3D Euler equations with inflow, outflow

Jim Kelliher
University of California Riverside

in collaboration with

Gung-Min Gie
University of Louisville

&

Anna Mazzucato
Penn State University

University of California San Diego

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Euler equations

3D Euler equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega \end{cases}$$

- $\Omega \subset \mathbb{R}^3$, bounded domain
- $T > 0$ to-be-determined
- $Q = (0, T) \times \Omega$
- $\operatorname{div} \mathbf{f} = 0$

Classically:

$$\mathbf{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma := \partial\Omega.$$

Classical existence a la McGrath/Kato

Here is the approach of McGrath [1966, 1967] and Tosio Kato [1967] [McGrath was Kato's student] to proving existence of Euler on all of $\mathbb{R}^2$ or $\mathbb{R}^3$:

Define an operator A mapping a $C^{1,\alpha}$ velocity field $\mathbf{u}$ to the $C^{1,\alpha}$ velocity field $A\mathbf{u}$:

1. Pushforward (transport in 2D) $\boldsymbol{\omega}_0 := \operatorname{curl} \mathbf{u}(0)$ by the flow map for $\mathbf{u}$, giving $\boldsymbol{\omega}$.
   - Equivalently, solve the linear problem, $\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \mathbf{g}$, $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$, $\mathbf{g} := \operatorname{curl} \mathbf{f}$, which comes from the vorticity equation.
   - Turns out that $\boldsymbol{\omega}(t)$ is a curl for all time because $\operatorname{div} \boldsymbol{\omega}$ is transported by the flow.
2. Let $A\mathbf{u}$ be the unique divergence-free vector field whose curl is $\boldsymbol{\omega}$.

Show that A has a fixed point using Schauder's fixed point theorem and that the fixed point solves the Euler equations:

- Show the nonlinear operator A is bounded in $C^{1,\alpha}$
- Show A is continuous in a slightly weaker norm (interpolation helps a lot).

Bootstrapping can give higher regularity, taking advantage of the symmetry of the Biot-Savart kernel in the whole space (Majda approach).

No-penetration boundary conditions

Koch [2002], following same scheme, obtains $C^{1,\alpha}$ solutions in a 2D/3D bounded domain with no-penetration conditions:

- Uses pushforward of the velocity field as a 1-form to define A, though uses pushforward of the vorticity as vector field in obtaining bounds on A.

- In ❷ $A\mathbf{u}$ is the unique divergence-free vector field whose curl is $\boldsymbol{\omega}$—and for which $A\mathbf{u} \cdot \boldsymbol{n} = 0$ on Γ.

Koch obtains higher regularity for 2D only, as the 3D Biot-Savart kernel in a bounded domain has insufficient symmetry to take the full space approach.

Inflow, outflow in brief

Replace the classical no-penetration condition with

$$\begin{cases} u^{\boldsymbol{n}} = \mathcal{U}^{\boldsymbol{n}} & \text{on } \Gamma, \\ \mathbf{u}^{\boldsymbol{\tau}} = \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}} & \text{on } \Gamma_+, \end{cases}$$

where Γ_+ is inflow portion of the boundary on which $\mathcal{U}^{\boldsymbol{n}} < 0$ (more details soon).

Here, $\boldsymbol{\mathcal{U}}$ is a given divergence-free "background flow."

Key difficulty: The flow (pushforward) of vorticity from the inflow boundary meets flow of vorticity from time zero:

> *The two flows must meet seamlessly enough to insure regularity. Moreover, this has to be done at the level of the linear solution so that the operator A has sufficient regularity to make the convergence argument. Requires compatibility conditions on the initial data, $\mathcal{U}^{\boldsymbol{n}}$, $\boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}}$, and forcing.*

Serious difficulty: Higher regularity cannot be obtained by bootstrapping. Instead, have A operate directly on $C^{N+1,\alpha}$ vector fields, $N \geq 0$.

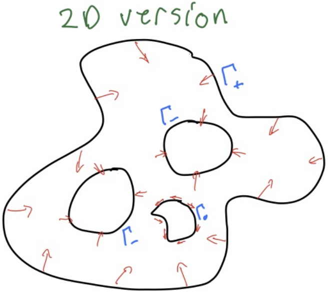
Another difficulty: The pressure boundary conditions yield one less derivative of regularity than in the classical case, yet the pressure is involved in generating the vorticity on the inflow boundary.

The setup for inflow, outflow

Fix $\alpha \in (0,1)$, assume $\partial\Omega$ is $C^{2,\alpha}$:

- $\partial\Omega = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$ (components only)
- Inflow on Γ_+: $\mathbf{u} \cdot \boldsymbol{n} < 0$
- Outflow on Γ_-: $\mathbf{u} \cdot \boldsymbol{n} > 0$
- Characteristic on Γ_0: $\mathbf{u} \cdot \boldsymbol{n} = 0$
- fix $\boldsymbol{\mathcal{U}} \in C^{2,\alpha}(Q)$ divergence-free and set

$$\begin{aligned} \mathbf{u}^{\boldsymbol{n}} &= \mathcal{U}^{\boldsymbol{n}} && \text{on } \Gamma, \\ \mathbf{u} &= \boldsymbol{\mathcal{U}} && \text{on } \Gamma_+ \end{aligned}$$



$$\operatorname{div} \boldsymbol{\mathcal{U}} = 0 \implies \int_{\Gamma_+} \mathcal{U}^{\boldsymbol{n}} = -\int_{\Gamma_-} \mathcal{U}^{\boldsymbol{n}}$$

What has been known

In three papers from 1980 by Kažihov and Ragulin,

with a combined version appearing in Chapter 4 of a 1983 (English translation 1990) text by Antontsev, Kazhikhov, and Monakhov [AKM],

obtained local-in-time well-posedness of 3D Euler equations with inflow, outflow.

This result was for $C^{1,\alpha}$ vector fields, and had a compatibility condition on the data.

They followed the McGrath/Kato plan.

They assumed a simply connected domain.

We address:

- What compatibility conditions are required for higher regularity, and how to obtain that regularity.
- Multiply connected domains.

Our main result

Define the affine spaces,

$$C_\sigma^{N+1,\alpha}(\Omega) := \left\{ \mathbf{u} \in C^{N+1,\alpha}(\Omega) \colon \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{n} = \mathcal{U}^{\boldsymbol{n}} \text{ on } \partial\Omega \right\},$$

$$C_\sigma^{N+1,\alpha}(Q) := \left\{ \mathbf{u} \in C^{N+1,\alpha}(Q) \colon \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{n} = \mathcal{U}^{\boldsymbol{n}} \text{ on } [0,T] \times \Gamma \right\}.$$

We say that the data has regularity N for an integer $N \geq 0$ if [$Q_\infty := [0,\infty) \times \Omega)$]

- Γ is $C^{N+2,\alpha}$, $\boldsymbol{\mathcal{U}} \in C_\sigma^{N+2,\alpha}(Q_\infty)$, $\mathbf{f} \in C^{\max\{N,1\},\alpha}(Q_\infty)$;
- $\mathbf{u}_0 \in C_\sigma^{N+1,\alpha}(\Omega)$, $\mathbf{u}_0^{\boldsymbol{\tau}} = \boldsymbol{\mathcal{U}}_0^{\boldsymbol{\tau}}$ on Γ_+.

Theorem (Gie, K, Mazzucato 2022)

Assume that the data has regularity N for some integer $N \geq 0$ and satisfies compatibility condition N (cond$_N$, below). There is a $T > 0$ such that there exists a solution $(\mathbf{u}, p)$ to the 3D Euler equations with inflow, outflow with $(\mathbf{u}, \nabla p) \in C_\sigma^{N+1,\alpha}(Q) \times C^{N,\alpha}(Q)$, which is unique up to an additive constant for the pressure.

The $N = 0$ result (for simply connected domains) is [AKM].

The key difficulty

The proof of our main result is long and technical, spread over two papers of some length. So I will focus on trying to give some idea of how we handle the key difficulty of insuring that the vorticity coming from the inflow boundary and the initial time meet seamlessly.

At the lowest level of regularity, ($N = 0$) as in [AKM], the issue if much the same as insuring that, using the method of characteristics, one obtains a weak solution to a first-order PDE. In fact, the $N = 0$ condition is the same as the Rankine-Hugoniot condition that insures a weak solution.

Higher regularity is not quite the same, however.

First, we need to explore how vorticity is generated on the inflow boundary.

Vorticity generation on inflow

We start with the Gromeka-Lamb form of the Euler equations,

$$\partial_t \mathbf{u} + \nabla \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \boldsymbol{\omega} - \mathbf{f} = 0.$$

Using the identity

$$[\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}} [\boldsymbol{\omega}^{\boldsymbol{\tau}}]^{\perp} - \omega^{\boldsymbol{n}} [\mathbf{u}^{\boldsymbol{\tau}}]^{\perp},$$

yields

$$u^{\boldsymbol{n}} \boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) + \mathbf{f} \right]^{\perp} - \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \qquad \omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}.$$

Here, $\mathbf{v}^{\perp} = \boldsymbol{n} \times \mathbf{v}$ is the tangential vector field $\mathbf{v}$ on Γ rotated 90 degrees counterclockwise around the normal vector $\boldsymbol{n}$ when viewed from outside Ω, ∇_Γ is the tangential derivative, and $\operatorname{curl}_\Gamma$ is the curl operator on the boundary.

Since $u^{\boldsymbol{n}} = \mathcal{U}^{\boldsymbol{n}} < 0$ on Γ_+, we have on Γ_+

$$\begin{aligned} \boldsymbol{\omega}^{\boldsymbol{\tau}} &:= \frac{1}{\mathcal{U}^{\boldsymbol{n}}} \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) + \mathbf{f} \right]^{\perp} - \frac{1}{u^{\boldsymbol{n}}} \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \\ \omega^{\boldsymbol{n}} &:= \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}. \end{aligned}$$

Vorticity generation and the linear problem

We have a formula for vorticity generation, but what do we do with it?

First, if $\mathbf{u}$ is a fixed point of A—and solves 3D Euler with inflow, outflow—then

$$\begin{aligned}\boldsymbol{\omega}^{\boldsymbol{\tau}} &:= \frac{1}{\mathcal{U}^{\boldsymbol{n}}}\left[-\partial_t \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(p + \frac{1}{2}|\boldsymbol{\mathcal{U}}|^2\right) + \mathbf{f}\right]^\perp - \frac{1}{u^{\boldsymbol{n}}}\operatorname{curl}_\Gamma \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}}\boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}}, \\ \omega^{\boldsymbol{n}} &:= \operatorname{curl}_\Gamma \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}}.\end{aligned}$$

But in solving the linear problem that defines the operator A, we can only specify that $\mathbf{u}^{\boldsymbol{n}} = \boldsymbol{\mathcal{U}}^{\boldsymbol{n}}$ on Γ : specifying the value of the tangential component would make the problem of recovering the velocity from the vorticity overdetermined.

This is good and bad:

- Good, because we don't need to worry about trying to do so for the linear problem.
- Bad, because we need to show that for a fixed point of A, the full boundary conditions are satisfied.

We will focus on the good! And the bad can be dealt with as in [AKM].

Vorticity and the flow map

Flow map for $\mathbf{u}$:

$\eta(t_1, t_2; \mathbf{x})$ = the position that a particle at $\mathbf{x} \in \overline{\Omega}$ at time t_1 will be as it moves, forward or backward, to time t_2:

$$(t_1, \mathbf{x}) \mapsto (t_2, \eta(t_1, t_2; \mathbf{x}))$$

Taking curl of the Euler equations (with $\mathbf{g} := \operatorname{curl} \mathbf{f}$):

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \mathbf{g},$$

Vorticity is transported by flow map in 2D (and stretched in 3D)

For impermeable boundary conditions, flow lines start and remain within $\overline{\Omega}$, so $\omega(t, \mathbf{x}) := \omega_0(\eta(t, 0; \mathbf{x}))$ is a Lagrangian solution in 2D (for $\mathbf{g} \equiv 0$)

But if $\mathbf{u}(x) \cdot \boldsymbol{n} < 0$:

- The flow (somehow) brings vorticity into the domain
- Once in Ω, it is transported in the usual way, until it perhaps exits the domain
- Transport of initial vorticity and transport of vorticity generated on the boundary must meet seamlessly to obtain any regularity (even continuity)

The linear problem in more detail

We will analyze the slightly more general situation:

$$\begin{cases} \partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u} + \mathbf{g} & \text{in } Q, \\ \mathbf{Y} = \mathbf{H} & \text{on } [0, T] \times \Gamma_+, \\ \mathbf{Y}(0) = \mathbf{Y}_0 & \text{on } \Omega. \end{cases}$$

Here, $\mathbf{u}$ is a given divergence-free vector field and $\mathbf{H}$ is a given inflow value.

If $\mathbf{Y}_0 = \operatorname{curl} \mathbf{v}$ for some divergence-free vector field $\mathbf{v}$ ("$\mathbf{Y}_0$ is in the range of the curl") we would like that $\mathbf{Y}(t)$ remains in the range of the curl for all time. This is not, in general, the case—it requires an additional condition on $\mathbf{H}$.

Were $u^{\boldsymbol{n}} = 0$ the flow lines would stay within Ω, and the solution (for $\mathbf{g} \equiv 0$) would be

$$\mathbf{Y}(t, \mathbf{x}) = \nabla \eta(0, t; \eta(t, 0; \mathbf{x}))\mathbf{Y}_0(\eta(t, 0; \mathbf{x})).$$

This is the pushforward of $\mathbf{Y}_0$ by the flow map.

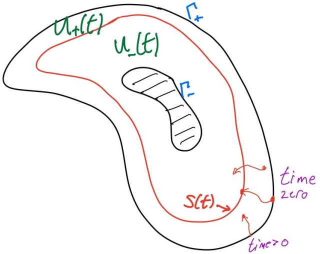
Two issues:

- Fundamental: Collision of flow lines from Γ_+ and time zero
- Technical: If $N = 0$ regularity need to treat $\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u} + \mathbf{g}$ weakly

Partition of $Q := [0, T] \times \Omega$ by the flow map

Recall: $(t_1, \mathbf{x}) \mapsto (t_2, \eta(t_1, t_2; \mathbf{x}))$

- U_- = points in Q whose flow line originated inside Ω
- U_+ = points in Q with vorticity generated on Γ_+:
  - $\gamma(t, \mathbf{x})$ = *point on* Γ_+ *at which the flow line through* $(t, \mathbf{x})$ *hits* Γ_+
  - $\tau(t, \mathbf{x})$ = *time that intersection occurs*
- $S := \{(t, \mathbf{x}) \in Q : \tau(t, \mathbf{x}) = 0\}$ hypersurface
- $S(t), U_-(t), U_+(t)$ sections
- Note $S(t) = \eta(0, t; \Gamma_+)$—$\Gamma_+$ transported by flow



Fact: γ, τ, and the hypersurface S inherit the regularity of the flow

Let $\mathbf{Y}_0 \in C^\alpha(\Omega)$ and $\mathbf{H} \in C^\alpha([0, T] \times \Gamma_+)$. Define the pushforward of $\mathbf{Y}_0$ by η on Q with boundary value $\mathbf{H}$ by

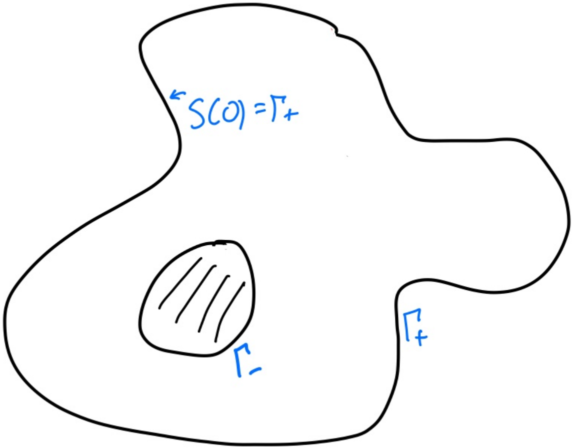
$$\mathbf{Y}(t, \mathbf{x}) := \begin{cases} \nabla\eta(0, t; \eta(t, 0; \mathbf{x}))\mathbf{Y}_0(\eta(t, 0; \mathbf{x})) & \text{on } U_-, \\ \nabla\eta(\tau(t, \mathbf{x}), t; \eta(t, \tau(t, \mathbf{x}); \mathbf{x}))\mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) & \text{on } U_+. \end{cases}$$

This gives the Lagrangian form of the solution, $\mathbf{Y}$, without forcing.

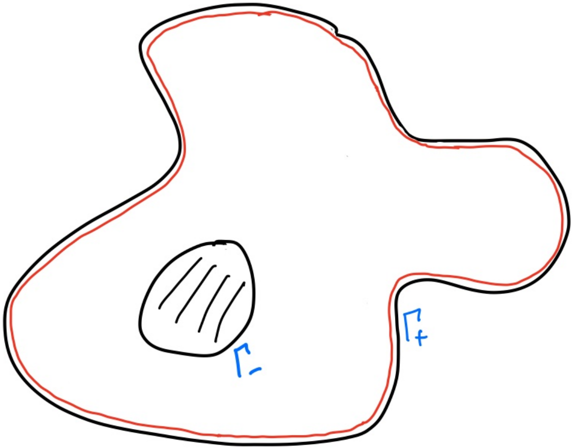
We remain silent on the value of $\mathbf{Y}$ along S. Even if $\mathbf{Y}_\pm$ will not be continuous along S, we still call $\mathbf{Y}$ a Lagrangian solution.

$\mathbf{Y}$ will not, however, yield an Eulerian solution on Q unless it has enough regularity along S.

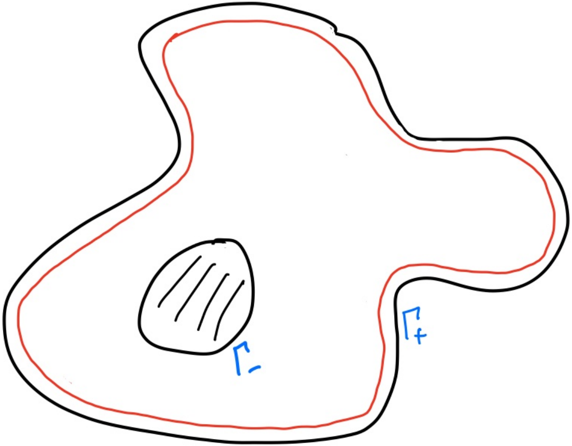
$N = 0$ regularity in pictures



$N = 0$ regularity in pictures



$N = 0$ regularity in pictures



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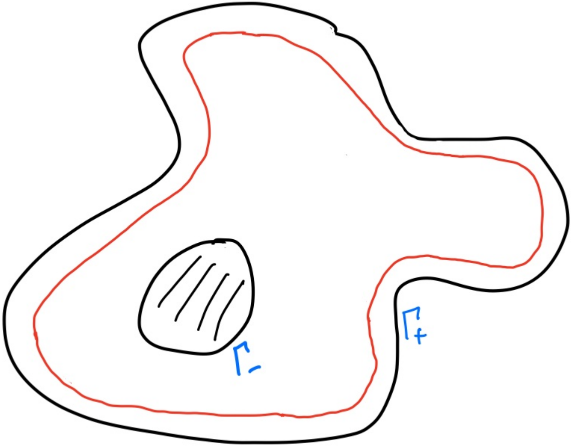
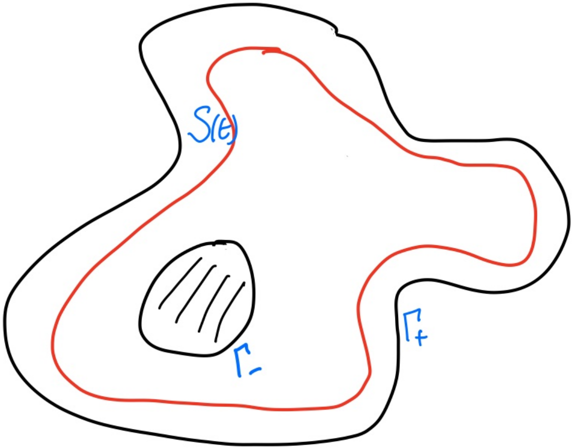


$N = 0$ regularity in pictures



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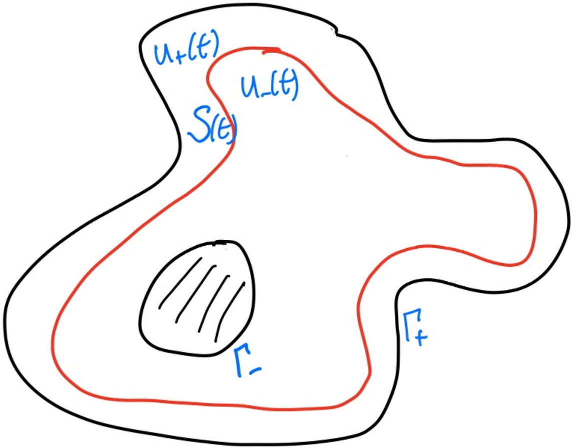

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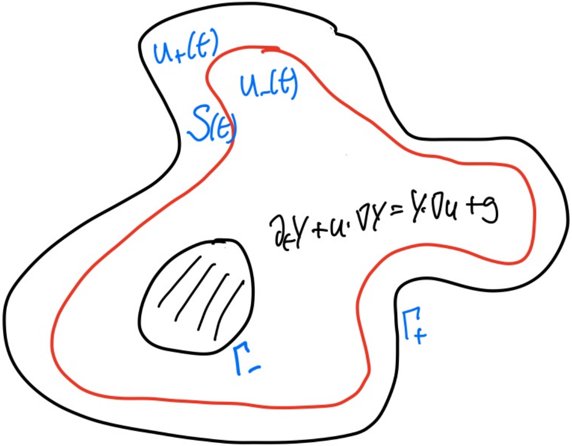

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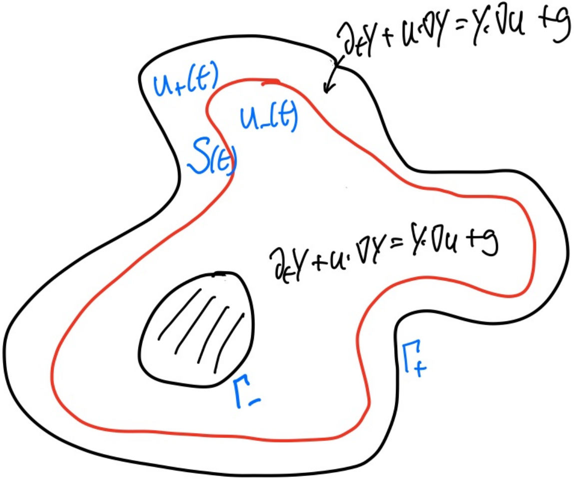

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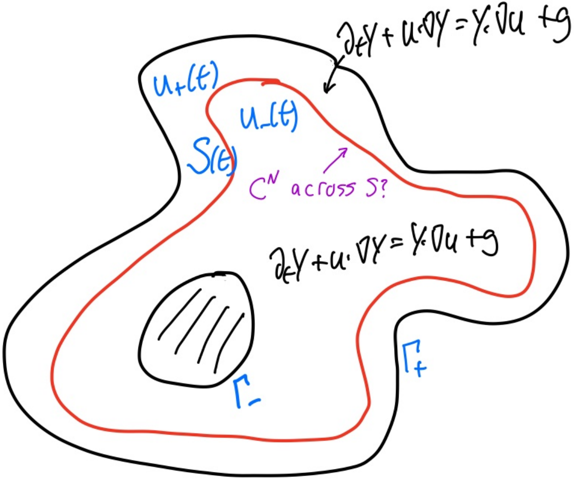
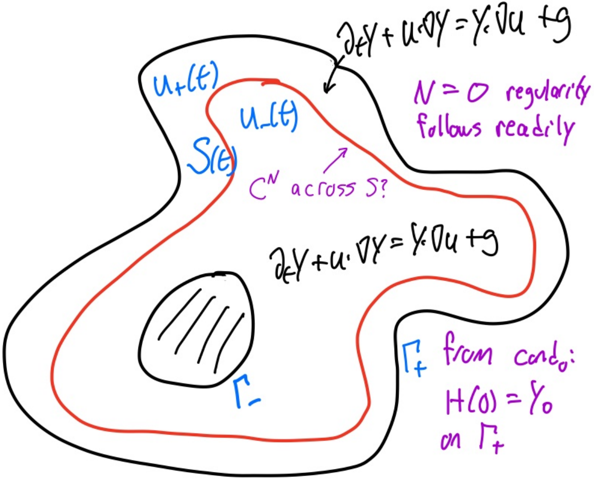


$N = 0$ regularity in pictures



Bootstrapping at the linear level

Let $N = 1$: so we have $C^{2,\alpha}$ regularity of $\mathbf{u}$ and seek $C^{1,\alpha}$ regularity of $\mathbf{Y}$.

It can be readily shown that $\mathbf{Y} \in C^{1,\alpha}(Q)$ *is* an Eulerian solution on $U_\pm$.

Then $\mathbf{Z} := \partial_t \mathbf{Y} \in C^\alpha(U_\pm)$ is an Eulerian solution on $U_\pm$ to

$$\begin{cases} \partial_t \mathbf{Z} + \mathbf{u} \cdot \nabla \mathbf{Z} - \mathbf{Z} \cdot \nabla \mathbf{u} = \mathbf{h} := \partial_t \mathbf{g} - \partial_t \mathbf{u} \cdot \nabla \mathbf{Y} + \mathbf{Y} \cdot \nabla \partial_t \mathbf{u}, \\ \mathbf{Z}(0) := \partial_t \mathbf{Y}(0). \end{cases}$$

Equality holds on $U_\pm$, so, in fact, $\mathbf{Z}$ is the Lagrangian solution with forcing $\mathbf{h} \in C^\alpha(U_\pm)$, initial value $\partial_t \mathbf{Y}(0)$, and $\mathbf{Z} = \partial_t \mathbf{H}$ on $[0, T] \times \Gamma_+$.

But, we only have $h \in C^{\alpha-1}(Q)$, a negative Hölder space. If we could handle such weak forcing, we could bootstrap, if we defined cond_1 to be

$$(\text{cond}_1): \qquad \text{cond}_0 \text{ and } \partial_t \mathbf{H}|_{t=0} = \partial_t \mathbf{Y}|_{t=0} := \mathbf{g}(0) - \mathbf{u}_0 \cdot \nabla \mathbf{Y}_0 + \mathbf{Y}_0 \cdot \nabla \mathbf{u}_0 \text{ on } \Gamma_+.$$

But we can't handle negative Hölder space forcing (this is why we assumed that $\mathbf{f}$ is $C^{1,\alpha}$ for $N = 0$) so we must treat $N = 1$ explicitly.

But $N \geq 2$ reduces to $N-1$ case

Such bootstrapping does work for $N \geq 2$, however, and leads to

$$(\mathrm{cond}_N): \qquad \mathrm{cond}_{N-1} \text{ and } \partial_t^N \mathbf{H}|_{t=0} = \text{“}\partial_t^N \mathbf{Y}|_{t=0}\text{”} \text{ on } \Gamma_+.$$

For the regularity of the spatial derivatives $\partial_j \mathbf{Y}$, the main story is that

$$\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u} + \mathbf{g} \in C^{N-1,\alpha}$$

holds from the $N-1$ result, so once $\partial_t \mathbf{Y} \in C^{N-1,\alpha}$ is established, we know that $\mathbf{u} \cdot \nabla \mathbf{Y} \in C^{N-1,\alpha}$.

But one can show that $\mathbf{u}(t)$ remains transversal to $S(t)$ for at least a short time, and we can obtain $\nabla \mathbf{Y} \in C^{N-1,\alpha}$.

This kind of argument is good to prove sufficiency of cond_N, but necessity requires a more direct analysis of the spatial derivatives as well.

Lagrangian solution revisited

Recall: We define the pushforward of $\mathbf{Y}_0$ by η on Q with boundary value $\mathbf{H}$ by

$$\mathbf{Y}(t,\mathbf{x}) := \begin{cases} \nabla\eta(0,t;\eta(t,0;\mathbf{x}))\mathbf{Y}_0(\eta(t,0;\mathbf{x})) & \text{on } U_-, \\ \nabla\eta(\tau(t,\mathbf{x}),t;\eta(t,\tau(t,\mathbf{x});\mathbf{x}))\mathbf{H}(\tau(t,\mathbf{x}),\gamma(t,\mathbf{x})) & \text{on } U_+, \end{cases}$$

or, with $G_\pm$ coming from forcing via Duhamel's principle,

$$\mathbf{Y} := \begin{cases} \mathbf{Y}_- := B_-\mathbf{Y}_0(\gamma_0) + G_- & \text{on } U_-, \\ \mathbf{Y}_+ := B_+\mathbf{H}(\tau,\gamma) + G_+ & \text{on } U_+, \end{cases}$$

where

$$\begin{aligned} \gamma_0 &= \gamma_0(t,\mathbf{x}) := \eta(t,0;\mathbf{x}) \text{ on } \overline{U}_-, \\ B_- &= B_-(t,\mathbf{x}) := \nabla\eta(0,t;\eta(t,0;\mathbf{x})) = \nabla\eta(0,t;\gamma_0) \text{ on } \overline{U}_-, \\ B_+ &= B_+(t,\mathbf{x}) := \nabla\eta(\tau,t;\eta(t,\tau;\mathbf{x})) = \nabla\eta(\tau,t;\gamma) \text{ on } \overline{U}_+ \setminus \{0\} \times \Gamma_+. \end{aligned}$$

On S, $\tau = 0$ and $B_- = B_+$.

For $N = 0$ solutions, the pushforward terms as well as $G_\pm$ are each continuous across S.

For $N \geq 1$ solutions, all derivatives are separately discontinuous across S, though derivatives of their sums, $\mathbf{Y}_\pm$, are continuous (up to N^{th} order).

$N = 1$ regularity

What we need is to establish C^1 regularity across the hypersurface S:

Proposition

Assume the data has $N = 1$ regularity and cond_0 *holds. Let $\mathbf{Y}$ be the Lagrangian solution. Then $D\mathbf{Y}_- = D\mathbf{Y}_+$ on S if and only if* cond_1 *holds, where $D := (\partial_t, \nabla)$.*

Recall:

$$(\mathrm{cond}_1): \qquad \mathrm{cond}_0 \text{ and } \partial_t \mathbf{H}|_{t=0} = \partial_t \mathbf{Y}|_{t=0} := \mathbf{g}(0) - \mathbf{u}_0 \cdot \nabla \mathbf{Y}_0 + \mathbf{Y}_0 \cdot \nabla \mathbf{u}_0 \text{ on } \Gamma_+.$$

Note that $D\mathbf{Y}_\pm$ is a 3×4 matrix.

This can be proved by a long, direct, brute-force calculation. But a little finesse goes a long way.

What makes the direct approach involved is that $\mathbf{Y}_\pm$ have distinct domains (except for S), so separate calculations must be made for each, then restricted to S.

Extending $\mathbf{Y}_-$

Instead, extend $\mathbf{Y}_0$ to all of $\mathbb{R}^3$ and $\mathbf{u}$ and $\mathbf{g}$ to all of $[0, \infty) \times \mathbb{R}^3$. This extends the flow map η, and hence $B_- = \nabla\eta(0, t; \eta(t, 0; \mathbf{x}))$, along with G_-. So

$$\mathbf{Y}_- := B_- \mathbf{Y}_0(\gamma_0) + G_-$$

makes sense and has the desired regularity on all of $[0, \infty) \times \mathbb{R}^3$, and, in particular, on $\overline{U}_+$, which includes the domain of $\mathbf{Y}_+$.

The $N = 1$ argument

The calculation is still too involved to give in full. I will try to just give the flavor of it.

First, let $\mathbf{Z}_\pm := \mathbf{Y}_\pm - \mathbf{G}_\pm$. Then

$$B_-^{-1}\mathbf{Z}_-(t, \mathbf{x}) = \mathbf{Y}_0(\gamma_0), \quad B_+^{-1}\mathbf{Z}_+(t, \mathbf{x}) = \mathbf{H}(\tau, \gamma).$$

A calculation gives

$$D(\mathbf{G}_- - \mathbf{G}_+)(t, \mathbf{x}) = B_-\mathbf{g}(0, \gamma_0) \otimes D\tau \text{ on } S.$$

Because B_-^{-1} is invertible, $D\mathbf{Y}_- = D\mathbf{Y}_+$ on S iff $B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) = 0$. Then,

$$\begin{aligned} B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) &= B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) + B_-^{-1}D(\mathbf{G}_+ - \mathbf{G}_-) \\ &= B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) - (B_-^{-1}B_-)\mathbf{g}(0, \gamma_0) \otimes D\tau \\ &= B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) - \mathbf{g}(0, \gamma_0) \otimes D\tau \end{aligned}$$

Also on S, by a form of the product rule,

$$B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) = D(B_-^{-1}(\mathbf{Z}_+ - \mathbf{Z}_-)) - (DB_-^{-1})(\mathbf{Z}_+ - \mathbf{Z}_-) = D(B_-^{-1}(\mathbf{Z}_+ - \mathbf{Z}_-)),$$

since $\mathbf{Z}_+ - \mathbf{Z}_- = \mathbf{H}(\tau, \gamma) - \mathbf{Y}_0(\gamma_0) = \mathbf{H}(0, \gamma_0) - \mathbf{Y}_0(\gamma_0)0$ on S by cond_0.

So,

$$(*) \qquad B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-)) = D(B_-^{-1}(\mathbf{Z}_+ - \mathbf{Z}_-)) \\ = D(B_+^{-1}\mathbf{Z}_+ - B_-^{-1}\mathbf{Z}_-) + D((B_-^{-1} - B_+^{-1})\mathbf{Z}_+) \\ = D[\mathbf{H}(\tau,\gamma) - \mathbf{Y}_0(\gamma_0)] + D(M\mathbf{Z}_+) \text{ on } S,$$

where $M := B_-^{-1} - B_+^{-1}$.

The argument is completed using the following observations:

1. The $N = 0$ condition that $\mathbf{H}(0) = \mathbf{Y}_0$ insured that $\mathbf{Z}_+ = \mathbf{Z}_-$ along S, and the continuity of the forcing terms insures that $\mathbf{G}_+ = \mathbf{G}_-$ along S. Hence, $\mathbf{Y}_+ = \mathbf{Y}_-$ along S. Then all derivatives along S also agree without additional compatibility conditions.
2. This leads to $D[\mathbf{H}(\tau,\gamma) - \mathbf{Y}_0(\gamma_0)] = [\partial_t\mathbf{H}(\tau,\gamma) + \nabla\mathbf{Y}_0(\gamma_0)\mathbf{u}_0(\gamma_0)] \otimes D\tau$.
3. An analysis of M, which vanishes on S, gives

$$D(M\mathbf{Z}_+) = -\nabla\mathbf{u}_0(\gamma_0)\mathbf{Y}_0(\gamma_0) \otimes D\tau \text{ on } S.$$

4. Going back to $(*)$ and invoking cond$_1$ gives $B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-)) = \mathbf{g}(0,\gamma_0) \otimes D\tau$ on S, so that $B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) = 0$, as desired.

Nonlinear compatibility conditions

The linear compatibility conditions depend only upon the initial data and forcing so they can be re-rexpressed in velocity and pressure, rather than vorticity. Here is what we get:

To define the compatibility conditions for data regularity $N \geq 0$, we first recover a unique mean-zero pressure q as the solution to 3D Euler with $\partial_t u^{\boldsymbol{n}}$ replaced by $\partial_t \mathcal{U}^{\boldsymbol{n}}$:

$$(*) \qquad \begin{cases} \Delta q = -\nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T & \text{in } \Omega, \\ \nabla q \cdot \boldsymbol{n} = \partial_t \mathcal{U}^{\boldsymbol{n}} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases}$$

(Note that $\mathbf{f} \cdot \boldsymbol{n} = 0$ on the boundary.) We know that if $(\mathbf{u}, p)$ solves 3D Euler then $\partial_t \mathcal{U}^{\boldsymbol{n}}(0) = \partial_t u^{\boldsymbol{n}}(0)$ on Γ, so $q_0 := q(0) = p(0)$ and

$$(**) \qquad \partial_t \mathbf{u}(0) = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla q_0 + \mathbf{f}(0).$$

We then define the N-th compatibility condition to be (making cond_{-1} vacuous)

$$\text{cond}_N : \text{cond}_{N-1} \text{ and } \partial_t^{N+1} \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}}|_{t=0} + \partial_t^N \left[\mathbf{u} \cdot \nabla \mathbf{u} + \nabla q - \mathbf{f}\right]_{t=0}^{\boldsymbol{\tau}} = 0 \text{ on } \Gamma_+,$$

after making the substitutions given by $(**)$, including in recovering $\partial_t^N \nabla q$ from $(*)$, applied inductively for $N \geq 1$.

The End

3D Euler equations with inflow, outflow

Gung-Min Gie, Jim Kelliher, Anna Mazzucato

Motivation: Boundary Layer Analysis

Open question: Do solutions to incompressible Navier-Stokes equations with no-slip conditions converge to a solution to the Euler equations as viscosity vanishes?

Inflow/Outflow boundary conditions: Yes, when one injects fluid in some boundary components and suctions it out of other boundary components.

Adding inflow/outflow changes the balance of terms:

- Temam & Wang 2002: First showed this for a 3D channel
- Gie, Hamouda, & Temam 2010: 3D domain with curved boundary
- Gie, K, & Mazzucato [in progress]: Linearized Navier-Stokes for a domain with corners

But. . .

> *These results all rely upon the existence of very regular solutions to the Euler equations with injection/suction, while only $C^{1,\alpha}$ solutions (for velocity) have ever been established, and that only for simply connected domains.*

What boundary conditions are possible?

3D Euler equations (zero-forcing for this talk):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0 & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega \end{cases}$$

- $\Omega \subset \mathbb{R}^3$, bounded domain
- $T > 0$ to-be-determined
- $Q = (0, T) \times \Omega$
- $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$

Will explore boundary conditions:

- Heuristically
- In 2D—but all essentials (except one) are the same in 3D

Boundary relations

Using only $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$ $\quad(\omega := \partial_1 u^2 - \partial_2 u^1 = \text{vorticity})$

$$\nabla p \cdot \boldsymbol{\tau} = -\partial_t \mathbf{u} \cdot \boldsymbol{\tau} - \mathbf{u} \cdot \boldsymbol{n}\, \omega - \frac{1}{2}\partial_{\boldsymbol{\tau}} (\mathbf{u} \cdot \boldsymbol{n})^2 - \frac{1}{2}\partial_{\boldsymbol{\tau}} (\mathbf{u} \cdot \boldsymbol{\tau})^2 \quad \text{on } \partial\Omega$$

One constraint among four parameters: $\underset{❶}{\omega}$, $\underset{❷}{\nabla p \cdot \boldsymbol{\tau}}$, $\underset{❸}{\mathbf{u} \cdot \boldsymbol{n}}$, $\underset{❹}{\mathbf{u} \cdot \boldsymbol{\tau}}$

But also

$$\begin{cases} \Delta p = -\nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = \partial_t \mathbf{u} \cdot \boldsymbol{n} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} & \text{on } \partial\Omega \end{cases}$$

we view as a constraint among [❷, ❸, ❹].

Two constraints among 4 parameters

Need 2 more constraints

First additional constraint

On $\partial\Omega$,

$$\nabla p \cdot \boldsymbol{\tau} = -\partial_t \mathbf{u} \cdot \boldsymbol{\tau} - \mathbf{u} \cdot \boldsymbol{n}\,\omega - \frac{1}{2}\partial_{\boldsymbol{\tau}}(\mathbf{u} \cdot \boldsymbol{n})^2 - \frac{1}{2}\partial_{\boldsymbol{\tau}}(\mathbf{u} \cdot \boldsymbol{\tau})^2$$
$$\nabla p \cdot \boldsymbol{n} = \partial_t \mathbf{u} \cdot \boldsymbol{n} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$$

Classical choice:

- $\mathbf{u} \cdot \boldsymbol{n} \equiv 0$ (impermeable boundary conditions)
- ω and $\mathbf{u} \cdot \boldsymbol{n}$ are eliminated, so we need no more constraints

In all cases:

- Constraint changes character when $\mathbf{u} \cdot \boldsymbol{n}$ changes sign
- Pragmatically, first constraint must be the value of $\mathbf{u} \cdot \boldsymbol{n}$
- This leaves one constraint

Second additional constraint: Choices

Vorticity is generated at inflow points, where $\mathbf{u} \cdot \boldsymbol{n} < 0$, as

$$\omega := -\frac{1}{(\mathbf{u} \cdot \boldsymbol{n})} \left[\nabla p \cdot \boldsymbol{\tau} + \partial_t \mathbf{u} \cdot \boldsymbol{\tau} + \frac{1}{2} \partial_{\boldsymbol{\tau}} (\mathbf{u} \cdot \boldsymbol{n})^2 + \frac{1}{2} \partial_{\boldsymbol{\tau}} (\mathbf{u} \cdot \boldsymbol{\tau})^2 \right] =: \mathbf{H}$$

Non-exhaustive choices at inflow:

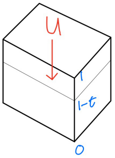
1. Specify $\mathbf{u} \cdot \boldsymbol{\tau}$: Done in 3D for Ω simply connected and $C^{1,\alpha}$ regularity of the velocity by Kazhikhov (Kažihov) and Ragulin 1980. Writeup in Antontsev, Kazhikhov, and Monakhov 1983, translated to English 1990: [AKM]
2. Specify vorticity: Done in 2D by Yudovich 1964
3. Independent relation between ω and $\mathbf{u} \cdot \boldsymbol{\tau}$. This was done by Chemetov and Antontsev 2008 for 2D weak solutions (without uniqueness) for Navier friction boundary conditions

We are concerned with 1, but for higher regularity will need a limited form of 2 in 3D

> *In 3D, there is an additional relation between the tangential and normal components of* $\mathbf{u}$ *and those of* $\boldsymbol{\omega}$*, so only the tangential components of the vorticity would appear in the boundary condition*

Simple example

Assume [as in Petcu 2006]:



- $\Omega := (0, L_1) \times (0, L_2) \times (0, 1)$, periodic in x, y
- $\boldsymbol{\mathcal{U}} = (0, 0, -1)$ and $\mathbf{f} \equiv 0$

Add strong symmetry: $\mathbf{u} = (u^1(t, z), u^2(t, z), -1)$

Direct calculation shows such solutions exist with:

- $\nabla p \equiv 0$
- $\implies$ vorticity generated on Γ_+ $(z = 1)$ must be zero
- $$\begin{cases} \boldsymbol{\omega}(t, z) = 0 & \text{if } z > 1 - t, \\ \boldsymbol{\omega}(t, z) = \boldsymbol{\omega}_0(t + z) & \text{if } z < 1 - t \end{cases}$$

For $\boldsymbol{\omega}$ to be continuous, must have $\boldsymbol{\omega}_0 = 0$ on Γ_+

To have $\boldsymbol{\omega} \in C^N(Q)$ must have $D^N \boldsymbol{\omega}_0 = 0$ on Γ_+

Compatibility Conditions

Define $\mathbf{H}$ on $[0, T] \times \Gamma_+$ by Recall (2D): $\mathbf{H} = -\frac{1}{U^n}\left[\nabla p \cdot \boldsymbol{\tau} + \partial_t \mathbf{u} \cdot \boldsymbol{\tau} + \frac{1}{2}\partial_{\boldsymbol{\tau}}(\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n})^2 + \frac{1}{2}\partial_{\boldsymbol{\tau}}(\mathbf{u} \cdot \boldsymbol{\tau})^2\right]$

$$\mathbf{H}^{\boldsymbol{\tau}} = -\left(\nabla_\Gamma \mathcal{U}^{\boldsymbol{n}} + \frac{1}{\mathcal{U}^{\boldsymbol{n}}}\left[\boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}} + \partial_t \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}} + \nabla_\Gamma p\right]\right)^\perp,$$

$$\mathbf{H}^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \boldsymbol{\mathcal{U}}^{\boldsymbol{\tau}},$$

p is recovered as though $(\mathbf{u}, p)$ were a solution to the Euler equations

For $N \geq 0$, define the following compatibility conditions cond_N on Γ_+:

cond_0 : $\mathbf{H}(0) = \boldsymbol{\omega}_0$, (as in [AKM])

cond_1 : cond_0 and $\partial_t \mathbf{H}(0) = \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0$,

cond_2 : cond_1 and $\partial_t^2 \mathbf{H}(0) = \partial_t \left[\boldsymbol{\omega} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega}\right]_{t=0}$,

$\ldots.$

cond_0 is equivalent to $\nabla p(0) = -\partial_t \boldsymbol{\mathcal{U}}(0) - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0$. For cond_1,

$$\partial_t \nabla p(0) = -\partial_{tt} \boldsymbol{\mathcal{U}}(0) - \partial_t \boldsymbol{\mathcal{U}}(0) \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \partial_t \mathbf{u}(0),$$

$$\nabla \partial_t \mathbf{u}(0) = -\nabla(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) - \nabla\nabla p(0),$$

we see that $\partial_t \nabla p(0)$ and so $\partial_t \mathbf{H}(0)$ are known from the initial data. Inductively, cond_N is a valid compatibility condition.

Uniqueness: the easy part

Let $(\mathbf{u}_1, \nabla p_1)$, $(\mathbf{u}_2, \nabla p_2)$ be two solutions with the same initial data for $N \geq 0$. Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Use that

- $\mathbf{w} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$
- $\mathbf{w} = 0$ on Γ_+
- $\mathbf{u}_1 \cdot \boldsymbol{n} > 0$ on Γ_-.

Same energy argument as for impermeable boundary conditions until the last step:

$$\int_\Omega (\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_2 + \mathbf{u}_1 \cdot \nabla \mathbf{w} + \nabla(p_1 - p_2)) \cdot \mathbf{w} = 0,$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 &= -\int_\Omega (\mathbf{w} \cdot \nabla \mathbf{u}_2) \cdot \mathbf{w} - \frac{1}{2}\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2 \\ &\leq \|\mathbf{u}_2\|_{C^1(Q)}\|\mathbf{w}\|^2 - \frac{1}{2}\int_{\partial\Omega} (\mathbf{u}_1 \cdot \boldsymbol{n})|\mathbf{w}|^2 \\ &= C\|\mathbf{w}\|^2 - \frac{1}{2}\int_{\Gamma_-} (\mathbf{u}_1 \cdot \boldsymbol{n})|\mathbf{w}|^2 \leq C\|\mathbf{w}\|^2.\end{aligned}$$

Gronwall's gives $\mathbf{w} \equiv 0$.

Recovering velocity from vorticity

$H := \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$

If $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ for $\mathbf{u} \in C^{1,\alpha}_{\sigma}(\Omega)$ then

$$\mathbf{u} = K[\boldsymbol{\omega}] + \boldsymbol{\mathcal{V}},$$

where $K[\boldsymbol{\omega}]$ is the Biot-Savart law, and $\operatorname{div} \boldsymbol{\mathcal{V}} = 0$, $\operatorname{curl} \boldsymbol{\mathcal{V}} = 0$, $\boldsymbol{\mathcal{V}} \cdot \boldsymbol{n} = \boldsymbol{\mathcal{U}} \cdot \boldsymbol{n}$ on $\partial\Omega$.

Let $\boldsymbol{\mathcal{V}} = \nabla\varphi$, where φ is the unique mean-zero solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega, \\ \nabla\varphi \cdot \boldsymbol{n} = \mathcal{U}^{\boldsymbol{n}} & \text{on } \partial\Omega. \end{cases}$$

Then $\|\boldsymbol{\mathcal{V}}\| \leq C\|\boldsymbol{\mathcal{U}}\|$ in all the pertinent norms. (If $\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n} = 0$ then $\boldsymbol{\mathcal{V}} \equiv 0$.)

(If multiply connected, also need to deal with an harmonic part.)

Strategy for $N = 0$

Develop an operator A: velocity $\to$ velocity. Given input velocity $\mathbf{u}$:

1. Write $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$, $\quad \mathbf{u} = K[\omega] + \boldsymbol{\mathcal{V}}$
2. Recover p from $\mathbf{u}$ as though the Euler equations hold [with adjustment]
3. Obtain a weak Eulerian solution to the linearized vorticity equation,

$$\begin{cases} \partial_t \overline{\boldsymbol{\omega}} + \mathbf{u} \cdot \nabla \overline{\boldsymbol{\omega}} = \overline{\boldsymbol{\omega}} \cdot \nabla \mathbf{u}, \\ \overline{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0, \\ \overline{\boldsymbol{\omega}}|_{\Gamma_+} = \mathbf{H} \end{cases}$$

4. Show that $\mathbf{v} = K[\omega] + \boldsymbol{\mathcal{V}}$ satisfies

$$\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{u} \cdot (\nabla \mathbf{v})^T + \nabla \pi = 0$$

5. Set output $A\mathbf{u} = \mathbf{v}$

The operator Λ defined by $\boldsymbol{\omega} \mapsto \mathbf{u} = K[\omega] \mapsto \overline{\boldsymbol{\omega}}$ is at the heart of [AKM]

Goal: Find a fixed point of A

Pushforward of the vorticity

At least formally,

$$\boldsymbol{\omega}(t, \mathbf{x}) = \begin{cases} \boldsymbol{\omega}_0(\eta(t, 0; \mathbf{x})) \cdot \nabla\eta(0, t; \eta(t, 0; \mathbf{x})) & \text{on } U_-, \\ \mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) \cdot \nabla\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x})) & \text{on } U_+ \end{cases}$$

This is the pushforward of the vorticity from time zero or off of Γ_+

The central issue is obtaining regularity across S

- Can be viewed as transport and stretching
- With enough regularity, $\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u}$
- For $\mathbf{u} \in C^{1,\alpha}(Q)$, $\mathbf{Y}_0 = \boldsymbol{\omega}_0 \in C^\alpha(\Omega)$ so $\mathbf{Y} = \eta(0, t)_* \mathbf{Y}_0$,

$$\partial_t \mathbf{Y} + \operatorname{div}(\mathbf{Y} \otimes \mathbf{u}) = \mathbf{Y} \cdot \nabla \mathbf{u} \text{ in } \mathcal{D}'(\mathring{U}_-)$$

- In 2D, for pure transport and weak solutions, inflow, outflow like this was treated by Boyer and Fabrie 2006, 2013

Key difficulties beyond regularity across S:

- Obtaining estimates on $\mathbf{H}$—uses clever pressure estimates obtained by AKM
- Obtaining estimates on the operator A incorporating the pressure estimates and estimates on the pushforward

$N = 1$ regularity for inflow, outflow

- Assume we have $N = 1$ regularity of the initial data
- Let $\mathbf{u}_1$ be the $N = 0$ solution with inflow, outflow (but choose some $\alpha > \frac{1}{2}$)
- Let $\mathbf{u}_2$ be the $N = 1$ solution with vorticity boundary conditions having vorticity on $[0, T] \times \Gamma_+$ given by $\boldsymbol{\omega}_1 = \operatorname{curl} \mathbf{u}_1$.
- Formally, the energy argument we just made for uniqueness holds, so $\mathbf{u}_1 = \mathbf{u}_2$, which would give the $N = 1$ result for inflow, outflow
- We lack sufficient regularity of $\mathbf{u}_1$ to do this, but an approximation argument works
- The key points are that $\mathbf{u}_1, \boldsymbol{\omega}_1$ satisfy the vorticity equation in weak Eulerian form and that $\nabla \boldsymbol{\omega}_2 \in L^\infty(Q)$