{|z+w|^2}-\frac{z^1}{|z|^2}\right| \leq \frac{40}{|z|^2}.$$

Proof.

$$\begin{aligned}&\left|\frac{z^1+w^1}{|z+w|^2}-\frac{z^1}{|z|^2}\right| \\ &\quad= \left|\frac{|z|^2\left(z^1+w^1\right)-|z+w|^2 z^1}{|z+w|^2}\right|\frac{1}{|z|^2} \\ &\quad= \left|z^1\frac{|z|^2-|z+w|^2}{|z+w|^2}+w^1\frac{|z|^2}{|z+w|^2}\right|\frac{1}{|z|^2} \\ &\quad\leq \left(\left|z^1\frac{|z|^2-|z|^2-2z\cdot w-|w|^2}{|z+w|^2}\right|+\left|w^1\frac{|z|^2}{|z+w|^2}\right|\right)\frac{1}{|z|^2} \\ &\quad= \left(\left|z^1\frac{2z\cdot w-1}{|z+w|^2}\right|+\left|\frac{|z|^2}{|z+w|^2}\right|\right)\cdot\frac{1}{|z|^2}\end{aligned}$$

But for $|z| \geq 2$,

$$\begin{aligned}\left| z^1 \frac{2z \cdot w - 1}{|z+w|^2} \right| &\leq |z| \frac{2|z||w| + 1}{(|z| - |w|)^2} = |z| \frac{2|z| + 1}{(|z| - 1)^2} \\ &= \frac{|z|}{|z| - 1} \frac{2|z| + 1}{|z| - 1} \leq 2 \cdot 5 = 10, \quad \text{and}\end{aligned}$$

$$\left| \frac{|z|^2}{|z+w|^2} \right| \leq \left| \frac{|z|^2}{(|z| - |w|)^2} \right| = \left| \frac{|z|}{(|z| - |w|)} \right|^2 \leq 2^2 = 4.$$

Therefore, for $|z| \geq 2$,

$$\left| \frac{z^1 + w^1}{|z+w|^2} - \frac{z^1}{|z|^2} \right| \leq \frac{10 \cdot 4}{|z|^2} = \frac{40}{|z|^2}.$$

□

7. WEAK CONVERGENCE OF $\{v_n^1 v_n^2\}$

Theorem 7.1 gives the weak convergence of $\{v_n^1 v_n^2\}$ to $v_1 v_2$, given certain assumptions on the initial data. As a result of what we proved in Section 3, these assumptions are met for initial data satisfying the hypotheses of Theorem 1.1. Thus, by proving Theorem 7.1 we complete the proof of Theorem 1.1. (Note that since v belongs to $L^\infty_{loc}(\mathbb{R}; E_m)$, it follows by Theorem 1.3.1(i) of [1] p. 13 that the pressure p associated to v belongs to $L^\infty_{loc}(\mathbb{R}; \mathcal{F}^{-1}(L^2 + L^\infty))$.)

Theorem 7.1. *Let m be a real number and let $\{v_n\}$ be a bounded sequence in the space $L^\infty_{loc}(\mathbb{R}; E_m)$ converging weakly to v. Suppose that the sequence $\{\omega_n\}$ is a bounded sequence in the space $L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))$ converging weakly to ω.*

If there exists a continuous measure ω^p such that $|\omega_n|$ converges weakly to ω^p then we have, in $\mathcal{D}'(\mathbb{R}^3)$,

$$v_n^1 v_n^2 \rightharpoonup v^1 v^2.$$

Weak convergence of a measure here means, as always, that any linear functional in the dual space evaluated on the sequence approaches that same functional evaluated on the weak limit of the function. The dual space to bounded measures is the set of functions vanishing at infinity. Chemin only deals with compactly supported continuous functions, though, in his proof, and I am not sure whether this is simply because they are dense in the set of continuous functions that vanish at infinity.

Before proving Theorem 7.1, we need to establish an important lemma.

Lemma 7.2. *Let X be a metric space, locally compact and σ-compact [a countable union of compact sets—Chemin calls this countable at infinity]. Let $\{\mu_n\}$ be a bounded sequence of bounded measures converging weakly to a measure μ.*

If the sequence $\{|\mu_n|\}$ converges weakly to ν, then, for any bounded Borel function f, vanishing at infinity and continuous outside a closed set N that is $\nu -$ negligible [that is, $\nu(N) = 0$], we have

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, \mu. \tag{7.1}$$

Proof. Let $\epsilon > 0$. Because f vanishes at infinity, we can choose a compactly supported continuous function, τ, so that[8]

$$\|(1-\tau)f\|_{L^\infty} \le \frac{\epsilon}{8 \sup_n \|\mu_n\| + 1}, \tag{7.2}$$

where the measure-norm $\|\mu_n\|$ is as we defined in Equation (5.2).

---

[8] This is a slight correction to Chemin's Equation (6.6) p. 113 of [1].

Let

$$N_p = \{x \in X : d(x, N \cap \operatorname{supp} \tau) < 1/p\},$$

where d is the metric on X. Define the sequence $\{\theta_n\}$ by[9]

$$\theta_p(x) = \frac{d(x, N_p^C)}{d(x, N_p^C) + d(x, N_{2p})}. \tag{7.3}$$

For $x \in N \cap \operatorname{supp} \tau$, $d(x, N_{2p}) = 0$ while $d(x, N_p^C) \neq 0$, so $\theta_p(x) = 1$. For $x \in (N \cap \operatorname{supp} \tau)^C$, for large enough value of p, $d(x, N_{2p}) \neq 0$ while $d(x, N_p^C) = 0$, so $\theta_p(x) = 0$. Also, the terms in the denominator of Equation (7.3) can never both be zero[10], so θ_p is well-defined. We conclude from this that

$$\lim_{p \to \infty} \theta_p(x) = \mathbf{1}_{N \cap \operatorname{supp} \tau}(x). \tag{7.4}$$

Because $\nu(N) = 0$, it must be true that after some value of p, $\int \theta_p \, d\nu$ can be made as small as we desire. Specifically, there is some value of p so that if we set $\alpha = \theta_p$, then

$$\int \alpha \, d\nu \leq \frac{\epsilon}{8 \|f\|_{L^\infty} + 1}. \tag{7.5}$$

But by assumption, $\{\mu_n\}$ converges weakly to ν, so there exists and integer n_0 so that

$$n \geq n_0 \implies \int \alpha \, d|\mu_n| \leq \frac{2\epsilon}{8 \|f\|_{L^\infty} + 1}. \tag{7.6}$$

Then[11],

$$\begin{aligned} &\int f \, d\mu - \int f \, d\mu_n \\ &\quad = \int (1 - \tau) f (d\mu - d\mu_n) \qquad (7.7) \\ &\qquad + \int \tau \alpha f (d\mu - d\mu_n) + \int \tau (1 - \alpha) f (d\mu - d\mu_n). \end{aligned}$$

We need the following two inequalities, which, to avoid disrupting the flow of the proof with too many details, we state now and prove later:

$$\left| \int f \, d\mu - \int f \, d\mu_n - \int \tau (1 - \alpha) f (d\mu - d\mu_n) \right| \leq \frac{3\epsilon}{4}, \tag{7.8}$$

and there exists an integer n_1 such that

$$n \geq n_1 \implies \left| \int \tau (1 - \alpha) f (d\mu - d\mu_n) \right| \leq \frac{\epsilon}{4}. \tag{7.9}$$

Together, these two inequalities imply Equation (7.1), completing the proof of the lemma.

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[9] This is a slight correction to the equation following Equation (6.6) p. 113 of [1].

[10] The slight change we made in the definition of θ_p was to avoid this problem.

[11] This is a slight correction of the equation before Equation (6.9) p. 114 of [1].

Proof of Equation (7.8). We have from Equation (7.7) that

$$\begin{aligned}&\left|\int f\,d\mu - \int f\,d\mu_n - \int(1-\alpha)\tau f(d\mu - d\mu_n)\right|\\&\qquad\le \|(1-\tau)f\|_{L^\infty}\left|\int\,d\mu - \int\,d\mu_n\right| + \|f\|_{L^\infty}\left|\int\alpha\,d\mu - \int\alpha\,d\mu_n\right|,\end{aligned}$$

where we used $|\tau| \le 1$.

But,

$$\begin{aligned}\left|\int\,d\mu - \int\,d\mu_n\right| &\le \left|\int\,d\mu\right| + \left|\int\,d\mu_n\right|\\&\le 2\sup_n\left|\int\,d\mu_n\right| = 2\sup_n\|\mu_n\|.\end{aligned}$$

In the last equality we used the weak convergence of μ_n to μ.

Also,

$$\begin{aligned}\left|\int\alpha\,d\mu - \int\alpha\,d\mu_n\right| &\le \left|\int\alpha\,d\mu\right| + \left|\int\alpha\,d\mu_n\right|\\&\le \int\alpha\,d|\mu| + \int\alpha\,d|\mu_n| \le 2\limsup_n\int\alpha\,d|\mu_n|,\end{aligned}$$

where we used again the weak convergence of μ_n to μ along with the fact that α is nonnegative to conclude that $\int\alpha\,d|\mu| \le \limsup_n\int\alpha\,d|\mu_n|$.

Thus,

$$\begin{aligned}&\left|\int f\,d\mu - \int f\,d\mu_n - \int(1-\alpha)\tau f(d\mu - d\mu_n)\right|\\&\quad\le 2\|(1-\tau)f\|_{L^\infty}\sup_n\|\mu_n\| + 2\|f\|_{L^\infty}\limsup_n\int\alpha\,d|\mu_n|\\&\quad\le 2\frac{\epsilon}{8\sup_n\|\mu_n\|+1}\sup_n\|\mu_n\| + 2\|f\|_{L^\infty}\frac{2\epsilon}{8\|f\|_{L^\infty}+1}\\&\quad\le 2\frac{\epsilon}{8\sup_n\|\mu_n\|}\sup_n\|\mu_n\| + 2\|f\|_{L^\infty}\frac{2\epsilon}{8\|f\|_{L^\infty}}\\&\quad= \frac{\epsilon}{4} + \frac{\epsilon}{2} = \frac{3\epsilon}{4},\end{aligned}$$

where we substituted Equation (7.2) and Equation (7.6). We also assumed that $\|f\|_{L^\infty}$ and $\sup_n\|\mu_n\|$ were nonzero, but the inequality holds, trivially, if either one is zero as well.

Proof of Equation (7.9). The function τ is compactly supported and continuous on all of X, the function α is continuous on all of X, and f is continuous on $X \setminus N$. Therefore, the function $(1-\alpha)\tau f$ is compactly supported and continuous on $X \setminus N$. But, in fact, $(1-\alpha)\tau f$ is continuous on all of X.

To see this, first observe that for any x in $N \setminus \operatorname{supp}\tau$, $(1-\alpha)\tau f$ is zero in some open neighborhood of x and hence is continuous at x[12].

Suppose x is in $N \cap \operatorname{supp}\tau$. As we observed in the proof of Equation (7.4), $\theta_p = 1$ on $N \cap \operatorname{supp}\tau$ for all p, and α was chosen to be one such θ_p; therefore, $\alpha(x) = 1$. Now let y be anywhere in X (not just in N). Then

$$\begin{aligned}
&|((1-\alpha)\tau f)(y) - ((1-\alpha)\tau f)(x)| \\
&\quad = |(1-\alpha(y))\tau(y)f(y)) - (1-\alpha(x))\tau(x)f(x))| \\
&\quad = |(1-\alpha(y))\tau(y)f(y))| \le \|\tau\|_{L^\infty} \|f\|_{L^\infty} |1-\alpha(y)| \\
&\quad = \|f\|_{L^\infty} |\alpha(x) - \alpha(y)| .
\end{aligned}$$

Since f is bounded, $\|f\|_{L^\infty}$ is finite and $(1-\alpha)\tau f$ is continuous on $N \cap \operatorname{supp}\tau$. All these facts about $(1-\alpha)\tau f$ are enough to conclude that it is a compactly supported continuous function on all of X. It follows, then, that there exists an integer n_1 such that Equation (7.9) holds. ☐

Proof. **Of Theorem 7.1.** We can write μ_n of Equation (2.3) as

$$\mu_n = \omega_n(t) \otimes \omega_n(t) \otimes dt.$$

Then

$$|\mu_n| = |\omega_n(t)| \otimes |\omega_n(t)| \otimes dt,$$

where dt, being Lebesgue measure, has no singular part.

Since $|\omega_n|$ converges weakly to ω^p by assumption, it follows that $|\mu_n|$ converges weakly to

$$\nu = \omega_t^p \otimes \omega_t^p \otimes dt.$$

Then

$$\nu(\mathbf{D}) = \int_{\mathbf{D}} d\nu = \int_{\mathbb{R}\times\mathbb{R}^2} \left(\int_{\{x\}} d\omega_t^p(y) \right) d\omega_t^p(x)\, dt. \tag{7.10}$$

But

$$\int_{\{x\}} d\omega_t^p(y) = 0,$$

by the assumption that ω^p is continuous. Therefore, $\nu(\mathbf{D}) = 0$.

We now know that the function G meets the hypotheses for the function f of Lemma 7.2 with the ν-negligible set being $\mathbf{D}$. It follows from that lemma that

$$\begin{aligned}
\lim_{n\to\infty} \langle v_n^1 v_n^2, g\rangle &= \lim_{n\to\infty} \int G(t,x,y)\, d\mu_n(t,x,y) \\
&= \int G(t,x,y)\, d\mu(t,x,y). \qquad (7.11)
\end{aligned}$$

---

[12] Chemin ignores this point in his text.

Our goal is to show that

$$\langle v^1 v^2, g\rangle = \int G(t,x,y)\, d\mu(t,x,y), \tag{7.12}$$

which combined with Equation (7.11) shows that $v_n^1 v_n^2 \rightharpoonup v^1 v^2$, completing the proof of Theorem 7.1. So all that remains is to prove Equation (7.12).

The demonstration of Equation (7.12) is a two step process. The first step will lead to Equation (7.13), the second to Equation (7.15). These two equalities, together with Equation (7.11), establish Equation (7.12), completing the proof.

First step: Having already regularized the initial velocity to obtain a series of approximate solutions $\{v_n\}$, we now take the weak limit velocity v and regularize it using, for convenience, the same approximation to the identity $\{\rho_n\}$. We call the resulting sequences of velocities and vorticities $\{\widetilde{v}_n\}$ and $\{\widetilde{\omega}_n\}$. (Note that $\widetilde{\omega}_n := \omega(\widetilde{v}_n) = \omega(\rho_n * v) = \rho_n * \omega(v) = \rho_n * \omega$. That is, the vorticity of the regularization equals the regularization of the vorticity.)

We know from Section 6 that G vanishes at infinity and, outside a set of zero Lebesgue and μ measure, is bounded and continuous. Let $\chi_r : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a smooth compactly supported function equal to 1 on a ball of radius r centered at the origin and taking values in $[0,1]$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \langle \widetilde{v}_n^1 \widetilde{v}_n^2, g\rangle \\
&\quad= \lim_{n\to\infty} \int G(t,x,y)(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\, dx\, dy\, dt \\
&\quad= \lim_{n\to\infty} \int \chi_r G(t,x,y)(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\, dx\, dy\, dt \\
&\qquad + \lim_{n\to\infty} \int (1-\chi_r) G(t,x,y)(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\, dx\, dy\, dt \\
&\quad= \int \chi_r G(t,x,y)\, d\mu(t,x,y) \\
&\qquad + \lim_{n\to\infty} \int (1-\chi_r) G(t,x,y)(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\, dx\, dy\, dt,
\end{aligned}$$

where we used Theorem C.7. But,

$$\begin{aligned}
&\left| \int (1-\chi_r) G(t,x,y)(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\, dx\, dy\, dt \right| \\
&\qquad \le \|(1-\chi_r)G\|_{L^\infty} \|(\rho_n\otimes\rho_n) * (\omega\otimes\omega)\|_{L^1([0,T]\times\mathbb{R}^5)} \\
&\qquad \le \|(1-\chi_r)G\|_{L^\infty}\, T\, \|\rho_n\otimes\rho_n\|_{L^1} \|\omega\otimes\omega\|_{\mathcal{M}} \\
&\qquad \le T \left\|\omega^0\right\|_{\mathcal{M}}^2 \|(1-\chi_r)G\|_{L^\infty},
\end{aligned}$$

which approaches zero as r approaches infinity, since G vanishes at infinity. A similar argument shows that

$$\int \chi_r G(t,x,y)\,d\mu(t,x,y) \to \int G(t,x,y)\,d\mu(t,x,y),$$

and we conclude that

$$\lim_{n\to\infty} \left\langle \widetilde{v}^1_n \widetilde{v}^2_n, g \right\rangle = \int G(t,x,y)\,d\mu(t,x,y). \tag{7.13}$$

Todo: We have really just proved Lemma 7.2 in a less general setting, and in fact in as general a setting, I believe, as was needed when we applied it. If this bears out, change the statement and proof of Lemma 7.2.

Second step: To complete the proof of Theorem 7.1, we use Lemma 7.3, and its definitions of $\alpha, \alpha_n, \beta, \beta_n$, to conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \left| \left\langle \widetilde{v}^1_n \widetilde{v}^2_n, g \right\rangle - \left\langle v^1 v^2, g \right\rangle \right| \\
&\qquad = \lim_{n\to\infty} \left| \left\langle \widetilde{v}^1_n \widetilde{v}^2_n - v^1 v^2, g \right\rangle \right| = \lim_{n\to\infty} \left| \langle \alpha_n + \beta_n - (\alpha + \beta), g \rangle \right| \\
&\qquad \le \lim_{n\to\infty} \left| \langle \alpha_n - \alpha, g \rangle \right| + \left| \langle \beta_n - \beta, g \rangle \right| \\
&\qquad \le \lim_{n\to\infty} \int_0^T \left\| (\alpha_{n,t} - \alpha_t) g \right\|_{L^1} + \left\| (\beta_{n,t} - \beta_{n,t}) g \right\|_{L^1} \, dt \\
&\qquad \le \|g\|_{L^\infty} \lim_{n\to\infty} \int_0^T \left\| \alpha_{n,t} - \alpha_t \right\|_{L^1} \qquad\qquad (7.14) \\
&\qquad\qquad + \|g\|_{L^2([0,T]\times\mathbb{R}^2)} \lim_{n\to\infty} \int_0^T \left\| \beta_{n,t} - \beta_{n,t} \right\|_{L^2} \, dt \\
&\qquad = \|g\|_{L^\infty} \int_0^T \lim_{n\to\infty} \left\| \alpha_{n,t} - \alpha_t \right\|_{L^1} \\
&\qquad\qquad + \|g\|_{L^2([0,T]\times\mathbb{R}^2)} \int_0^T \lim_{n\to\infty} \left\| \beta_{n,t} - \beta_{n,t} \right\|_{L^2} \, dt \\
&\qquad = 0,
\end{aligned}$$

where the time projection of $\operatorname{supp} g$ is contained in $[0,T]$. We were able to bring the limits inside the integral since Lemma 7.3 gives uniform convergence to zero of the two integrands. Also, the two norms in g are finite because g is continuous and compactly supported.

It follows from Equation (7.14) that

$$\lim_{n\to\infty} \left\langle \widetilde{v}^1_n \widetilde{v}^2_n, g \right\rangle = \left\langle v^1 v^2, g \right\rangle. \tag{7.15}$$

As we stated following Equation (7.12), this completes the proof.

□

Lemma 7.3. *Let $\widetilde{v}_n$ be as in the proof of Theorem 7.1. There exist functions α_n, α, β_n, β on $[0,T] \times \mathbb{R}^2$ such that*

$$\widetilde{v}^1_n \widetilde{v}^2_n = \alpha_n + \beta_n, \quad v^1 v^2 = \alpha + \beta,$$

and

$$\lim_{n\to\infty} \|\alpha_{n,t} - \alpha_t\|_{L^1} + \|\beta_{n,t} - \beta_t\|_{L^2} = 0,$$

convergence being uniform over any compact interval in time. That is, for any $\epsilon > 0$ *there exists an integer* N *independent of the time, such that for all* $n > N$*,* $\|\alpha_{n,t} - \alpha_t\|_{L^1} + \|\beta_{n,t} - \beta_t\|_{L^2} < \epsilon$*.*

Proof. To reduce the amount of notation, we fix the time, but suppress the time parameter in the argument that follows.

Since

$$v^1 v^2 = \sigma^1\sigma^2 + w^1\sigma^2 + w^2\sigma^1 + w^1 w^2$$

and

$$\begin{aligned}\widetilde{v}_n^1\widetilde{v}_n^2 &= (\rho_n * \sigma^1)(\rho_n * \sigma^2) + (\rho_n * w^1)(\rho_n * \sigma^2)\\ &\quad + (\rho_n * w^2)(\rho_n * \sigma^1) + (\rho_n * w^1)(\rho_n * w^2),\end{aligned}$$

a logical decomposition is the following:

$$\alpha = w^1 w^2, \quad \beta = \sigma^1\sigma^2 + w^1\sigma^2 + w^2\sigma^1$$

and

$$\begin{aligned}\alpha_n &= (\rho_n * w^1)(\rho_n * w^2),\\ \beta &= (\rho_n * \sigma^1)(\rho_n * \sigma^2) + (\rho_n * w^1)(\rho_n * \sigma^2) + (\rho_n * w^2)(\rho_n * \sigma^1).\end{aligned}$$

Writing,

$$\begin{aligned}A &= (\rho_n * \sigma^1)(\rho_n * \sigma^2) - \sigma^1\sigma^2,\\ B &= (\rho_n * \sigma^1)(\rho_n * w^2) - \sigma^1 w^2,\\ C &= (\rho_n * \sigma^2)(\rho_n * w^1) - \sigma^2 w^1,\\ D &= (\rho_n * w^1)(\rho_n * w^2) - w^1 w^2,\end{aligned}$$

if we can show that each of $\|A\|_{L^2}$, $\|B\|_{L^2}$, $\|C\|_{L^2}$, and $\|D\|_{L^1}$ approach 0 in the limit, then the lemma will be proved.

A and D are the easiest to deal with. We have

$$\begin{aligned}\|A\|_{L^2} &= \left\|(\rho_n * \sigma^1)(\rho_n * \sigma^2) - (\rho_n * \sigma^1)\sigma^2 + (\rho_n * \sigma^1)\sigma^2 - \sigma^1\sigma^2\right\|_{L^2}\\ &\le \left\|(\rho_n * \sigma^1)(\rho_n * \sigma^2) - (\rho_n * \sigma^1)\sigma^2\right\|_{L^2} + \left\|(\rho_n * \sigma^1)\sigma^2 - \sigma^1\sigma^2\right\|_{L^2}\\ &= \left\|(\rho_n * \sigma^1)(\rho_n * \sigma^2 - \sigma^2)\right\|_{L^2} + \left\|(\rho_n * \sigma^1 - \sigma^1)\sigma^2\right\|_{L^2}\\ &\le \left\|(\rho_n * \sigma^1)\right\|_{L^\infty} \left\|\rho_n * \sigma^2 - \sigma^2\right\|_{L^2} + \left\|\sigma^2\right\|_{L^\infty} \left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^2}\\ &\le \|\rho_n\|_{L^1} \left\|\sigma^1\right\|_{L^\infty} \left\|\rho_n * \sigma^2 - \sigma^2\right\|_{L^2} + \left\|\sigma^2\right\|_{L^\infty} \left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^2}\\ &\to 0,\end{aligned}$$

since $\|\rho_n\|_{L^1} = 1$, σ is in L^∞, and since $\left\|\rho_n * \sigma^k - \sigma^k\right\|_{L^2} \to 0$, $k = 1, 2$, by Lemma A.6.

As for D, we start off by arguing as in the first three steps of our argument for A, then proceed using Hölder's inequality, Young's convolution inequality, and Theorem C.8:

$$\begin{aligned}\|D\|_{L^1} &\le \left\|(\rho_n * w^1)(\rho_n * w^2 - w^2)\right\|_{L^1} + \left\|(\rho_n * w^1 - w^1)w^2\right\|_{L^1}\\ &\le \left\|\rho_n * w^1\right\|_{L^2} \left\|\rho_n * w^2 - w^2\right\|_{L^2} + \left\|\rho_n * w^1 - w^1\right\|_{L^2} \left\|w^2\right\|_{L^2}\\ &\le \|\rho_n\|_{L^1} \left\|w^1\right\|_{L^2} \left\|\rho_n * w^2 - w^2\right\|_{L^2} + \left\|\rho_n * w^1 - w^1\right\|_{L^2} \left\|w^2\right\|_{L^2}\\ &\to 0.\end{aligned}$$

Starting with the same first three steps of our argument for A, we have

$$\begin{aligned}\|B\|_{L^2} &\le \left\|(\rho_n * \sigma^1)(\rho_n * w^2 - w^2)\right\|_{L^2} + \left\|(\rho_n * \sigma^1 - \sigma^1)w^2\right\|_{L^2}\\ &\le \left\|\rho_n * \sigma^1\right\|_{L^\infty} \left\|\rho_n * w^2 - w^2\right\|_{L^2} + \left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^\infty} \left\|w^2\right\|_{L^2}.\end{aligned}$$

But by Young's convolution inequality,

$$\left\|\rho_n * \sigma^1\right\|_{L^\infty} \le \|\rho_n\|_{L^1} \left\|\sigma^1\right\|_{L^\infty} = \left\|\sigma^1\right\|_{L^\infty},$$

which is finite since σ is bounded (see p. 11 Chemin).

Also, we claim that $\left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^\infty} \to 0$. To see this, let K be any compact subset of $\mathbb{R}^d$. Then by Theorem C.6, for all $\epsilon > 0$ there exists a positive integer N such that $\left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^\infty(K)} < \epsilon$ for all $n \ge N$. Also,

$$\begin{aligned}\left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^\infty(\mathbb{R}^d - K)} &\le \left\|\rho_n * \sigma^1\right\|_{L^\infty(\mathbb{R}^d - K)} + \left\|\sigma^1\right\|_{L^\infty(\mathbb{R}^d - K)},\\ &\le 2\left\|\sigma^1\right\|_{L^\infty(\mathbb{R}^d - K)},\end{aligned}$$

where in the last inequality we use Young's convolution inequality. But σ decays at infinity like $C/|x|$, so by choosing K large enough, we can insure that $2\left\|\sigma^1\right\|_{L^\infty(\mathbb{R}^d - K)} < \epsilon$. Hence for all $n \ge N$, $\left\|\rho_n * \sigma^1 - \sigma^1\right\|_{L^\infty}$ is less than ϵ, so it approaches zero.

Finally, the uniformness of the convergence follows from the fact that by an hypothesis of Theorem 7.1, $\{v_n\}$, and hence its weak limit v, is a bounded sequence in the space $L^\infty_{loc}(\mathbb{R}; E_m)$, so each of the norms in our bounds above can be assumed uniform over any compact interval of time. □

Corollary 7.4. **D** *is μ-negligible, where μ is the limit measure of Section 5.*

Proof. From Corollary 3.5, ω and hence μ are continuous. But that was precisely the property of ω^p that was used in Equation (7.10) to conclude that **D** was ν-negligible. Making the same argument with ω in place of ω^p, we conclude that **D** is μ-negligible □

APPENDIX A. THE SPACE E_m

The most natural function space for a velocity field v that is a solution to (NS) or (E) is probably $L^2_{sol}(\mathbb{R}^2)$, the space of all divergence-free (in the sense of a distribution) vector fields in $L^2(\mathbb{R}^2)$, since this space is exactly the space of solutions of finite energy. (One could argue that also assuming some level of smoothness is more natural or even required.) A deficiency of this space, however, is that, under fairly weak additional assumptions on v in $L^2_{sol}(\mathbb{R}^2)$, it necessarily follows that $\int_{\mathbb{R}^2} \omega(v) = 0$ (see Theorem A.5). There are, however, applications in which one wishes to consider nonnegative or nonpositive measures; for instance, when studying vortex patches.

A vortex patch is the solution to (E) in which the initial vorticity ω^0 is the characteristic function of a bounded domain. The velocity v^0 associated to such a vorticity is given by the Biot-Savart law:

$$v^0 = K * \omega^0,$$

where $K(x) = (-x^2, x^1)/\left|x\right|^2$. Since vorticity is transported by the flow for a solution to (E), $\omega(t)$ remains nonnegative for all time, and $v(t)$ is never in $L^2_{sol}(\mathbb{R}^2)$. Thus, we need a larger space than $L^2_{sol}(\mathbb{R}^2)$ when dealing with vortex patches. It turns out, as we will see below, that the velocity associated with a vortex patch is in the space E_m, where m is the total vorticity—that is, $\int_{\mathbb{R}^2} \omega$.

Because the E_m spaces are central to all of our results in the plane, we chose to give a self-contained and careful elucidation of all of their properties that we will use. This appendix can be seen as a fleshing out of the account of these spaces given by Chemin in Chapter 1 of [1].

E_m is defined in Chapter 1 of [1] p. 12 as the space of all divergence-free vector fields that are the sum of a stationary vector field σ of total vorticity m (that is, $\int_{\mathbb{R}^2} \omega(\sigma) = m$) and an L^2 vector field. A stationary vector field is defined in [1] p. 11 as a vector field in the plane of the form

$$\sigma = \left(-\frac{x^2}{r^2} \int_0^r \rho g(\rho)\, d\rho, \frac{x^1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right), \tag{A.1}$$

where $g \in C_0^\infty(\mathbb{R})$.

Majda and Bertozzi in [7] p. 93 call this way of decomposing a vector in E_m the *radial-energy decomposition*, though they only make the assumption that g is smooth, not that it is compactly supported. We use whichever of Chemin's or Majda and Bertozzi's terminology seems more convenient at the time.

The following lemma is Proposition 1.3.1 p. 9 of [1]:

Lemma A.1. *Two vector fields whose coefficients are tempered distributions and whose divergence and vorticity are equal, equal each other up to a vector field with harmonic polynomials as coefficients.*

An immediate corollary of Lemma A.1 is the following:

Corollary A.2. *Two vector fields whose coefficients are tempered distributions that vanish at infinity and whose divergence and vorticity are equal, are equal to each other.*

Proof. Let v and v' be two such vector fields. It follows from Lemma A.1 that they differ by a vector field with harmonic polynomials as coefficients. But $v - v'$ vanishes at infinity; hence, the polynomial they differ by is zero, so $v = v'$. ☐

Remark: It follows, in particular, from Corollary A.2 that a divergence-free vector field whose coefficients are tempered distributions that vanish at infinity is uniquely determined by its vorticity.

The following theorem is a combination of Corollary A.2 and Proposition 1.3.2 p. 11 of [1]:

Theorem A.3. *Let σ be a stationary vector field. Then σ has the following properties:*

(1) *σ is smooth;*
(2) *$\omega(\sigma)$ is in $C_0^\infty(\mathbb{R}^2)$ and is radially symmetric;*
(3) $\operatorname{div}\sigma = 0$*;*
(4) *$|\sigma|$ is radially symmetric and for all sufficiently large r,*

$$|\sigma|(r) = \frac{m}{2\pi r},$$

where $m = \int_{\mathbb{R}^2} \omega(\sigma)$;

(5) *σ is a solution to the time-independent Euler equations*

Conversely, properties (1) through (4) are enough to insure that a vector field is a stationary vector field with $g = \omega(\sigma)$; in fact, this still holds if property (4) is weakened to the assumption that the vector field vanishes at infinity. (To be more explicit, if properties (1) through (3) hold and σ vanishes at infinity, then σ is given by Equation (A.1) with $g = \omega(\sigma)$, and all five properties hold.)

Proof. Let σ be a stationary vector field. Write

$$\sigma = \left(-x^2 f, x^1 f\right),$$

where

$$f = r^{-2} \int_0^r \rho g(\rho)\, d\rho.$$

It is clear that σ is smooth away from the origin, and smoothness at the origin follows from an argument using Taylor's remainder theorem. Thus, we have established property (1).

The vorticity of σ is

$$\begin{aligned}\partial_1\sigma^2 - \partial_2\sigma^1 &= \partial_1(x^1 f) - \partial_2(-x^2 f) = f + x^1\partial_1 f + f + x^2\partial_2 f\\ &= 2f + \partial_r f(x^1\partial_1 r + x^2\partial_2 r) = 2f + \partial_r f(x^1\frac{x^1}{r} + x^2\frac{x^2}{r})\\ &= 2f + r\partial_r f.\end{aligned}$$

But,

$$\partial_r f = -\frac{2}{r^3}\int_0^r \rho g(\rho)\,d\rho + r^{-2}rg = -\frac{2}{r}f + \frac{g}{r},$$

so

$$\omega(\sigma) = \partial_1\sigma^2 - \partial_2\sigma^1 = 2f + r(-\frac{2}{r}f + \frac{g}{r}) = g,$$

establishing property (2).

We have

$$\operatorname{div}\sigma = \partial_1\left(-\frac{x^2}{r^2}f\right) + \partial_2\left(\frac{x^1}{r^2}f\right) = 0,$$

by the obvious symmetry in x^1 and x^2, establishing property (3).

The prove the decay in property (4), let B be the ball of radius r centered at the origin. Then

$$\begin{aligned}\int_B \omega(\sigma) &= -\int_B \operatorname{div}\sigma^\perp = -\int_{\partial B}\sigma^\perp\cdot\mathbf{n} = -\int_{\partial B}\sigma\cdot\mathbf{n}^\perp\\ &= -\int_{\partial B}(-x^2 f, x^1 f)\cdot(-x^2, x^1)/r = -\int_{\partial B} r^2 f/r = -r\int_{\partial B} f\\ &= -2\pi r^2 f,\end{aligned}$$

where $v^\perp := (-v^2, v^1)$. But, $|\sigma| = \left|(-x^2 f, x^1 f)\right| = r\,|f|$, so

$$|\sigma| = \frac{1}{2\pi r}\int_B \omega(\sigma).$$

Then, since $\omega(\sigma) = g$ and g is compactly supported, for all sufficiently large r,

$$|\sigma| = \frac{m}{2\pi r}.$$

Finally, we prove that property (5) holds, via a fairly lengthy calculation. Because $\sigma = (-x^2 f, x^1 f)$,

$$\begin{aligned}\sigma\cdot\nabla\sigma &= \sum_i \sigma^i\partial_i\sigma = \sigma^1\partial_1\sigma + \sigma^2\partial_2\sigma\\ &= -x^2 f(\partial_1(-x^2 f), \partial_1(x^1 f)) + x^1 f(\partial_2(-x^2 f), \partial_2(x^1 f))\\ &= \begin{pmatrix} -x^2 f\partial_1(-x^2 f) + x^1 f\partial_2(-x^2 f)\\ -x^2 f\partial_1(x^1 f) + x^1 f\partial_2(x^1 f))\end{pmatrix}.\end{aligned}$$

Notice that the two coordinates for $\sigma \cdot \nabla\sigma$ are the same, but with the indexes 1 and 2 transposed. But,

$$\begin{aligned}
&-x^2 f\partial_1(-x^2 f) + x^1 f\partial_2(-x^2 f) = -x^2 f(-x^2\partial_1 f) + x^1 f(-f - x^2\partial_2 f)\\
&\quad = -x^1 f^2 + f\left[(x^2)^2\partial_1 f - (x^1x^2)\partial_2 f\right]\\
&\quad = -x^1 f^2 + f\left[(x^2)^2\partial_1 r - (x^1x^2)\partial_2 r\right]\partial_r f\\
&\quad = -x^1 f^2 + f\left[(x^2)^2\frac{x^1}{r} - (x^1x^2)\frac{x^2}{r}\right]\partial_r f\\
&\quad = -x^1 f^2,
\end{aligned}$$

where we used the fact that

$$\partial_i r = \partial_i((x^1)^2 + (x^2)^2)^{1/2} = \frac{1}{2}((x^1)^2 + (x^2)^2)^{-1/2} 2x^i = \frac{x^i}{r}.$$

Similarly,

$$-x^2 f\partial_1(x^1 f) + x^1 f\partial_2(x^1 f) = -x^2 f^2,$$

because of the symmetry in the coordinates of $\sigma \cdot \nabla\sigma$ that we observed above. Thus,

$$\sigma \cdot \nabla\sigma = f^2 \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

Now let[13]

$$h(r) = \int_0^r \rho f(\rho)^2\, d\rho.$$

Then

$$\begin{aligned}
-\nabla(h(r)) &= \begin{pmatrix} -\partial_1 h \\ -\partial_2 h \end{pmatrix} = \begin{pmatrix} -\partial_r h\partial_1 r \\ -\partial_r h\partial_2 r \end{pmatrix} = -\partial_r h \begin{pmatrix} \partial_1 r \\ \partial_2 r \end{pmatrix}\\
&= -rf^2 \begin{pmatrix} \frac{x^1}{r} \\ \frac{x^2}{r} \end{pmatrix} = -f^2 \begin{pmatrix} \frac{x^1}{r} \\ \frac{x^2}{r} \end{pmatrix} = \sigma \cdot \nabla\sigma.
\end{aligned}$$

Then $\partial_t\sigma = 0$ (as we would expect for a stationary vector field), and so σ satisfies Equation 1.6 p. 7 of Chemin—the strong version of (E) of p. 8 of Chemin—with the pressure p equal to h. This establishes property (5).

To argue the converse, suppose that v is a vector field satisfying properties (1) through (3) that also vanishes at infinity (which allows, of course, property (4) as one possible way to vanish at infinity). Let g be its vorticity. Then we know by what we proved above that the stationary vector field σ given by Equation (A.1) satisfies properties (1) through (4) as well with the same vorticity. Thus, $\omega(\sigma) = \omega(v) = g$, so v and σ are two vector fields vanishing at infinity with the same divergence (namely, zero) and curl and so, by Corollary A.2, they are equal. ☐

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[13] This is a correction of an expression in [1].

Remark: Let σ be a stationary vector field with total vorticity m. Then because σ divergence-free by Theorem A.3, it follows that if $\sigma + v$ is in E_m, then v is also divergence-free.

Corollary A.4. *The space E_m is independent of the particular choice of stationary vector field of the form Equation (A.1) that is used to define it.*

Proof. If σ and σ' are two stationary vector fields with the same total vorticity m, then it follows from property (4) of Theorem A.3 that $\sigma - \sigma'$ is in $L^2(\mathbb{R}^2)$ (in fact, in $C_0^\infty(\mathbb{R}^2)$). Even if the two stationary vector fields do not share the same origin about which their vorticities are circularly symmetric, property (4) insures that their difference decays like $1/r^2$ and so is in $L^2(\mathbb{R}^2)$. Hence, the space E_m does not depend upon the specific choice of the stationary vector field. $\square$

Remark: The independence in Corollary A.4 would not follow simply from assuming that the function g vanishes at infinity: compact support of g is stronger than required, but an exact statement of the required decay is not immediately clear.

The following is Lemma 1.3.1 p. 12 of [1]:

Theorem A.5. *Let μ be a finite measure such that $(1+|x|)\,|\mu|$ is also finite. If μ is in $H^{-1}(\mathbb{R}^2)$, then there exists a unique divergence-free vector field v in E_m, where*

$$m = \int_{\mathbb{R}^2} d\mu,$$

and such that $\omega(v) = \mu$.

Proof. We first prove uniqueness. Suppose v and v' both satisfy the conclusion of the theorem. Then v and v' are both divergence-free and have the same vorticity. It follows from Lemma A.1 that they differ by a vector field with harmonic polynomials as coefficients. But $v - v'$ is in $L^2(\mathbb{R}^2)$ and so vanishes at infinity; hence, the polynomial they differ by is zero, so $v = v'$.

We now prove existence. Let σ be any stationary vector field such that $\int_{\mathbb{R}^2} \omega(\sigma) = m$. Let

$$v = \sigma + \mathcal{F}^{-1}\left[\overline{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right], \tag{A.2}$$

where $\overline{\xi} := -i\xi^{\perp} = (i\xi^2, -i\xi^1)$ and i is $\sqrt{-1}$, not an index. We will show that v satisfies the conclusion of the theorem by showing that $\operatorname{div} v = 0$, that the vorticity of v equals μ, and that $v - \sigma$ is in $L^2(\mathbb{R}^2)$. (We motivate the definition of v in Equation (A.2) following the proof.)

The first two of these facts follow from straightforward calculations. Since $\partial_k \mathcal{F}^{-1} u = \mathcal{F}^{-1}(i\xi^k u)$ for any $u : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned}
\operatorname{div} v &= \partial_1 v^1 + \partial_2 v^2 \\
&= \operatorname{div} \sigma + \partial_1 \mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_1 \\
&\qquad + \partial_2 \mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_2 \\
&= \mathcal{F}^{-1}\left[-i\xi^1(i\xi^2)\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right] \\
&\qquad + \mathcal{F}^{-1}\left[i\xi^2(-i\xi^1)\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right] = 0,
\end{aligned}$$

and

$$\begin{aligned}
\omega(v) &= \omega(\sigma) + \partial_1 \mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_2 \\
&\qquad - \partial_2 \mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_1 \\
&= \omega(\sigma) + \mathcal{F}^{-1}\left[i\xi^1(-i\xi^1)\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right] \\
&\qquad - \mathcal{F}^{-1}\left[i\xi^2(i\xi^2)\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right] \\
&= \omega(\sigma) + \mathcal{F}^{-1}\left[((\xi^1)^2 + (\xi^2)^2)\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right] \\
&= \omega(\sigma) + \mathcal{F}^{-1}\left[\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right] \\
&= \omega(\sigma) + \mu - \omega(\sigma) = \mu.
\end{aligned}$$

Finally, we need to prove that v belongs to $\sigma + L^2(\mathbb{R}^2)$; that is, that

$$\left\|\mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_k\right\|_{L^2} < \infty,$$

for $k = 1, 2$. We have, with $l = 3 - k$,

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_k\right\|_{L^2} &= \left\|\left[\bar{\xi}\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right]_k\right\|_{L^2} \\
= \left\|\xi^l\,|\xi|^{-2}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right\|_{L^2} &\le \left\||\xi|^{-1}\left(\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)\right)\right\|_{L^2} \\
= \int_B \frac{|\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)|^2}{|\xi|^2}\,d\xi &+ \int_{\mathbb{R}^d \setminus B} \frac{|\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)|^2}{|\xi|^2}\,d\xi, \qquad \text{(A.3)}
\end{aligned}$$

where B is the unit ball centered at the origin. We bound these last two integrals separately.

The derivatives of $\widehat{\mu}$ are bounded since

$$|\partial_k \widehat{\mu}(\xi)| = |\widehat{x\mu}|\,(\xi) = \left|\int_{R^d} x e^{2\pi i x\xi}\,d\mu(x)\right| \le \int_{R^d} |x|\;d\,|\mu|\,(x),$$

which is finite by the assumption on μ. Also, $\partial_k \widehat{\omega}(\sigma)(\xi)$ is finite by similar reasoning, but using the fact that $\omega(\sigma)$ is compactly supported. Hence,

$|\nabla(\widehat{\mu} - \widehat{\omega}(\sigma))| \le C$. Then, since

$$(\widehat{\mu} - \widehat{\omega}(\sigma))(0) = \int_{\mathbb{R}^2} \mu - \int_{\mathbb{R}^2} \omega(\sigma) = m - m = 0,$$

it follows that $|\widehat{\mu} - \widehat{\omega}(\sigma)|(\xi) \le C|\xi|$, and we conclude that the quotient $|\widehat{\mu}(\xi) - \widehat{\omega}(\sigma)(\xi)| / |\xi|$ is bounded. Thus, the first integral in Equation (A.3) is finite, since its integrand is finite.

As for the second integral in Equation (A.3), it is bounded above by

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus B} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2}\, d\xi + \int_{\mathbb{R}^d \setminus B} \frac{|\widehat{\omega}(\sigma)(\xi)|^2}{|\xi|^2}\, d\xi \\
&\quad \le 2 \int_{\mathbb{R}^d \setminus B} \frac{|\widehat{\mu}(\xi)|^2}{1 + |\xi|^2}\, d\xi + \int_{\mathbb{R}^d \setminus B} |\widehat{\omega}(\sigma)(\xi)|^2\, d\xi \\
&\quad \le 2 \|\mu\|^2_{H^{-1}(\mathbb{R}^2)} + \|\omega(\sigma)\|^2_{L^2}.
\end{aligned}$$

Here, the first term is finite by assumption, and the second term is finite since $\omega(\sigma)$ is compactly supported. ☐

The definition of v in Equation (A.2) can be motivated by the following formal calculation, in which, for clarity, we assume that $\sigma = 0$. Starting with Equation (A.2), if it is to be true that $\omega(v) = \mu$, then

$$\begin{aligned}
v = \mathcal{F}^{-1}\left(\frac{\overline{\xi}}{|\xi|^2}\widehat{\mu}\right) &\implies \widehat{v} = \frac{\overline{\xi}}{|\xi|^2}\widehat{\mu} = -i\frac{\xi^\perp}{|\xi|^2}\widehat{\mu} \implies |\xi|^2 \widehat{v} = -i\xi^\perp \widehat{\mu} \\
&\implies \widehat{\Delta v} = \widehat{\nabla^\perp \mu} \\
&\implies \Delta v = \nabla^\perp \mu = (-\partial_2, \partial_1)(\partial_1 v^2 - \partial_2 v^1) \\
&\qquad\quad = (-\partial_1\partial_2 v^2 + \partial_2^2 v^1, \partial_1^2 v^2 - \partial_2\partial_1 v^1) = \Delta v,
\end{aligned}$$

where in the last step we used $\operatorname{div} v = 0$ so $\partial_2 v^2 = -\partial_1 v^1$. Thus, working backwards from the final identity, we deduce, formally, Equation (A.2).

Remark: Given a vector in E_m, all we can immediately say about its vorticity is that it lies in $H^{-1}(\mathbb{R}^2)$ (in particular, it is generically not in any L^p-spaces). Using Theorem A.5 we could define a space, or at least a set, E'_m of all vectors in E_m whose vorticity satisfies the conditions of Theorem A.5. These sets would have the properties that $E'_m \cap E'_{m'} = \emptyset$ when $m \ne m'$, and that if $v = \sigma + \overline{v}$ is a radial-energy decomposition of v, then $\int_{\mathbb{R}^2} \overline{v} = 0$. Only the first property is shared by the spaces E_m.

Remark: We can use Equation (A.2) to define an operator from the space of measures satisfying the conditions of Theorem A.5 to a specific E_m or even to the union of all E_m. In the latter form, we can view it as a complement to the Biot-Savart law which Chemin shows, in Proposition 3.1.1 p. 45 of [1], maps a vorticity in L^a to its associated vector field lying in $L^a + L^b$, where $a < 2$ and $b > 2a/(2 - a)$ (in two dimensions). Our new operator would partially address the case $a = 2$, mapping a measure-valued vorticity satisfying the conditions of Theorem A.5 to a vector in weak-$L^2 + L^2$. It might be interesting to investigate the properties of this operator.

Remark: Let v be in E_m. If $\omega(v)$ is compactly supported and smooth, then it follows from the Biot-Savart law that v decays like $m/((2\pi)r)$. (For instance, see [7] p. 92.) It follows that such a v will be in $L^2(\mathbb{R}^2)$ if and only if $m = 0$; that is, if and only if its total vorticity is zero. Theorem A.5 can be seen as a broad generalization of this result in which the condition of smoothness and compact support of $\omega(v)$ are replaced with much weaker conditions.

The following is Lemma 5.1.2 p. 89 of [1], along with its proof. The proof is an expansion of that in [1].

Lemma A.6. *Let (ρ_n) be an approximation to the identity and σ a stationary vector field in E_m. Then $\sigma - \rho_n * \sigma$ is in L^2 and*

$$\lim_{n\to\infty} \|\sigma - \rho_n * \sigma\|_{L^2} = 0.$$

Proof. Being an approximation to the identity means that $\rho_n(x) = (1+n)^2\rho((1+n)x)$ where ρ is in $\mathcal{S}(\mathbb{R}^2)$ and integrates to 1. We let $\sigma_n = \rho_n * \sigma$.

By the mean value theorem, given $\xi \in \mathbb{R}^2$, there exists an η on the line segment between the origin and $\xi/(n+1)$ (see, for instance, Theorem 8.4 p. 254 of [8]) such that

$$\frac{\widehat{\rho}(\xi/(n+1)) - \widehat{\rho}(0)}{|\xi/(n+1)|} = \widehat{\xi} \cdot \nabla\widehat{\rho}(\eta),$$

where $\widehat{\xi}$ is a unit vector in the direction of ξ. From this it follows that

$$\left|1 - \widehat{\rho}\left(\frac{\xi}{n+1}\right)\right| \le \frac{|\xi|}{1+n} \|D\widehat{\rho}\|_{L^\infty}.$$

Even though σ is not in L^2, we still have

$$\begin{aligned}
\|\widehat{\sigma} - \widehat{\sigma_n}\|_{L^2} &= \left\|\widehat{\sigma} - \widehat{\rho_n * \sigma}\right\|_{L^2} = \|\widehat{\sigma} - \widehat{\rho_n}\widehat{\sigma}\|_{L^2} = \|\widehat{\sigma}(1 - \widehat{\rho_n})\|_{L^2} \\
&= \left\|\widehat{\sigma}(\xi)\left(1 - \widehat{\rho}\left(\frac{\xi}{1+n}\right)\right)\right\|_{L^2} \le \left\|\widehat{\sigma}(\xi)\frac{|\xi|}{1+n}\|D\widehat{\rho}\|_{L^\infty}\right\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\xi\widehat{\sigma}(\xi)\|_{L^2} = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\left\|\widehat{\nabla\sigma}\right\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\nabla\sigma\|_{L^2} \le C\frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\omega(\sigma)\|_{L^2},
\end{aligned}$$

the last inequality being by Theorem 3.1.1 p. 45 of [1]. Letting $n \to 0$ completes the proof.

□

APPENDIX B. MEASURE THEORY

Most of the material in this appendix is adapted from [12] and [13].

Definition B.1. We assume throughout this section that (X, Σ, μ) is a measure space with X the underlying topological space, Σ the σ-algebra of measurable subsets of X, and μ the measure. Any sequence μ_n of measures we will always assume shares the same same underlying space X and the same σ-algebra Σ. We also assume that X is locally compact and that all measures are σ-finite and regular.

(1) $|\mu|$, the total *variation measure* of μ, is defined for $E \in \Sigma$ by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|,$$

the supremum being over all partitions $\{E_i\}$ of E.

(2) A measure is called *finite* if $\mu(X) < \infty$.

(3) A measure is called *bounded* if there exists some $M > 0$ such that $|\mu(E)| \leq M$ for all E in Σ.

(4) Let $\mathcal{M}$ be the set of all bounded Borel measures on X. Then for μ in $\mathcal{M}$, $\|\mu\|_{\mathcal{M}} := |\mu|(X)$ is a norm on $\mathcal{M}$ (this is really a theorem). The norm $\|\mu\|_{\mathcal{M}}$ is also called the *total mass* of μ.

(5) A *signed* or *real* measure is a complex measure that only takes real values (positive, negative, or zero). It is a finite measure, as all complex measures are.

(6) If μ and λ are two signed measures, then $\mu \leq \lambda$ means that for all measurable sets E, $\mu(E) \leq \lambda(E)$.

(7) See Section 3 for the definition of weak convergence of one measure to another.

Theorem B.1 (Section 6.6 p. 119 and Corollary p. 126 of [12])**.** *If μ is a signed measure, then $\mu = \mu^+ - \mu^-$, where*

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Both μ^+ and μ^- are bounded, positive measures. This decomposition is called the Jordan decomposition.

Moreover, if $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \leq \mu^-$. Also,

$$|\mu| \leq \lambda_1 + \lambda_2.$$

Proof. We prove the last statement only, the rest of the proof being in [12]. We have by the first two parts that

$$|\mu| = \mu^+ + \mu^- \leq \lambda_1 + \lambda_2.$$

□

Lemma B.2. *Decompose the measure μ as*

$$\mu = f^+ - f^- + \mu^s,$$

where $f^+ - f^-$ is the Jordan decomposition (see Theorem B.1) of the absolutely continuous part of μ with respect to Lebesgue measure, and μ^s is the singular part with respect to Lebesgue measure. (There is nothing special about Lebesgue measure in this theorem: any other positive measure would do.) If μ^s is nonnegative, then

$$|\mu| = f^+ + f^- + \mu^s.$$

Proof. Let $\mu^a = f^+ - f^-$. Then $\mu = \mu^a + \mu^s$ is the Lebesgue decomposition of μ into an absolutely continuous and a singular part, which are mutually singular; that is, there exists a measurable set A such that for any measurable set E,

$$\mu(E) = \mu^a(E \cap A) + \mu^s(E \cap A^C),$$

where $A^C = X \setminus A$. (See Theorem 6.10 p. 121 of [12], for instance.)

But because $f^+ - f^-$ is the Jordan decomposition of μ^a, by the Hahn decomposition theorem (Theorem 6.14 p. 125 of [12]) there exists a measurable set B—lying in A—such that

$$f^+(E) = \mu(E \cap B) \quad \text{and} \quad f^-(E) = \mu(E \cap (A \setminus B)).$$

Let $C = B \cup A^C$ and let $\lambda^+ = f^+ + \mu^s$, $\lambda^- = f^-$. It follows that

$$\lambda^+(E) = \mu(E \cap C) \quad \text{and} \quad \lambda^-(E) = \mu(E \cap C^C).$$

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ, meaning, again by the Hahn decomposition theorem, that there exists a measurable set D such that

$$\mu^+(E) = \mu(E \cap D) \quad \text{and} \quad \mu^-(E) = \mu(E \cap D^C).$$

Then,

$$\begin{aligned} \mu \le \lambda^+ &\Rightarrow \mu^+(E) = \mu(E \cap D) \le \lambda^+(E \cap D) \le \lambda^+(E) \\ \mu \le \mu^+ &\Rightarrow \lambda^+(E) = \mu(E \cap C) \le \mu^+(E \cap C) \le \mu^+(E), \end{aligned}$$

so $\mu^+ = \lambda^+$. A similar argument give $\mu^- = \lambda^-$. But then

$$|\mu| = \mu^+ + \mu^- = f^+ + f^- + \mu^s.$$

□

We switch between viewing a bounded measure μ on a subset Ω of R^d as a measure in the usual sense and as a distribution in $\mathcal{D}'(\Omega)$ in the sense that if φ is in $\mathcal{D}(\Omega)$ (a continuous compactly supported smooth function on Ω), then

$$\varphi \mapsto \int_\Omega \varphi \, d\mu$$

defines a linear functional on $\mathcal{D}(\Omega)$, and so a distribution in $\mathcal{D}'(\Omega)$. (See Section 6.11 p. 157 of [13].) (All bounded measures are distributions, but not all distributions are bounded measures.)

We also switch in the usual way between viewing μ as a distribution and as a regular function (when it is, indeed, a regular function), and between viewing μ as a bounded measure and a regular function. We will not always comment on our shifts of view.

Theorem B.3 (Theorem 6.13 p. 125 of [12]). *Suppose μ is a positive measure and f is in $L^1(\mu)$. If the measure λ is defined for any measurable set E by*

$$\lambda(E) = \int_E f \, d\mu,$$

then

$$|\lambda| (E) = \int_E |f| \, d\mu.$$

Corollary B.4. *Let m be Lebesgue measure on $\mathbb{R}^d$ and suppose that f is in $L^1(\mathbb{R}^d)$. Then*

$$\|f\|_{\mathcal{M}} = \|f\|_{L^1} ,$$

viewing f as a measure and as a function.

Proof. Apply Theorem B.3 with μ being Lebesgue measure and E being all of $\mathbb{R}^d$. Then

$$\|f\|_{\mathcal{M}} = |f| (\mathbb{R}^d) = \int_{\mathbb{R}^d} |f| \, d\mu = \|f\|_{L^1} .$$

□

Theorem B.5 (Exercise 5(a) and 5(b) p. 175 of [12]). *Let μ and λ be two measures in $\mathcal{M}$. Then $\mu * \lambda$ can be given a meaning that coincides with the usual meaning of convolution when μ and λ are in L^1, and*

$$\|\mu * \lambda\|_{\mathcal{M}} \leq \|\mu\|_{\mathcal{M}} \|\lambda\|_{\mathcal{M}} .$$

Comment: When one or the other of the measures in Theorem B.5 is absolutely continuous with respect to Lebesgue measure, we can replace the measure norm for that measure with the L^1 norm. This follows from Corollary B.4. Also, in this case, which is the most general case we need, the inequality follows easily from Hölder's inequality on an abstract measure space, so we have made life more complicated than necessary.

Comment: When one or the other of the measures in Theorem B.5 is smooth, then so is $\mu * \lambda$ (as we can see from our distribution view of the measures), so both the smooth measure and $\mu * \lambda$ can have its norm replaced by the L^1-norm.

Lemma B.6. *Let* $\{\mu_n\}$ *be a bounded sequence of positive measures,* μ *a positive measure, with* $\mu_n \rightharpoonup \mu$*, let* K *be a compact subset of* X*, and let* U *be an open subset of* X*. Then*

$$\mu(U) \leq \liminf_{n\to\infty} \mu_n(U),$$

and

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K).$$

Proof. To prove[14] the first inequality, let U be open and choose a compact K in X. By Urysohn's lemma, there is a continuous function supported in U taking values in $[0,1]$ and identically equal to 1 on K. Then

$$\mu(K) \leq \int f\, d\mu = \lim_{n\to\infty} \int f\, d\mu_n \leq \liminf_{n\to\infty} \mu_n(U).$$

Thus,

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\, K \text{ compact}\} \leq \liminf_{n\to\infty} \mu_n(U).$$

To prove the second inequality, let K be compact and choose an open set U containing it. Define the function f as before. Then

$$\mu(U) \geq \int f\, d\mu = \lim_{n\to\infty} \int f\, d\mu_n \geq \limsup_{n\to\infty} \mu_n(K).$$

Thus,

$$\mu(K) = \inf\{\mu(U) : K \subseteq U,\, U \text{ open}\} \geq \limsup_{n\to\infty} \mu_n(K).$$

□

Comment: More is proved in [11]. In fact, weak convergence in the sense of integrating against compactly supported test functions and the two inequalities in Lemma B.6 are shown to be equivalent, as is the statement that $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for every bounded Borel set with $\mu(\partial B) = 0$. And this is all proved in the setting of Radon measures, which are Borel regular measures for which the measure of every compact set is finite.

Lemma B.7. *Let* $\{\mu_n\}$ *and* $\{\lambda_n\}$ *be sequences of bounded positive measures converging weakly to the bounded positive measures* μ *and* λ*, respectively. Then:*

(1) *If* $\mu_n(E) \leq \alpha$ *for all* n *and for one fixed measurable set* E*, then* $\mu(E) \leq \alpha$*.*

---

[14] This proof is from the proof of Theorem 1.9.1 p. 54 of [11].

(2) *If* $\mu_n(E) \leq \lambda_n(E)$ *for all* n *and for one fixed measurable set* E*, then* $\mu(E) \leq \lambda(E)$*.*
(3) *If* $\mu_n(E) \leq \lambda_n(E)$ *for all measurable sets* E*, or even just for all compact sets* E*, then* $\mu \leq \lambda$*.*

Proof. Since (1) is a special case of (2), we need only prove (2) and (3). To prove (2), assume first that E is compact. Then by Lemma B.6, observing that $\lambda_n - \mu_n \geq 0$,

$$(\lambda - \mu)(E) \geq \limsup(\lambda_n - \mu_n)(E) \geq 0.$$

Since we assumed all our measures are regular, for an arbitrary measurable set it follows that

$$(\lambda - \mu)(E) = \sup\{(\lambda - \mu)(K) : K \subseteq E,\, K \text{ compact}\} \geq 0.$$

Applying (2) to every measurable set, (3) follows immediately. ☐

Appendix C. Other Useful Results

We collect here the statements of results from fluids, as well as standard theorems of analysis, for easy reference.

Theorem C.1 (Theorem 4.2.4 p. 82 of [1]). *Let m and r be real numbers with $r > 1$. Let v_0 be a divergence-free vector field in $E_m \cap C^r$. Then there exists a unique global strong solution (v, p) of the Euler equations lying in*

$$L^\infty_{loc}(\mathbb{R}; C^r) \cap C(\mathbb{R}; E_m) \times L^\infty_{loc}(\mathbb{R}; H^1 \cap C^{r+1}).$$

Theorem C.2 (Equation (4.33) p. 83 of [1]). [15] *Let m and v be as in Theorem C.1, and σ any stationary solution in E_m. Then*

$$\|v(t) - \sigma\|^2_{L^2} \le \|v_0 - \sigma\|^2_{l^2} \, e^{2t\|\nabla\sigma\|_{L^\infty}}.$$

Theorem C.3 (Proposition 2.2.1 p. 20 of [1]). *$H^s(\mathbb{R}^d)$ is dual to $H^{-s}(\mathbb{R}^d)$ for all real s, in the sense that the map*

$$\varphi \mapsto (u \mapsto \langle u, \overline{\varphi}\rangle),$$

is a conjugate-linear bijective isometry from the dual of $H^s(\mathbb{R}^d)$ to $H^{-s}(\mathbb{R}^d)$. (Since all our functions are real, we truly do have the dual.)

Theorem C.4 (Corollary 1.2.2 p. 7 of [1]). *Let w be a divergence-free vector field in $\mathbb{R}^2$ whose coefficients are tempered distributions. Then there exists a tempered distribution f such thatl*

$$w = \nabla^\perp f := (-\partial_2 f, \partial_1 f).$$

Theorem C.5 (p. 44 of p [1]). *Let $1 \le a < 2$, and assume that the vorticity is in $L^a(\mathbb{R}^2)$. Then the vector field whose components are given by*

TODO: Check whether you are off by a sign.

$$v^i(x) = \frac{(-1)^{m-1}}{2\pi} \sum_m \int_{\mathbb{R}^2} \frac{x^m - y^m}{|x - y|^2} \omega(y)\, dy$$

is divergence-free and has vorticity ω.

More can be said. Let b be a real number such that

$$b > \frac{2a}{2 - a}.$$

Then the vector v given above lies in $L^a(\mathbb{R}^2) + L^b(\mathbb{R}^2)$ and can be uniquely decomposed into a sum $v = v_1 + v_2$ with v_1 in $L^a(\mathbb{R}^2)$ and v_2 in $L^b(\mathbb{R}^2)$. Also, there exists a constant C depending only on a and b such that

$$\|v_1\|_{L^a} + \|v_2\|_{L^b} \le C \|\omega\|_{L^a}.$$

---

[15] Chemin does not have a 2 in the exponent of this estimate; however, in working through his calculations, it seems that he is missing a factor of 2. The 2 or the lack of it is immaterial for us, in any case.

TODO: Include the proof.

Theorem C.6 (Lemma 3.5(i) p. 131 of [7]). *Let* v *be a bounded continuous function on* $\mathbb{R}^2$. *Then* $\rho_n * v \to v$ *uniformly on any compact subset of* $\mathbb{R}^2$.

Theorem C.7 (Theorem 6.32(b) p. 173 of [13]). *Let* ρ_n *be an approximation to the identity on* $\mathbb{R}^d$. *Then for any distribution* u *in* $\mathcal{D}'(\mathbb{R}^d)$,

$$\rho_n * u \rightharpoonup u.$$

Theorem C.8 (Theorem 9.6 p. 148 of [4]). *Let* ρ *be in* $L^1(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} \rho = 1$, *and define*

$$\rho_n(x) = (n+1)^d \rho((n+1)x).$$

If f *is in* $L^p(\mathbb{R}^d)$ *then for* $1 \leq p < \infty$,

$$\|\rho_n * f - f\|_{L^p(\mathbb{R}^d)} \to 0 \quad \textit{as} \quad n \to \infty.$$

Theorem C.9 (Banach-Alaoglu, p. 68 of [13]). *If* V *is a neighborhood of 0 in a topological vector space* X *and if*

$$K = \{\Lambda \in X^* : |\Lambda x| \leq 1 \, \textit{for all} \, x \in V\},$$

then K *is weak-* compact.*

Proof. See [9] p. 49, [13] p. 68-69, or [16] p. 22. □

Corollary C.10. *If* $\{x_n\}$ *is a bounded sequence in a topological vector space, then it contains a weakly convergent subsequence.*

Theorem C.11 (Theorem 3.12 p. 66 of [13]). *Let* E *be a convex subset of a locally convex space. Then the closure of* E *in the weak-* topology is the same as the closure in the strong topology.*

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