

PROPERTIES OF THE FLOW FOR SOLUTIONS TO THE EULER EQUATIONS IN THE PLANE

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ABSTRACT. We construct a class of examples of initial vorticities for which the solution to the Euler equations in the plane has an associated flow that lies in no Hölder space of positive exponent for any positive time. Our initial vorticities have L^p -norms that do not grow much faster than $\log p$, which Yudovich showed ensures the uniqueness of solutions to the Euler equations ([8]). Our class of examples extends an example of Bahouri's and Chemin's ([1]) with bounded initial vorticity, for which the flow lies in no Hölder space of exponent greater than e^{-t} .

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1. INTRODUCTION

Yudovich showed in [7] that there exists a unique solution to the Euler equations in a bounded domain of the plane with bounded initial vorticity. Also, there exists a unique continuous flow, and this flow lies in the Hölder space of exponent e^{-Ct} for all positive time t . Yudovich's result is easily modified to apply to solutions in the whole plane. Bahouri and Chemin in [1] showed that this regularity of the flow was in a sense optimal by constructing an example for which the flow lies in no Hölder space of exponent higher than e^{-t} .

In [8] Yudovich extended his uniqueness result to a certain class \mathbb{Y} of unbounded vorticities and showed that there exists as well a unique flow. There is an upper bound on the modulus of continuity of this flow that can be arbitrarily poor, depending upon how unbounded the initial vorticity is. It would be useful to know whether the lower bound on the modulus of continuity can also be arbitrarily poor for certain initial vorticities, as this might cast some light on how near to the "edge of uniqueness" the class \mathbb{Y} has brought the Euler equations.

We give some partial information by extending the example of Bahouri and Chemin to a class of initial vorticities in \mathbb{Y} having a point singularity. We show that for some such initial vorticities the flow lies in no Hölder space of positive exponent for any positive time. We also discuss why it is difficult, though perhaps not impossible, to extend our approach to obtain an arbitrarily poor lower bound on the modulus of continuity of the flow.

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2. YUDOVICH'S THEOREM FOR UNBOUNDED VORTICITY

We first define the function spaces in which our initial velocity will lie. We let E_m be as in [2] (although we could also work in the larger spaces of [3]). For any real number m , a vector v belongs to E_m if it is divergence-free and can be written in the form $v = \sigma + v'$, where v' is in $L^2(\mathbb{R}^2)$ and where σ is a *stationary vector field*, meaning that σ is of the form

$$\sigma = \left(-\frac{x_2}{r^2} \int_0^r \rho g(\rho) d\rho, \frac{x_1}{r^2} \int_0^r \rho g(\rho) d\rho \right) \quad (2.1)$$

for some g in $C_0^\infty(\mathbb{R})$. E_m is an affine space; fixing an origin, σ , in E_m we can define a norm by $\|\sigma + v'\|_{E_m} = \|v'\|_{L^2(\Omega)}$. Convergence in E_m is equivalent to convergence in the L^2 -norm to a vector in E_m .

We will further restrict our initial velocities to have vorticities that are “only slightly unbounded” in a sense we now make precise.

Definition 2.1. Let $\theta : [p_0, \infty) \rightarrow \mathbb{R}^+$ for some p_0 in $[1, 2)$. We say that θ is *admissible* if the function $\beta : (0, \infty) \rightarrow [0, \infty)$ defined, for some $M > 0$, by¹

$$\beta_M(x) := C \inf \{ (M^\epsilon x^{1-\epsilon} / \epsilon) \theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \}, \quad (2.2)$$

where C is a fixed absolute constant, satisfies

$$\int_0^1 \frac{dx}{\beta(x)} = \infty. \quad (2.3)$$

Because

$$\begin{aligned} \beta_M(x) &= CM \frac{x}{M} \inf \{ ((x/M)^{-\epsilon} / \epsilon) \theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \} \\ &= M \beta_1(x/M), \end{aligned}$$

this definition is independent of the value of M . Also, β_M is a monotonically increasing continuous function, with $\lim_{x \rightarrow 0^+} \beta_M(x) = 0$.

Definition 2.2. We say that a vector field v has *Yudovich vorticity* if for some admissible function $\theta : [p_0, \infty) \rightarrow \mathbb{R}^+$ with p_0 in $[1, 2)$, $\|\omega(v)\|_{L^p} \leq \theta(p)$ for all p in $[p_0, \infty)$.

Examples of admissible bounds on vorticity are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \dots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p, \quad (2.4)$$

where \log^m is log composed with itself m times. These admissible bounds are described in [8] (see also [4].) Roughly speaking, the L^p -norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$ and still be admissible. Such growth in the L^p -norms arises, for example, from a point singularity of the type $\log \log(1/|x|)$ (see Lemma A.1).

¹The definition of β in Equation (2.2) differs from that in [4] in that it directly incorporates the factor of p that appears in the Calderón-Zygmund inequality; in [4] this factor is included in the equivalent of Equation (2.3).

Given an admissible function θ and fixing m in \mathbb{R} , we define the space

$$\mathbb{Y}_\theta = \{v \in E_m : \|\omega(v)\|_{L^p} \leq C\theta(p) \text{ for all } p \text{ in } [p_0, \infty)\},$$

for some constant C . We define the norm on \mathbb{Y}_θ to be

$$\|v\|_{\mathbb{Y}_\theta} = \|v\|_{E_m} + \sup_{p \in [p_0, \infty)} \|\omega(v)\|_{L^p} / \theta(p). \quad (2.5)$$

This space is not separable, because $L^{p_0} \cap L^\infty$, which is not separable, is compactly embedded in it.

Finally, we define the space

$$\mathbb{Y} = \{v \in Y_\theta : \theta \text{ is admissible}\},$$

but place no norm on this space.

The final thing we must do before stating Yudovich's theorem is to define what we mean by a weak solution to the Euler equations.

Definition 2.3 (Weak Euler Solutions). Given an initial velocity v^0 in \mathbb{Y}_θ , v in $L^\infty([0, T]; \mathbb{Y}_\theta)$ is a weak solution to the Euler equations (without forcing) if $v(0) = v^0$ and

$$(E) \quad \frac{d}{dt} \int_{\Omega} v \cdot \varphi + \int_{\Omega} (v \cdot \nabla v) \cdot \varphi = 0$$

for all divergence-free φ in $(H^1(\mathbb{R}^2))^2$.

Our form of the statement of Yudovich's theorem is a generalization of the statement of Theorem 5.1.1 of [2] from bounded to unbounded vorticity.

Theorem 2.4 (Yudovich's Theorem for Unbounded Vorticity). ***First part:** For any v^0 in \mathbb{Y} there exists a unique weak solution v of (E). Moreover, v is in $C(\mathbb{R}; E_m) \cap L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$. Also,*

$$\|\omega(t)\|_{L^p(\mathbb{R}^2)} = \|\omega^0\|_{L^p} \text{ for all } 1 \leq p \leq \infty. \quad (2.6)$$

***Second part:** The vector field has a unique continuous flow. More precisely, if v^0 is in \mathbb{Y}_θ then there exists a unique mapping ψ , continuous from $\mathbb{R} \times \mathbb{R}^2$ to \mathbb{R}^2 , such that*

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x)) ds.$$

Let

$$\mu(r) = (C/r)\beta_1(r^2/4), \quad (2.7)$$

where β_1 is the function of Definition 2.1 associated with θ and let $\Gamma_t: [0, \infty) \rightarrow [0, \infty)$ be defined by $\Gamma_t(0) = 0$ and for $s > 0$ by

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = t \text{ or equivalently } \int_{s^2/4}^{\Gamma_t(s)^2/4} \frac{dr}{\beta_1(r)} = t. \quad (2.8)$$

Then $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the modulus of continuity of the flow at time $t > 0$; that is, for all x and y in \mathbb{R}^2

$$|\psi(t, x) - \psi(t, y)| \leq \Gamma_t(|x - y|).$$

Also, for all x and y in \mathbb{R}^2

$$|v(t, x) - v(t, y)| \leq \mu(|x - y|), \quad (2.9)$$

Existence in the first part of Yudovich's theorem can be established, for instance, by modifying the approach on p. 311-319 of [6], which establishes existence under the assumption of bounded vorticity; the uniqueness argument is given by Yudovich in [8]. The second part is Theorem 2 of [8], the bound on the modulus of continuity of the flow following from working out the details of Yudovich's proof (see Sections 5.2 through 5.4 of [5]).

The function μ of Equation (2.7) will have all the important properties of β_1 : $\mu(0) = 0$, μ is nondecreasing, and $\int_0^1 (\mu(r))^{-1} dr = \infty$. Equation (2.8) defines Γ_t in terms of μ , but it can equally well be used to define μ in terms of any modulus of continuity Γ_t , and such a μ will have the three properties just described. Thus, Equation (2.8) can give an arbitrarily poor upper bound on the modulus of continuity of the flow.

The factors of $1/4$ in Equation (2.8) are inconsequential, since if we define $\Gamma_t(s)$ by

$$\int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{\beta_1(r)} = t \quad (2.10)$$

we will simply obtain a slightly weaker upper bound on the modulus of continuity of the flow.

3. SQUARE-SYMMETRIC VORTICITIES

Ignoring for the moment the Euler equations, we will assume that the vorticity has certain symmetries, and from these symmetries deduce some useful properties of the divergence-free velocity having the given vorticity. In Section 4, we will then consider what happens to a solution to the Euler equations whose initial vorticity possesses these symmetries.

For convenience, we number the quadrants in the plane Q_1 through Q_4 , starting with

$$Q_1 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\},$$

and moving counterclockwise through the quadrants.

Definition 3.1. We say that a Yudovich vorticity (vorticity as in Definition 2.2) is *symmetric by quadrant*, or SBQ, if ω is compactly supported and $\omega(x) = \omega(x_1, x_2)$ is odd in x_1 and x_2 ; that is, $\omega(-x_1, x_2) = -\omega(x_1, x_2)$ and $\omega(x_1, -x_2) = -\omega(x_1, x_2)$ —so also $\omega(-x) = \omega(x)$.

Lemma 3.2. *Let ω be SBQ. Then there exists a unique vector field v in $E_0 \cap \mathbb{Y}$ with $\omega(v) = \omega$, and v satisfies the following:*

- (1) $v_2(x_1, 0) = 0$ for all x_1 in \mathbb{R} ;

- (2) $v_1(0, x_2) = 0$ for all x_2 in \mathbb{R} ;
(3) $v(0, 0) = 0$.

If, in addition, $\omega \geq 0$ in Q_1 , then

- (4) $v_1(x_1, 0) \geq 0$ for all $x_1 \geq 0$.

Proof. Let p be in $[1, 2)$ and let $q > 2p/(2-p)$. By Proposition 3.1.1 p. 44 of [2], for any vorticity ω in L^p there exists a unique divergence-free vector field v in $L^p + L^q$ whose curl is ω , with v being given by the Biot-Savart law,

$$v = K * \omega. \quad (3.1)$$

Here, K is the Biot-Savart kernel, $K(x) = (1/2\pi)x^\perp/|x|^2$, which decays like $1/|x|$ with a singularity of order $1/|x|$ at the origin.

Because ω is compactly supported and lies in $L^2(\mathbb{R}^2)$, ω is in $L^p(\mathbb{R}^2)$, and Equation (3.1) gives our velocity v , unique in all the spaces $L^p + L^q$. Also, because $\int_{\mathbb{R}^2} \omega = 0$, v is in $(L^2)^2 = E_0$ (see the comment following Definition 1.3.3 of [2], for instance).

Then

$$\begin{aligned} v_1(x_1, 0) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{|x-y|^2} \omega(y) dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y) dy \\ &= \frac{1}{2\pi} \sum_{j=1}^4 \int_{Q_j} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y) dy. \end{aligned}$$

Making the changes of variables, $u = (-y_1, y_2)$, $u = -y$, and $u = (y_1, -y_2)$ on Q_2 , Q_3 , and Q_4 , respectively, in all cases the determinant of the Jacobian is ± 1 , and we obtain

$$\begin{aligned} v_1(x_1, 0) &= \frac{1}{2\pi} \left[\int_{Q_1} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y) dy - \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u) du \right. \\ &\quad \left. + \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u) du - \int_{Q_1} \frac{u_2}{(x_1 - u_1)^2 + u_2^2} \omega(u) du \right] \end{aligned}$$

or

$$v_1(x_1, 0) = \frac{1}{\pi} \int_{Q_1} (f_1(x_1, y) - f_2(x_1, y)) \omega(y) dy, \quad (3.2)$$

where

$$f_1(x_1, y) = \frac{y_2}{(x_1 - y_1)^2 + y_2^2}, \quad f_2(x_1, y) = \frac{y_2}{(x_1 + y_1)^2 + y_2^2}. \quad (3.3)$$

It follows from $(x_1 - y_1)^2 + y_2^2 \leq (x_1 + y_1)^2 + y_2^2$ on Q_1 that $f_1(x_1, y) > f_2(x_1, y)$ for all $x_1, y_1 > 0$. Conclusion (4) then follows from Equation (3.2).

By the Biot-Savart law of Equation (3.1),

$$\begin{aligned}
v_2(x_1, -x_2) &= (K_2 * \omega)(x_1, -x_2) \\
&= \int_{\mathbb{R}^2} K_2(x_1 - y_1, -x_2 - y_2) \omega(y_1, y_2) dy \\
&= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 + y_2) \omega(y_1, -y_2) dy \\
&= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 - (-y_2)) \omega(y_1, -y_2) dy \\
&= -v_2(x_1, x_2).
\end{aligned}$$

Here we used $K_2(x_1, -x_2) = -K_2(x_1, x_2)$ and the symmetry of ω . A similar calculation shows that $v_1(-x_1, x_2) = -v_1(x_1, x_2)$. Thus, the velocity along the x -axis is directed along the x -axis and the velocity along the y -axis is directed along the y -axis, so the axes are preserved by the flow. In particular, the origin is fixed. This gives conclusions (1)-(3). \square

The following lemma is a slight generalization of Proposition 2.1 of [1] (see also Proposition 5.3.1 of [2]):

Lemma 3.3. *Let ω be SBQ with*

$$\omega = 2\pi \mathbf{1}_{[0,r] \times [0,r]} \quad (3.4)$$

on Q_1 for some r in $(0, 1)$. Then for any λ in $(0, 1)$ and λ' in $(0, \lambda)$ there is a right neighborhood $\mathcal{N} = \mathcal{N}_{\lambda, \lambda'}$ of 0 for which

$$v_1(x_1, 0) \geq 4(1 - \lambda')x_1 \log(1/x_1) \quad (3.5)$$

for all x_1 in \mathcal{N} for which $x_1^\lambda \leq r$.

Remark 3.1. The neighborhood \mathcal{N} depends only upon λ and λ' ; in particular, it is independent of r .

Proof. Equation (3.2) gives

$$\begin{aligned}
v_1(x_1, 0) &= 2\pi \frac{1}{\pi} \int_{[0,r] \times [0,r]} (f_1(x_1, y) - f_2(x_1, y)) dy \\
&= 2 \int_0^r \int_0^r \frac{y_2}{(x_1 - y_1)^2 + y_2^2} dy_2 dy_1 - 2 \int_0^r \int_0^r \frac{y_2}{(x_1 + y_1)^2 + y_2^2} dy_2 dy_1.
\end{aligned}$$

Both of the inner integrals above are of the form

$$\begin{aligned}
\int_0^r \frac{y_2}{a^2 + y_2^2} dy_2 &= \frac{1}{2} \int_0^{r^2} \frac{du}{a^2 + u} du = \frac{1}{2} [\log(a^2 + u)]^{r^2} \\
&= \frac{1}{2} \log(a^2 + r^2) - \frac{1}{2} \log(a^2),
\end{aligned}$$

where a is $x_1 - y_1$ and $x_1 + y_1$ in the two integrals. This gives

$$\begin{aligned} v_1(x_1, 0) &= 2 \int_0^r \left[\frac{1}{2} \log((x_1 - y_1)^2 + r^2) - \frac{1}{2} \log((x_1 - y_1)^2) \right. \\ &\quad \left. - \frac{1}{2} \log((x_1 + y_1)^2 + r^2) + \frac{1}{2} \log((x_1 + y_1)^2) \right] dy_1 \quad (3.6) \\ &= \tilde{v}_1(x_1, 0) + \bar{v}_1(x_1, 0), \end{aligned}$$

where

$$\begin{aligned} \tilde{v}_1(x_1, 0) &= \int_0^r \log((x_1 + y_1)^2) dy_1 - \int_0^r \log((x_1 - y_1)^2) dy_1, \\ \bar{v}_1(x_1, 0) &= \psi_r(x_1) = \int_0^r \log \frac{(x_1 - y_1)^2 + r^2}{(x_1 + y_1)^2 + r^2} dy_1. \end{aligned} \quad (3.7)$$

But,

$$\int_0^r \log((x_1 - y_1)^2) dy_1 = 2(r - x_1) \log |r - x_1| + 2x_1 \log x_1 - 2r,$$

which follows by adding the integral from 0 to x_1 to the integral from x_1 to r . Also,

$$\int_0^r \log((x_1 + y_1)^2) dy_1 = 2(r + x_1) \log(r + x_1) - 2x_1 \log x_1 - 2r,$$

so

$$\begin{aligned} \tilde{v}_1(x_1, 0) &= -4x_1 \log x_1 + 2(r + x_1) \log(r + x_1) - 2(r - x_1) \log |r - x_1| \\ &= 4x_1 \log(1/x_1) + \phi_r(x_1), \end{aligned}$$

where

$$\phi_r(x_1) = 2(r + x_1) \log(r + x_1) - 2(r - x_1) \log |r - x_1|. \quad (3.8)$$

(So far, our analysis is much as in the proof of Proposition 2.1 of [1], the significant difference being that for Bahouri and Chemin $r = 1$. This allows them to conclude that $v_1(x_1, 0) \geq -2x_1 \log x_1$ in some right-neighborhood of the origin, but we will find that for an arbitrary r in $(0, 1)$, the size of this neighborhood shrinks with r . This will be of importance in Lemma 3.5 where we consider an initial vorticity that is a sum of vorticities that are square-symmetric by quadrant. We seek both to refine Bahouri's and Chemin's bound slightly and to establish a lower bound on the size of the right-neighborhood.)

By Equation (3.6), Equation (3.7), and Equation (3.8),

$$v_1(x_1, 0) = -4x_1 \log x_1 + \phi_r(x_1) + \psi_r(x_1). \quad (3.9)$$

Suppose we can show for x_1 lying in $(0, 1)$ but also bounded by some function of r , that

$$\phi_r(x_1) + \psi_r(x_1) \geq -Cx_1 \log(1/x_1), \quad (3.10)$$

where $0 < C < 4$. This will insure that

$$v_1(x_1, 0) \geq C_1 x_1 \log(1/x_1) \quad (3.11)$$

for the given range of x_1 , where $C_1 = 4 - C$ is a positive constant.

We will first show that $\psi_r(x_1)$ is negligible compared to the other two terms in Equation (3.9). Then we will show that $\phi_r(x_1)$, while comparable in magnitude to $-4x_1 \log x_1$, can be made sufficiently smaller than it so that Equation (3.10) will hold, as long as we restrict x_1 so that $x_1 < r^\lambda$ for some λ in $(0, 1)$.

We have,

$$\begin{aligned} \frac{\partial}{\partial r} \psi_r(x_1) &= -2 \tan^{-1} \left(\frac{x_1 - r}{r} \right) - 2 \tan^{-1} \left(\frac{x_1 + r}{r} \right) + 4 \tan^{-1} \left(\frac{x_1}{r} \right) \\ &\quad + \log \left(\frac{x_1^2 + 2r^2 - 2x_1 r}{x_1^2 + 2r^2 + 2x_1 r} \right) \\ &= -2 \tan^{-1} (w - 1) - 2 \tan^{-1} (w + 1) + 4 \tan^{-1} (w) \\ &\quad + \log \left(\frac{w^2 + 2 - 2w}{w^2 + 2 + 2w} \right) \\ &=: h(w), \end{aligned}$$

where $w = x_1/r < 1$, since $0 \leq x_1 < r$. Also,

$$h'(w) = 4 \frac{w^2(w^2 - 4)}{(w^2 + 2 - 2w)(w^2 + 2 + 2w)(1 + w^2)},$$

which is negative for w in $(0, 2)$, is zero at $w = 2$, and is positive for w in $(2, \infty)$. This means that $h(w)$ decreases from a value of 0 at 0 to a negative value at $w = 2$, then monotonically increases. However, it is also true that $\lim_{w \rightarrow \infty} h(w) = 0$, which means that $h(w) < 0$ for all $w > 0$.

What we have shown is that that for all $x_1, r > 0$,

$$\frac{\partial}{\partial r} \psi_r(x_1) = h(w) < 0,$$

so for a fixed value of x_1 , $\psi_r(x_1)$ is a decreasing function of r . Restricting r to lie in the range $(0, 1]$, we can therefore bound $\psi_r(x_1)$ from below by $\psi_1(x_1)$. But $\psi_1(x_1)$ is a smooth function equal to zero at the origin, so

$$\psi_r(x_1) \geq \psi_1(x_1) \geq -C_3 x_1$$

for some positive constant C_3 in a fixed neighborhood of the the origin that is independent of the value of r . This means that by restricting x_1 to a possibly smaller neighborhood of the origin, we can insure that the contribution of $\psi_r(x_1)$ to the left-hand side of Equation (3.10) is as small as required. Hence, we can replace Equation (3.10) by the requirement that

$$\phi_r(x_1) \geq -C x_1 \log(1/x_1) \quad (3.12)$$

for all $x_1 < r$, where $0 < C < 4$, and from this Equation (3.11) will follow.

We have,

$$\begin{aligned}
\frac{\phi_r(x_1)}{2} &= (r+x_1)\log(r+x_1) - (r-x_1)\log(r-x_1) \\
&= r(1+w)\log(r(1+w)) - r(1-w)\log(r(1-w)) \\
&= r(1+w)\log r - r(1-w)\log r + r(1+w)\log(1+w) \\
&\quad - r(1-w)\log(1-w) \\
&= r\log r + rw\log r - r\log r + rw\log r \\
&\quad + r[\log(1+w) - \log(1-w)] + rw[\log(1+w) + \log(1-w)] \\
&= 2rw\log r + r\log\left(\frac{1+w}{1-w}\right) + rw\log(1-w^2).
\end{aligned}$$

Since

$$\begin{aligned}
&r\log\left(\frac{1+w}{1-w}\right) + rw\log(1-w^2) \\
&\geq r\log\left(\frac{1+w}{1-w}\right) + r\log(1-w^2) = r\log\left(\frac{1+w}{1-w}(1+w)(1-w)\right) \\
&= 2r\log(1+w) \geq 0,
\end{aligned}$$

it follows that

$$\phi_r(x_1) \geq 4rw\log r = 4x_1\log r.$$

We now add the restriction that

$$x_1^\lambda \leq r$$

for some λ in $(0, 1)$. Then

$$\phi_r(x_1) \geq 4x_1\log r \geq 4x_1\log x_1^\lambda = -4\lambda x_1\log(1/x_1),$$

which is Equation (3.12) with $C = 4\lambda$. It follows from Equation (3.11) that

$$v_1(x_1, 0) \geq (4 - 4\lambda)x_1\log(1/x_1) - C_3x_1. \quad (3.13)$$

Equation (3.5) then follows, since choosing $\lambda' < \lambda$ allows us to ignore the term $-C_3x_1$. \square

Definition 3.4. We say that ω is square-symmetric if ω is SBQ and $\omega(x_1, x_2) = \omega(\max\{x_1, x_2\}, 0)$ on Q_1 .

Being square-symmetric means that a vorticity is SBQ and is constant in absolute value along the boundary of any square centered at the origin.

Lemma 3.5. *Assume that ω is square-symmetric, finite except possibly at the origin, and $\omega(x_1, 0)$ is non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$*

$$v^1(x_1, 0) \geq C(1 - \lambda')\omega(x_1^\lambda, 0)x_1\log(1/x_1) \quad (3.14)$$

for all x_1 in the neighborhood $\mathcal{N} = \mathcal{N}_{\lambda, \lambda'}$ of Lemma 3.3, where $C = 2/\pi$.

Proof. We can write ω on Q_1 as

$$\omega(x) = 2\pi \int_0^1 \alpha(s) 1_{[0,s] \times [0,s]}(x) ds, \quad (3.15)$$

for some measurable, nonnegative function $\alpha: (0, 1) \rightarrow [0, \infty)$. This means that

$$\omega(x_1, 0) = 2\pi \int_{x_1}^1 \alpha(s) ds. \quad (3.16)$$

Let $V(s)$ be the value of $v_1(x_1, 0)$ that results from assuming that ω is given by Equation (3.4). By Lemma 3.2, $V(s) > 0$. Then because the Biot-Savart law of Equation (3.1) is linear, and using Lemma 3.3,

$$\begin{aligned} v_1(x_1, 0) &= \int_0^1 \alpha(s) V(s) ds \\ &= \int_0^{x_1^\lambda} \alpha(s) V(s) ds + \int_{x_1^\lambda}^1 \alpha(s) V(s) ds \\ &\geq \int_{x_1^\lambda}^1 \alpha(s) V(s) ds \\ &\geq 2\pi \left(\int_{x_1^\lambda}^1 \alpha(s) ds \right) \frac{4}{2\pi} (1 - \lambda') x_1 \log(1/x_1) \\ &= \frac{2}{\pi} (1 - \lambda') \omega(x_1^\lambda, 0) x_1 \log(1/x_1). \end{aligned}$$

□

Remark 3.2. Properly speaking, we must allow the function α of Equation (3.15) to be a distribution since, for instance, to obtain ω of Lemma 3.3, we would need $\alpha = \delta_r$. We could avoid this complication, however, by assuming that ω is strictly decreasing and that $\omega(x_1, 0)$ is sufficiently smooth as a function of $x_1 > 0$.

4. SQUARE-SYMMETRIC INITIAL VORTICITIES

We now assume that our initial vorticity is square-symmetric, and consider what happens to the solution to (E) over time.

Theorem 4.1. *Assume that ω^0 is square-symmetric, finite except possibly at the origin, and $\omega^0(x_1, 0)$ is non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$*

$$v_1(t, x_1, 0) \geq C(1 - \lambda') \omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1) \quad (4.1)$$

for all x_1 in the neighborhood $\mathcal{N} = \mathcal{N}_{\lambda, \lambda'}$ of Lemma 3.3 and all time $t \geq 0$, where Γ_t is defined as in Theorem 2.4. The constant $C = 2/\pi$.

Further, let $L(t, x_1)$ be any continuous lower bound on $v_1(t, x_1, 0)$, Equation (4.1) being one possibility. Then if $x_1(t)$ is the solution to

$$\frac{dx_1(t)}{dt} = L(t, x_1)$$

with $x_1(0) = a > 0$ in \mathcal{N} , then $\psi^1(t, a, 0) \geq x_1(t)$ for all $t \geq 0$.

Proof. Since $\omega^0(x_1, x_2) = -\omega^0(x_1, -x_2)$, if $\omega(t, x_1, x_2)$ is a solution to (E) then $-\omega(t, x_1, -x_2)$ is also a solution. But the solution to (E) is unique by Theorem 2.4, so we conclude that $\omega(t, x_1, x_2) = -\omega(t, x_1, -x_2)$. Similarly, $\omega(t, x_1, x_2) = -\omega(t, -x_1, x_2)$, and we see that ω is SBQ. By Lemma 3.2, then, it follows that the flow transports vorticity in Q_k , $k = 1, \dots, 4$, only within Q_k , because of the direction of v along the axes for all $t \geq 0$. Therefore, $\omega(t)$ is also nonnegative in Q_1 for all time.

Our approach then will be to produce a point-by-point lower bound $\bar{\omega}(t)$ on $\omega(t)$ that satisfies all the requirements of Lemma 3.5. In particular, it is SBQ, so $\omega(t) - \bar{\omega}(t)$ is SBQ and nonnegative in Q_1 . It follows from Lemma 3.2 that $v_1(t, x_1, 0) - \bar{v}_1(t, x_1, 0) \geq 0$ for all $t \geq 0$, where $\omega(\bar{v}(t)) = \bar{\omega}(t)$. Thus, the lower bound on $\bar{v}_1(t, x_1, 0)$ coming from Lemma 3.5 will also be a lower bound on $v_1(t, x_1, 0)$. We now determine $\bar{\omega}(t)$.

Because conclusion (3) of Lemma 3.2 holds for all time, ω being SBQ for all time, the origin is fixed by the flow; that is $\psi(t, 0) = \psi^{-1}(t, 0) = 0$ for all t . Also, the Euler equations are time reversible, and the function Γ_t of Equation (2.8) depends only upon the Lebesgue norms of the vorticity, which are preserved by the flow; therefore, Γ_t is a bound on the modulus of continuity of $\psi^{-1}(t, \cdot)$ as well. Thus,

$$|\psi^{-1}(t, x)| = |\psi^{-1}(t, x) - \psi^{-1}(t, 0)| \leq \Gamma_t(|x|).$$

In Q_1 , the value of $\omega(t, x)$, then, is bounded below by using the minimum value of ω^0 on the circle of radius $\Gamma_t(|x|)$ centered at the origin, since this is the furthest away from the origin that $\psi^{-1}(t, x)$ can lie, and ω decreases with the distance from the origin. That is,

$$\omega(t, x) = \omega^0(\psi^{-1}(t, x)) \geq \omega^0(\Gamma_t(|x|), 0)$$

because ω^0 is square-symmetric.

Since $\sqrt{2} \max\{x_1, x_2\} \geq |x|$, Γ_t is nondecreasing, and ω_0 is nonincreasing on Q_1 , $\omega^0(\Gamma_t(\sqrt{2} \max\{x_1, x_2\}), 0) \leq \omega^0(\Gamma_t(|x|), 0)$ on Q_1 . Letting

$$\bar{\omega}(t, x_1, x_2) = \omega^0(\Gamma_t(\sqrt{2} \max\{x_1, x_2\}), 0)$$

we see that $\bar{\omega}$ is square-symmetric, and on Q_1 , $\bar{\omega}(t, x) \leq \omega(t, x)$, so $\bar{\omega}$ is our desired lower bound on ω .

Then from Equation (3.14),

$$\begin{aligned} v^1(x_1, 0) &\geq C(1 - \lambda') \bar{\omega}(t, x_1^\lambda, 0) x_1 \log(1/x_1) \\ &= C(1 - \lambda') \omega^0(\Gamma_t((\sqrt{2} \max\{x_1, 0\})^\lambda), 0) x_1 \log(1/x_1) \\ &= C(1 - \lambda') \omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1). \end{aligned}$$

The lower bound on the flow, $\psi^1(t, a, 0) \geq x_1(t)$, follows from using the minimum possible value of $v^1(t, x_1, 0)$ in Equation (4.1), setting it equal to $dx_1(t)/dt$, and integrating over time. \square

5. BOUNDED VORTICITY

We now apply Theorem 4.1 to the first in the sequence of Yudovich's vorticity bounds in Equation (2.4) in which we have bounded vorticity. We assume that ω is square-symmetric with $\omega^0 = \mathbf{1}_{[0,1/2] \times [0,1/2]}$ in Q_1 so that $\|\omega^0\|_{L^1 \cap L^\infty} = 1$. We have,

$$\beta_1(r) = C \inf \{r^{1-\epsilon}/\epsilon : \epsilon \text{ in } (0, 1]\} = C \inf \{g(\epsilon) : \epsilon \text{ in } (0, 1]\},$$

where $g(\epsilon) = r^{1-\epsilon}/\epsilon$. Then

$$g'(\epsilon) = C \frac{r^{1-\epsilon}(\epsilon \log(1/r) - 1)}{\epsilon^2},$$

which is zero when $\epsilon = \epsilon_0 := 1/(\log(1/r))$ if $r < 1$ and $\epsilon_0 < 1$, and never zero otherwise. But

$$\begin{aligned} \epsilon_0 < 1 &\iff \frac{1}{\log(1/r)} < 1 \iff \log(1/r) > 1 \\ &\iff \frac{1}{r} > e \iff r < e^{-1}, \end{aligned}$$

so the condition $r < 1$ is redundant.

Assume that $r < e^{-1}$. Then $g(\epsilon)$ approaches infinity as ϵ approaches either zero or infinity; hence, ϵ_0 minimizes g . Thus,

$$\begin{aligned} \beta_1(r) &= Cr^{1-\epsilon_0}/\epsilon_0 = Cr(1/r)^{\epsilon_0} \log(1/r) \\ &= C r e^{\log(1/r)\epsilon_0} \log(1/r) = -C e r \log(r). \end{aligned}$$

Then from Equation (2.8),

$$\begin{aligned} \int_{x_1^2/4}^{\Gamma_t(x_1)^2/4} \frac{dr}{\beta_1(r)} &= -C [\log(-\log r)]_{x_1^2/4}^{\Gamma_t(x_1)^2/4} = t \\ &\implies \log(-\log(x_1^2/4)) - \log(-\log(\Gamma_t(x_1)^2/4)) = Ct \\ &\implies \Gamma_t(x_1)^2 = 4(x_1^2/4)e^{-Ct} \implies \Gamma_t(x_1) = 2(x_1/2)e^{-Ct} \end{aligned}$$

as long as $\Gamma_t(x_1)^2/4 < e^{-1}$.

Thus, Theorem 4.1 gives

$$\begin{aligned} v^1(t, x_1, 0) &\geq C(1 - \lambda')\omega^0(2(2^{\lambda/2}x_1^\lambda/2)^{e^{-Ct}}, 0)x_1 \log(1/x_1) \\ &\geq C(1 - \lambda')x_1 \log(1/x_1) \end{aligned}$$

as long as $2(2^{\lambda/2}x_1^\lambda/2)^{e^{-Ct}} < 1/2$.

Solving $dx_1(t)/dt = C(1 - \lambda')x_1 \log(1/x_1)$ with $x_1(0) = a$ gives

$$\psi^1(t, a, 0) \geq x_1(t) = a^{\exp(-C(1-\lambda')t)},$$

which applies for sufficiently small a .

Since $\psi(t, 0, 0) = 0$,

$$\frac{|\psi(t, a, 0) - \psi(t, 0, 0)|}{a^\alpha} = \frac{|\psi^1(t, a, 0)|}{a^\alpha} \geq a^{\exp(-C(1-\lambda')t) - \alpha},$$

which is infinite for any $\alpha > \exp(-C(1-\lambda')t)$. This shows that the flow can be in no Hölder space C^α for $\alpha > \exp(-C(1-\lambda')t)$. But this is true for any λ' in $(0, 1)$, so the flow can be in no Hölder space C^α for $\alpha > e^{-Ct}$, reproducing, up to a constant, the result of [1] (or see Theorem 5.3.1 of [2].)

6. YUDOVICH'S HIGHER EXAMPLES

Assume that $m \geq 2$ and let ω^0 have the symmetry described in Theorem 4.1 with

$$\omega^0(x_1, 0) = \log^m(1/x_1), \quad (6.1)$$

for x_1 in $(0, \exp^{-m}(0))$, and ω^0 equal to zero elsewhere in the first quadrant. Then by Lemma A.1

$$\theta(p) = \|\omega^0\|_{L^p} \leq C \log^{m-1} p$$

for all p larger than some p^* .

Using an observation of Yudovich's in [8], if β_1 is the function of Definition 2.1 associated with the admissible function θ , then letting $\epsilon_0 = 1/\log(1/r)$ for $r < e^{-p^*}$,

$$\begin{aligned} \beta_1(r) &\leq C(r^{1-\epsilon_0}/\epsilon_0)\theta(1/\epsilon_0) = -Crr^{1/\log r} \log r \theta(\log(1/r)) \\ &\leq Cr \log(1/r) \log^m(1/r) \leq Cr \log(1/r) \log^2(1/r) \cdots \log^m(1/r) \\ &= Cr\theta_{m+1}(1/r), \end{aligned}$$

with θ_{m+1} given by Equation (2.4).

Then, if we define the upper bound on the modulus of continuity of the flow by Equation (2.10) instead of Equation (2.8), we have

$$\begin{aligned} -C [\log^{m+1}(1/r)]_{s^2}^{\Gamma_t(s)^2} &= \int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{Cr\theta_{m+1}(1/r)} \\ &\leq \int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{\beta_1(r)} = t \\ \implies -C \log^{m+1}(1/\Gamma_t(s)^2) &\leq -C \log^{m+1}(1/s^2) + t \\ \implies \log^{m+1}(1/\Gamma_t(s)^2) &\geq \log^{m+1}(1/s^2) - Ct \\ \implies \log^m(1/\Gamma_t(s)^2) &\geq e^{-Ct} \log^m(1/s^2) \\ \implies \log^m(1/\Gamma_t(s)) &\geq (1/2)e^{-Ct} \log^m(1/s^2) \geq (1/2)e^{-Ct} \log^m(1/s) \end{aligned}$$

for sufficiently small s by Lemma 6.2.

From Equation (4.1), for $x_1 > 0$,

$$\begin{aligned}
v^1(t, x_1, 0) &\geq C(1 - \lambda')\omega^0(\Gamma_t(2^{\lambda/2}x_1^\lambda), 0)x_1 \log(1/x_1) \\
&\geq C(1 - \lambda') \log^m(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))x_1 \log(1/x_1) \\
&\geq C'e^{-Ct} \log^m(1/2^{\lambda/2}x_1^\lambda)x_1 \log(1/x_1) \\
&\geq C'e^{-Ct} \log^m(1/x_1)x_1 \log(1/x_1)
\end{aligned} \tag{6.2}$$

as long as x_1 is sufficiently small.

Specializing to the case $m = 2$, we have

$$v^1(t, x_1, 0) \geq Ce^{-Ct} \log^2(1/x_1)x_1 \log(1/x_1) = Ce^{-Ct}x_1\theta_2(1/x_1).$$

Solving for

$$\frac{dx_1(t)}{dt} = Ce^{-Ct}x_1\theta_2(1/x_1) \tag{6.3}$$

with $x_1(0) = a$, we get

$$\log^3(1/x_1(t)) = \log^3(1/a) + C(e^{-Ct} - 1),$$

so

$$\begin{aligned}
\psi^1(t, a, 0) \geq x_1(t) &= \exp\left(-(-\log a)^{\exp(C(e^{-Ct}-1))}\right) \\
&= e^{-(\log a)^\gamma},
\end{aligned}$$

where $\gamma = \exp(C(e^{-Ct} - 1))$.

Observe that $\gamma < 1$ for all $t > 0$. Thus, for any α in $(0, 1)$ and all $t > 0$,

$$\begin{aligned}
\|\psi\|_{C^\alpha} &\geq \lim_{a \rightarrow 0^+} \frac{\psi^1(t, a, 0) - \psi^1(t, 0, 0)}{a^\alpha} \geq \lim_{a \rightarrow 0^+} \frac{x_1(t)}{a^\alpha} \\
&= \lim_{a \rightarrow 0^+} \frac{e^{-(\log a)^\gamma}}{e^{(-\log a)^\alpha}} = \lim_{u \rightarrow \infty} \frac{e^{-u^\gamma}}{e^{-\alpha u}} = \lim_{u \rightarrow \infty} e^{\alpha u - u^\gamma} = \infty.
\end{aligned} \tag{6.4}$$

We conclude that the flow lies in no Hölder space of positive exponent for all positive time, a result that we state explicitly as a corollary of Theorem 4.1.

Corollary 6.1. *There exists initial velocities satisfying the conditions of Theorem 2.4 for which the unique solution to (E) has an associated flow lying, for all positive time, in no Hölder space of positive exponent.*

We used the following lemma above:

Lemma 6.2. *Let m be a positive integer. Then for sufficiently small positive x ,*

$$\log^m(1/x) \geq (1/2) \log^m(1/x^2).$$

Proof. The proof is by induction. For $x < 1$,

$$\log(1/x^2) = 2 \log(1/x),$$

which establishes the inequality (in fact, equality) for $m = 1$. So assume the inequality holds for all positive integers up to m . Then

$$\begin{aligned}\log^{m+1}(1/x^2) &= \log \log^{m-1}(1/x^2) \leq \log(2 \log^{m-1}(1/x)) \\ &= \log 2 + \log^m(1/x) \leq 2 \log^m(1/x)\end{aligned}$$

as long as x is sufficiently small that $\log 2 \leq \log^m(1/x)$. \square

7. DIFFICULTIES FOR $m > 2$

When $m > 2$, the right-hand side of Equation (6.3) is no longer exactly integrable, so it becomes difficult to obtain a lower bound on the modulus of continuity of the flow. And even if one could obtain such a lower bound, the severity of the singularity in $|\omega^0|$ decreases with increasing m , so we should not expect to obtain a worse bound on the modulus of continuity of the flow.

An alternate approach, and the one followed in [5], is to use

$$\omega^0(x_1, 0) = \log^2(1/x_1) \cdots \log^m(1/x_1),$$

in place of Equation (6.1), because one can show using the same approach as in the proof of Lemma A.1 that

$$\|\omega^0\|_{L^p} \sim \log p \cdots \log^{m-1} p.$$

The severity of the singularity in $|\omega^0|$ increases with increasing m , so there is hope of obtaining a worse bound on the modulus of continuity of the flow. The equivalent of Equation (6.2), however, becomes difficult to work with, because we must also bound $\log^k(1/\Gamma_t(s))$ for $k = 2, \dots, m-1$. Attempting to do this introduces powers of these lower logarithms with exponents that depend upon t , which again makes the analog of Equation (6.3) not exactly integrable.

Yet another approach is to assume that ω^0 is SBQ, but is constant along vertical line segments (in each quadrant), though supported in a square. This would still allow the inner integral in the first expression for $v_1(x_1, 0)$ in the proof of Lemma 3.3 to be calculated exactly. One would then need to choose the dependence of ω^0 on the x_1 coordinate carefully to produce the equivalent of Equation (4.1). It is not at all clear that this approach has any advantages, however.

Most fundamentally, if we are assuming SBQ, then the function $f = f_1 - f_2$, where f_1 and f_2 are defined in Equation (3.3), controls the bound, and this function is continuous except for a singularity at $y = (x_1, 0)$, where it goes to positive infinity (on Q_1). Thus, whatever lower bound we derive on $v_1(x_1, 0)$, it will blow up, and only blow up, at a singularity of $|\omega(t)|$ that lies along the x -axis (this follows from Equation (3.2)). The lower bound on the modulus of continuity of the velocity then follows from allowing a point $a = (a_1, 0)$ to approach the singularity and looking at how large the appropriate difference quotient becomes, as in Equation (6.4). But to do this, we need control on the position of the singularity of $|\omega(t)|$, and,

when assuming SBQ , the origin is the one point at which we have the most control—the singularity doesn't move at all.

Thus, the assumption of SBQ naturally leads us to assume a point singularity at the origin. Then, because it appears that we can only bound from below the effect on $v_1(x_1, 0)$ of the vorticity outside of the square on which a point lies (actually, an even large square because of the exponent λ in Lemma 3.5), we are naturally led to the assumption that $|\omega^0|$ decreases with the distance from the origin, which leads to Lemma 3.5. Perhaps, then, one possible way out of our difficulties is to obtain a lower bound on $v_1(x_1, 0)$ that accounts for the vorticity all the way to the origin, but this may simply be impossible.

Of course, another possibility is to break the SBQ symmetry, but then obtaining any lower bounds using the Biot-Savart law would probably become extremely difficult.

A final possibility is that the difficulties encountered are an indication that, in fact, the lower bound on the modulus of continuity does not become arbitrarily bad. Perhaps one should seek an upper bound, maybe looking first at Yudovich's examples for $m > 2$, for initial vorticities having isolated point singularities with (or maybe even without) local SBQ .

APPENDIX A. LOGARITHMIC SINGULARITIES

We show how to bound the L^p -norms of a function with a singularity like those we use in Section 6.

Lemma A.1. *Let ω^0 have the symmetry described in Theorem 4.1 with*

$$\omega^0(x_1, 0) = \log^m(1/x_1),$$

for x_1 in $(0, \exp^{-m}(0))$, and ω^0 equal to zero elsewhere in the first quadrant. Then

$$\|\omega^0\|_{L^p} \sim \log^{m-1} p$$

for large p , where $C = 1$ for $m \geq 2$.

Proof. Because of the symmetry of ω^0 ,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &= 4 \int_0^{\exp^{-m}(0)} 2 \int_0^{x_1} (\omega^0(x_1, 0))^p dx_2 dx_1 \\ &= 8 \int_0^{\exp^{-m}(0)} x_1 [\log^m(1/x_1)]^p dx_1. \end{aligned} \tag{A.1}$$

Making the change of variables, $u = \log(1/x_1) = -\log x_1$, it follows that $x_1 = e^{-u}$ and $du = -(1/x_1) dx_1$ so $dx_1 = -e^{-u} du$. Thus,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &= 8 \int_{\infty}^{\exp^{m-1}(0)} e^{-u} [\log^{m-1} u]^p (-e^{-u}) du \\ &= 8 \int_{\exp^{m-1}(0)}^{\infty} e^{-2u} [\log^{m-1} u]^p du. \end{aligned}$$

Making the further change of variables $x = u/p$, so that $u = px$ and $du = p dx$, we have

$$\|\omega^0\|_{L^p}^p = 8p \int_{\exp^{m-1}(0)/p}^{\infty} e^{-2xp} [\log^{m-1}(xp)]^p dx. \quad (\text{A.2})$$

Obtaining a lower bound on $\|\omega^0\|_{L^p(\mathbb{R}^2)}$ is easy. For $x \geq 1$,

$$\log^{m-1}(xp) \geq \log^{m-1} p,$$

so

$$\begin{aligned} \|\omega^0\|_{L^p}^p &\geq 8p \int_{\exp^{m-1}(0)/p}^{\infty} e^{-2xp} [\log^{m-1} p]^p dx \\ &= 8p [\log^{m-1} p]^p \int_{\exp^{m-1}(0)/p}^{\infty} e^{-2xp} dx \\ &= 8p [\log^{m-1} p]^p \left(-\frac{1}{2p}\right) [e^{-2xp}]_{\exp^{m-1}(0)/p}^{\infty} \\ &= 4 \exp^{-2(m-1)(0)} [\log^{m-1} p]^p. \end{aligned}$$

Thus, asymptotically, $\|\omega^0\|_{L^p} \geq \log^{m-1} p$.

We now obtain an upper bound on $\|\omega^0\|_{L^p}$. For $x \leq 1$,

$$\log^{m-1}(xp) \leq \log^{m-1} p,$$

while for $x \geq 1$ and sufficiently large p , Equation (A.3) holds. Thus,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &\leq 8p \left(\int_0^1 + \int_1^{\infty} \right) e^{-2xp} [\log^{m-1}(xp)]^p dx \\ &\leq 8p \int_0^1 e^{-2xp} [\log^{m-1} p]^p dx \\ &\quad + 8p \int_1^{\infty} e^{-2xp} [[\log^{m-1} p] e^{x-1}]^p dx \\ &\leq 8p [\log^{m-1} p]^p \left[\int_0^1 e^{-2xp} dx + e^{-p} \int_1^{\infty} e^{-xp} dx \right] \\ &= 8p [\log^{m-1} p]^p \left[\frac{1}{2p} (1 - e^{-2p}) + e^{-p} \frac{e^{-p}}{p} \right] \\ &\leq 8 [\log^{m-1} p]^p. \end{aligned}$$

It follows that for sufficiently large p , $\|\omega^0\|_{L^p} \leq 8^{1/p} \log^{m-1} p$, so asymptotically, $\|\omega^0\|_{L^p} \leq \log^{m-1} p$, which completes the proof. \square

Remark A.1. Essentially the same proof gives the same result for radial or other symmetries (except that e^{-2} might become a different constant), as well as for any spatial dimension. In dimension n with radial symmetry, for example, the factor of x_1 in the original integration in Equation (A.1) would become r^n and the factor of 8 would become the surface area of

the unit sphere. The r^n factor would become $e^{-n xp}$ (even for $n = 1$) in Equation (A.2), which does not materially affect the rest of the argument.

Lemma A.2. *Let m be a positive integer. Then for sufficiently large p ,*

$$\log^m(xp) \leq (\log^m p)e^{x-1} \quad (\text{A.3})$$

for all $x \geq 1$.

Proof. We prove this for $m = 2$, the proof for other values of m being entirely analogous. First, by taking the logarithm of both sides of Equation (A.3), that equation holds if and only if

$$f(x) := \log \log \log(xp) \leq g(x) := \log(\log p \log \log p) + x - 1.$$

Because equality holds for $x = 1$, our result will follow if we can show that $f' \leq g'$ for all $x \geq 1$ and sufficiently large p . This is, in fact, true, since

$$f' = \frac{1}{x \log(xp) \log \log(xp)} \leq 1 = g'$$

for all $x \geq 1$ and $p \geq e^e$. □

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