ANALYTIC CONTINUATION OF MULTIPLE DIRICHLET SERIES
OF EULER-ZAGIER TYPE

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Abstract. We define a class of multiple Dirichlet series, or $L$-functions, in $d$-complex dimensions generalizing Zagier’s multiple zeta function ([21]), and study their meromorphic continuation using the theory of distributions. Our approach is to view the multiple $L$-function as the pairing of a distribution $\Psi$ meromorphic on $\mathbb{C}^d$ with a test function $g$ defined on $\mathbb{R}^d$. We show that to continue the $L$-function, it is sufficient for $g$ to be in the Schwartz class $S((0,1) \times (0,\infty)^{d-1})$ by constructing an extension to $S(\mathbb{R}^d)$. We also show that we can compensate for certain types of singularities in $g$ by moving these singularities to $\Psi$, giving a meromorphic continuation of the $L$-function at the expense of additional poles. We adapt our results to the continuation of a multiple Hurwitz $L$-function. We apply our techniques to continue various single and multiple $L$-functions: 1) the single $L$-function associated to a twisted cusp form; 2) the single $L$-function whose coefficients are given by the harmonic numbers (also evaluating the $L$-function at the negative even integers); 3) multiple-dimensional $L$-functions with periodic and quasi-periodic coefficients; 4) the double $L$-function whose coefficients in each single $L$-function are the harmonic numbers.

1. Introduction

A $d$-dimensional multiple Dirichlet series, $d \geq 1$, is a function of the form

$$L(s) = \int a(n_1, \ldots, n_d)n_1^{-s_1} \cdots n_d^{-s_d} d\mu(n_1, \ldots, n_d)$$

(1.1)

where $s = (s_1, \ldots, s_d)$, $(X, \mu)$ is a measure space, $a$ is $\mu$–measurable, and the integral is over $X$. For certain choices of $a$, $L$ will map a domain of $C^d$ into $\mathbb{C}$. Our primary interest is the analytic continuation of $L$ beyond this domain, and our primary tools are the Mellin transform and the theory of the analytic continuation of tempered distributions.

Special cases of Equation (1.1) include classical single-dimensional Dirichlet series, where $\mu$ is a one-dimensional discrete measure, and the multiple Dirichlet series of [6], where $\mu$ is a product measure containing both discrete and continuous (Lebesgue) measures on $\mathbb{R}$. Another special case of Equation (1.1) is an $L$-function of Euler-Zagier type, which we study in this paper. In [21], Zagier defined the multiple zeta function of depth $d$,

$$\zeta_d(s_1, s_2, \ldots, s_d) = \sum_{0 < n_1 < \cdots < n_d} n_1^{-s_1} \cdots n_d^{-s_d}, \quad s = (s_1, \ldots, s_d) \in \mathbb{C}^d,$$
which is absolutely convergent and analytic in the region ([16])

\[ \text{Re}(s_k + \cdots + s_d) > d - k + 1 \quad \text{for} \quad k = 1, \ldots, d. \quad (1.2) \]

Through the work of many mathematicians, including Beilinson, Deligne, Goncharov, Kontsevich, and Zagier, the multiple zeta function has played an important role in the development of arithmetic geometry over the past decade (see for example [4, 8, 9, 18, 21, 22]). In this paper, we define a class of multiple \( L \)-functions generalizing the multiple zeta function, and use analytic continuation of tempered distributions to prove a sufficient condition for meromorphic continuation to \( \mathbb{C}^d \). Our theory is applied to prove meromorphic continuation of various single and multiple \( L \)-functions, including the double \( L \)-function whose coefficients are given by the harmonic numbers \( H_n \),

\[
\sum_{0 < m < n} H_m H_n m^{-s_1} n^{-s_2},
\]

which is absolutely convergent and analytic for \( \text{Re}(s_1 + s_2) > 2 \) and \( \text{Re}(s_2) > 1 \). In the knowledge of the authors, this is the first double \( L \)-function of this type shown to have a meromorphic continuation to \( \mathbb{C}^2 \) that is not derived from a double \( L \)-function at least one of whose coefficients is periodic or quasi-periodic.

More generally, let

\[
(a_{n,k})_{n=1}^\infty, \quad k = 1, \ldots, d,
\]

be \( d \) sequences of complex numbers, and assume that each sequence satisfies the growth condition

\[
|a_{n,k}| \leq C_k n^{r_k}, \quad k = 1, \ldots, d, \quad (1.3)
\]

for some nonnegative real constants \( C_k \) and \( r_k \). Also assume that no sequence \((a_{n,k})_{n=1}^\infty, \quad k = 1, \ldots, d\), is identically zero. We define the multiple \( L \)-function of depth \( d \),

\[
L(s) = \sum_{0 < n_1 < \cdots < n_d} a_{n_1,1} \cdots a_{n_d,d} n_1^{-s_1} \cdots n_d^{-s_d}.\quad (1.4)
\]

By the absolute region of convergence for the multiple zeta function in Equation (1.2) and a simple estimate using Equation (1.3), it follows that \( L \) is absolutely convergent and analytic in the region

\[
\text{Re}(s_k + \cdots + s_d) > d - k + 1 + r_k + \cdots + r_d \quad \text{for} \quad k = 1, \ldots, d. \quad (1.5)
\]

One can also view an \( L \)-function of depth \( d \) as being constructed from \( d \) single-dimensional \( L \)-functions, \( L_1, \ldots, L_d \), where the coefficients for \( L_k \) are \((a_{n,k})_{n=1}^\infty\).

Various examples of multiple \( L \)-functions of the form Equation (1.4) have appeared in the literature, including the multiple Dirichlet \( L \)-function in [9, 2, 19, 15], and the multiple Dedekind zeta function in [24, 15]. The first of these functions is obtained by letting each \( L_k \) be the \( L \)-function of a Dirichlet character, and the second is obtained by letting each \( L_k \) be the Dedekind zeta function of a number field.

It was believed for some time that the multiple zeta function had no unique meromorphic continuation to \( \mathbb{C}^d \). This is because certain points outside of its domain of absolute convergence can be approached along different paths to obtain distinct
finite limits. It was recognized, however, by Goncharov and Kontsevich in [9] and Zhao in [23] that such points are special types of singularities called points of indeterminacy (see [11]), and these authors established meromorphic continuation of the multiple zeta function using a distribution-theoretic approach. Meromorphic continuation was also established by Akiyama, Egami, and Tanigawa in [1] and Masri in [15] using different variants of the Euler-Maclaurin summation formula.

Our goal in this paper is to develop a unified approach to meromorphic continuation of multiple $L$–functions of the form Equation (1.4) (actually, of a slightly more general form, as described in Section 6). We use meromorphic continuation of tempered distributions (see [7]), with the construction of an extension operator on a certain Schwartz class of functions to determine a sufficient condition for Equation (1.4) to have a meromorphic continuation to $\mathbb{C}^d$. We also specialize to the one-dimensional case of Equation (1.4), where information about the poles and residues of the $L$–functions can be more easily determined. Many of the techniques we develop apply as well to the more general multiple Dirichlet series of Equation (1.1).

This paper is organized as follows: In Section 2 we briefly present the necessary background on the analytic continuation of tempered distributions. In Section 3, we describe our approach to meromorphic continuation and its genesis in [9] and [23]. In Section 4, we state our main results, which give techniques for meromorphic continuation of single and multiple $L$–functions of the form Equation (1.4), and in Section 5 we prove these results. In Section 6, we define a multiple Hurwitz $L$–function, and indicate how the results of Section 4 can be adapted to continue this new function.

In the next four sections we apply our techniques to the continuation of various single- and multiple-dimensional $L$–functions. In Section 7 we continue the $L$–function $L(s, \Phi \otimes \chi)$ associated to a cusp form $\Phi$ twisted by a Dirichlet character $\chi$, and in Section 8 we continue the single $L$–function whose coefficients are given by the harmonic numbers. In Section 9, we continue multiple $L$–functions with periodic and quasi-periodic coefficients, and in Section 10 we continue the double $L$–function $H$ whose coefficients in each single $L$–function are the harmonic numbers.

Finally, in Appendix A we prove some analytic results which are used in a fundamental way in the proofs in Section 5. Most important is Lemma A.2, where we construct an extension operator from the Schwartz class $S((0,1) \times (0,\infty)^{d-1})$ to $S(\mathbb{R}^d)$.

2. Background on tempered distributions

We give here a brief overview of the analytic continuation of tempered distributions, referring the reader to [7], especially Sections 3.1-3.3 of Chapter 1, for a more in depth discussion. Section 2 of [23] is also helpful in this regard.

Let $\Omega$ be an open subset of $\mathbb{R}^d$. Then $S(\Omega)$, the *Schwartz-class functions* on $\Omega$, are defined to be the set of all complex-valued $C^\infty$–functions $f$ on $\Omega$ such that

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \Omega} |x^\alpha D^\beta f(x)| < \infty$$

(2.1)
for all \((d\text{-dimensional})\) multi-indices \(\alpha\) and \(\beta\). A multi-index \(\alpha\) is an ordered pair of \(d\) nonnegative integers \((\alpha_1, \ldots, \alpha_d)\), \(x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}\), and

\[
D^\alpha f := \frac{\partial |\alpha| f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},
\]

where \(|\alpha| = \alpha_1 + \cdots + \alpha_d\). This definition is equivalent to that of [7] and [23] when \(\Omega = \mathbb{R}^d\), and it is only in this case that we use \(S(\Omega)\) as a space of test functions.

Endowed with the sufficient family of semi-norms, \(\{\rho_{\alpha, \beta}\}, S(\mathbb{R}^d)\) is a Fréchet space.

We will also make use of the space \(S^\gamma(\Omega)\), where \(\gamma\) is a multi-index in which we allow any \(\gamma_i\) to equal \(\infty\), defined to be the space of all complex-valued \(C^\infty\)-functions \(f\) on \(\Omega\) such that Equation (2.1) holds for all multi-indices \(\alpha\) and all multi-indices \(\beta\) such that \(\beta \leq \gamma\)—that is, \(\beta_k \leq \gamma_k\) for \(k = 1, \ldots, d\).

A \((tempered)\) distribution is an element of \(S'(\mathbb{R}^d)\), the dual space of \(S(\mathbb{R}^d)\); that is, the set of all continuous linear functionals on \(S(\mathbb{R}^d)\), continuity being with respect to all the semi-norms \(\rho_{\alpha, \beta}\) separately. A distribution \(\psi\) applied to a test function \(\varphi\) in \(S(\mathbb{R}^d)\) is written as \((\psi, \varphi)\), the operation \((\cdot, \cdot)\) or, more explicitly, \((\cdot, \cdot)_{S'(\mathbb{R}^d), S(\mathbb{R}^d)}\), defining a pairing of \(S'(\mathbb{R}^d)\) and \(S(\mathbb{R}^d)\). If, for some locally integrable function \(\psi\), \((\psi, \varphi) = \int_{\mathbb{R}^d} \overline{\psi} \varphi\) for all \(\varphi\) in \(S(\mathbb{R}^d)\), then the distribution is called a \(regular\) distribution, and \(\psi\) and \(\overline{\psi}\) are normally identified.

A distribution \(\psi\) is analytic (meromorphic) if for any test function \(\varphi\) in \(S(\mathbb{R}^d)\), \((\psi, \varphi)\) is analytic (meromorphic) in some domain in \(\mathbb{C}^d\). If \(\psi\) is regular and analytic on some domain of \(\mathbb{C}^d\) and, for any test function \(\varphi\) in \(S(\mathbb{R}^d)\), \((\psi, \varphi)\) analytically continues to an analytic or meromorphic function, then \(\psi\) is said to analytically continue to an analytic or meromorphic distribution. (For short, we will sometimes say that the distribution meromorphically continues.) A region on which \(\psi\) is regular and analytic is called a region of absolute convergence of \(\psi\).

We will also have a need for the tensor product of distributions, which we define as follows. Let \(\psi_1\) and \(\psi_2\) be distributions in \(S'(\mathbb{R}^{d_1})\) and \(S'(\mathbb{R}^{d_2})\), and let \(\varphi\) be in \(S(\mathbb{R}^{d_1+d_2})\). Then the functions

\[
\varphi_1(x_1) = (\psi_2(\cdot), \varphi(x_1, \cdot))_{S'(\mathbb{R}^{d_2}), S(\mathbb{R}^{d_1})} \quad \text{and} \quad \varphi_2(x_1) = (\psi_1(\cdot), \varphi(\cdot, x_2))_{S'(\mathbb{R}^{d_1}), S(\mathbb{R}^{d_2})}
\]

are in \(S(\mathbb{R}^{d_1+d_2})\); this follows from the general fact that for any fixed \(f\) in \(S'(\mathbb{R}^{d_1})\) and \(\varphi\) in \(S(\mathbb{R}^{d_1+d_2})\), \(x \mapsto (f(\cdot), \varphi(x, \cdot))\) lies in \(S(\mathbb{R}^{d_2})\), along with the symmetric relation with the order of the variables transposed. Then we define \(\psi_1 \otimes \psi_2\) by

\[
(\psi_1 \otimes \psi_2, \varphi)_{S'(\mathbb{R}^{d_1+d_2}), S(\mathbb{R}^{d_1+d_2})} := (\psi_1, \varphi_1)_{S'(\mathbb{R}^{d_1}), S(\mathbb{R}^{d_1})} = (\psi_2, \varphi_2)_{S'(\mathbb{R}^{d_2}), S(\mathbb{R}^{d_2})}. \quad (2.2)
\]

To show that this definition is consistent, we must show that equality holds in the last two expressions in Equation (2.2). So suppose first that \(\varphi = \varphi_1 \otimes \varphi_2\). Then, writing Equation (2.2) more concisely as \((\psi_1 \otimes \psi_2, \varphi) = (\psi_1, (\psi_2, \varphi)) = (\psi_2, (\psi_1, \varphi))\), we have

\[
(\psi_1, (\psi_2, \varphi)) = (\psi_1, (\psi_2, \varphi_1 \varphi_2)) = (\psi_1, \varphi_1(\psi_2, \varphi_2)) = (\psi_2, \varphi_2)(\psi_1, \varphi_1)
\]

\[
= (\psi_2, (\psi_1, \varphi)),
\]

where in the second and third equalities we used the linearity of the pairings involved, and in the final equality we used the symmetric equality with the order of \(\psi_1\) and \(\psi_2\).
transposed. This shows that Equation (2.2) is well-defined for test functions that are product-form and hence by linearity for all test functions in $S(\mathbb{R}^{d_1}) \otimes S(\mathbb{R}^{d_2})$. But $S(\mathbb{R}^{d_1}) \otimes S(\mathbb{R}^{d_2})$ is dense in $S(\mathbb{R}^{d_1+d_2})$ (a nontrivial fact) so the definition is, in fact, well-defined for all distributions in $S(\mathbb{R}^{d_1+d_2})$.

Equation (2.2) can also be seen as the analog of Fubini’s theorem for tempered distributions. In fact, it follows for regular distributions by an application of Fubini’s theorem, and hence is a natural definition of the tensor product of two distributions.

3. Description of our approach

As mentioned in Section 1, the special case of the multiple zeta function, in which $a_{n,k} = 1$ for all $n$ and $k$, is treated in [9] and [23], where it is shown that the zeta function continues to a meromorphic function on $\mathbb{C}^d$, and the locations of the poles, all of which are simple, are identified. The authors of [9] and [23] employ a multi-dimensional Mellin transform to show that the zeta function equals, in its region of absolute convergence, a meromorphic distribution applied to a test function. This gives an explicit continuation of the zeta function to all of $\mathbb{C}^d$, the poles being the same as those of the distribution.

The meromorphic distribution in [9] and [23] is

$$\Psi(x_1, \ldots, x_d; s_1, \ldots, s_d) = \frac{\Gamma(u_1) \cdots \Gamma(u_{d-1})}{(s_d - 1)} \psi(x_1, \ldots, x_d; s_1, \ldots, s_d), \quad (3.1)$$

where

$$u_k = s_k + \cdots + s_d - d + k - 1 \quad (3.2)$$

and

$$\psi(x_1, \ldots, x_d; s_1, \ldots, s_d) = \frac{(x_1)^{u_1-1}}{\Gamma(u_1)} \prod_{k=2}^{d} \frac{(1 - x_k)^{s_k-1} (x_k)^{u_k-1}}{\Gamma(s_k-1) \Gamma(u_k)}, \quad (3.3)$$

where $t_+ := t \mathbf{1}_{(0,\infty)}$. This is a regular distribution in the domain of $\mathbb{C}^d$ where $s_1 > 0, \ldots, s_{d-1} > 0$ and $u_1 > 0, \ldots, u_d > 0$. Because, $\psi$ continues to an entire distribution as we show in Lemma 5.2, the poles of $\Psi$ arise from the other factors in Equation (3.1).

The test function of [9] and [23] is defined on

$$R = (0, \infty) \times (0, 1)^{d-1} \quad (3.4)$$

by

$$g(x) = g(x_1, \ldots, x_d) = \frac{y_1(x) \cdots y_d(x)}{(e^{y_1(x)} - 1) \cdots (e^{y_d(x)} - 1)},$$

and

$$y_k = y_k(x) = x_1 \cdots x_k. \quad (3.5)$$

Since $\Psi$ is zero for $x$ outside of $R$, the value of the pairing $(\Psi, g)$ does not depend upon the value of $g$ outside of $R$; nonetheless, it is essential that $g$ extend to a test function on all of $\mathbb{R}^d$. (One way to see that $g$ extends for the zeta function is to view $g$ as a function of $y = y(x)$. It then factors into a product form, each factor of which is in $S((0, \infty))$ by the argument in the proof of Corollary 4.7, below; hence, $g(y)$ is in
$S(g^{-1}(R))$. By Lemma A.1, $g(x)$ is thus in $S(R)$, which we show in Lemma A.2 is a sufficient (and, of course, necessary) condition to insure that $g(x)$ extends to $S(R^d).$

If we apply the continuation argument of [9] and [23] to arbitrary L–functions then, as we show in the proof of Theorem 4.1, the only thing that changes is the nature of the test function $g$, which is now equal to

$$g(x) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+n_d,d} y_1(x) e^{-n_1 y_1(x)} \cdots y_d(x) e^{-n_d y_d(x)}.$$  

(3.6)

Although $g$ is always in $C^\infty(R)$ and in $S((\epsilon, \infty) \times (\epsilon', 1)^{d-1}$ for all $\epsilon > 0$ and $\epsilon'$ in (0, 1) (see the comment following the proof of Theorem 4.1), it will no longer necessarily extend to $S(R^d)$ because it need no longer lie in $S(R)$, its behavior on the boundary of $R$ (specifically, on $\{ x \in R^d : x_1 \cdots x_d = 0 \}$) not being controllable a priori. (Hence, we are abusing terminology when we refer to $g$ as a test function when it is not in $S(R)$ and so does not extend to a true test function in $S(R^d)$.)

Our main innovation is to circumvent this problem, in many cases, by shifting the poor behavior of the test function $g$ near the boundary of $R$ to the distribution $\Psi$ by dividing $g$ by a function $A$ and multiplying $\Psi$ by the same function $A$, giving a new test function and distribution. Since $\int_R (\Psi A)(g/A) = \int_R \Psi g$, if $A$ can be chosen so that $g/A$ is in $S(R)$ and $\Psi A$ is meromorphic (though with more poles than $\Psi$), then the continuation argument can be completed as in [9] and [23], though now there will be more poles not all of which need be simple. This approach leads to Theorem 4.1.

In one dimension, the distributional approach is not so different from the classical one-dimensional approach using the Mellin transform; the differences lie more in language than in mathematical content. It is in multiple dimensions where the advantage of the distributional approach becomes more evident, because of the greater complexity of the singularities in the multi-dimensional Mellin transform. Distributions provide a framework within which it is more natural to apply powerful tools from harmonic analysis, such as Stein’s extension argument, which we adapt in the proof of Lemma A.2.

4. Statements of the main results

Theorem 4.1. Assume that Equation (1.3) is satisfied, and let $A: R \rightarrow \mathbb{C}$ be such that the following two conditions hold:

1. $g/A$ is in $S(R)$, and
2. $\psi A$ is a meromorphic distribution on $\mathbb{C}^d$ (here $A$ is actually an arbitrary extension of $A$ to all of $\mathbb{R}^d$).

Then by necessity, the absolute region of convergence of $\psi A$ (that is, the region on which $\psi A$ is a regular distribution) contains the region defined by

$$\text{Re } u_k > r_k + \cdots + (d - k + 1)r_d, \ k = 1, \ldots, d \text{ and } \text{Re } s_k > 0, \ k = 1, \ldots, d - 1,$$

and $L$ meromorphically continues to all of $\mathbb{C}^d$ with the following possible poles:

1. a simple pole along the hyperplane $s_d = 1$;
(2) a simple pole along the hyperplane $u_k = n$ for all integers $n \leq 0$ for $k = 1, \ldots, d - 1$;

(3) the same poles and same orders as those of $\psi A$.

If, however, a possible pole in (3) matches one in (1) or (2), then the possible order of the pole increases by 1.

Remark: The term pole has a dual meaning. In Theorem 4.1 and the corollaries that follow, we are using the interpretation given on p. 168 of [11] of a pole as a holomorphic subvariety of dimension $d - 1$, the order of the pole being defined on p. 168 of [11] as well. The points of this subvariety may contain both points of indeterminacy and poles in the sense of the definition on p. 164 of [11]. In dimension 1, the two terms can be identified, and it is only in this case that we will use the second meaning of the term pole.

A pole is either empty or is a subvariety of dimension $d - 1$. The points of indeterminacy form a subvariety that is either empty or of dimension $d - 2$ (in dimension 1, then, there are no points of indeterminacy).

When we say that $S$ is the set of possible poles of a function, we mean that the set of all poles of the function is a subset of $S$. When we say that the possible order of a pole is $n$, it means that the pole has an order greater-than-or-equal-to $n$ (and so might not be a pole at all).

The region in Equation (4.1) is smaller than Matsumoto’s region of absolute convergence of $L$ in Equation (1.5).

There are, of course, practical difficulties in applying Theorem 4.1. The first difficulty one faces is in obtaining the original test function $g$, or at least sufficient information about it to understand its behavior on the boundary of $R$. A functional relationship on the coefficients or on their generating function is usually needed. If $g$ can be found, a function $A$ can be determined by the asymptotic behavior of $g$ near the boundary of $R$.

The second difficulty is in determining whether $A$ meets the second criterion of Theorem 4.1, and if so, the location and order of the poles of $\psi A$ that result. The conditions on $A$ in Theorem 4.2, Corollary 4.4, and Corollary 4.6 give fairly general and practicable sets of conditions to address the second difficulty.

Both of these difficulties are considerably lessened in one dimension, where it is often possible to obtain a closed form expression for the test function $g$, and where we need only understand the behavior of $g$ near zero.

Theorem 4.2. Condition (2) of Theorem 4.1 ($\psi A$ is meromorphic) is met if $A(x) = A_1(x_1) \cdots A_d(x_d)$ and each $\psi_k A_k$ is a meromorphic distribution on $u_1$ in $\mathbb{C}$ for $k = 1$ and on $(s_{k-1}, u_k)$ in $\mathbb{C}^2$ for $k > 1$, where

$$\psi_1 = \frac{(x_1)^{u_1-1}}{\Gamma(u_1)} \text{ and } \psi_k = \frac{(1 - x^u_k)^{(s_k-1)}(x_k)^{u_k-1}}{\Gamma(s_{k-1})\Gamma(u_k)}, k = 2, \ldots, d. \quad (4.2)$$

(Here, we are treating $u_1, \ldots, u_d$ and $s_1, \ldots, s_{d-1}$ as independent complex variables.)

Furthermore, the poles of $\psi A$ are precisely those of each $\psi_k A_k$, $k = 1, \ldots, d$, though now as codimension-one subvarieties of $\mathbb{C}^d$ rather than of $\mathbb{C}$ or $\mathbb{C}^2$. (For instance, if
\(d = 4\) and \(\psi_2A_2\) has a pole along the one-dimensional hyperplane \(s_1 = 2\) of \(\mathbb{C}^2\), then \(\psi A\) has a pole along the three-dimensional hyperplane \(s_1 = 2\).

**Definition 4.3.** The function \(g : \mathbb{R} \rightarrow \mathbb{R}\) has the singularity type of the function \(A : \mathbb{R} \rightarrow \mathbb{R}\) if \(g = fA\) for some \(f\) in \(S(R)\), where \(R\) is defined in Equation (3.4). We denote the set of all functions with the singularity type of \(A\) by \(\text{Sing } A\). (Observe that \(A\) is not in \(\text{Sing } A\), since the constant function 1 is not in \(S(R)\).)

Corollary 4.4 is a corollary of Theorem 4.1 and Theorem 4.2.

**Corollary 4.4.** Let \(p_1, \ldots, p_d, q_2, \ldots, q_d\) be real numbers, let \(v_1, \ldots, v_d, w_2, \ldots, w_d\) be nonnegative integers, and let
\[
A(x) = x_1^{-p_1} \cdots x_d^{-p_d}(1 - x_2)^{-q_2} \cdots (1 - x_d)^{-q_d}(\log x_1)^{v_1} \cdots (\log x_d)^{v_d}
\]
\[
(\log(1 - x_2))^{w_2} \cdots (\log(1 - x_d))^{w_d}.
\]
If \(g\) is in \(\text{Sing } A\), then \(L = (\Psi, g)\) can be continued to a function meromorphic on \(\mathbb{C}^d\). When \(p_1, \ldots, p_d\) and \(q_2, \ldots, q_d\) are nonnegative integers, \(L\) has possible poles along the following hyperplanes:

1. for \(k = 1, \ldots, d - 1\), poles of order \(w_k + 1\) along \(s_k = 1, \ldots, q_k\) and poles of order \(w_k\) along \(s_k = n\) for each integer \(n \leq 0\);
2. a simple pole along \(s_d = 1\);
3. for \(k = 1, \ldots, d - 1\), poles of order \(v_k + 1\) along \(u_k = n\) for each integer \(n \leq p_k\);
4. poles of order \(v_d + 1\) along \(u_d = 1, \ldots, p_d\) and poles of order \(v_d\) along \(u_d = n\) for each integer \(n \leq 0\).

As we will see (in Equation (5.5)), \(g\) is always bounded by a function like \(A\) of Corollary 4.4. It is a much stronger statement, however, to say that \(g\) is in \(\text{Sing } A\).

**Corollary 4.5.** Assume that Equation (1.3) is satisfied and define the function \(h : (0, \infty)^d \rightarrow \mathbb{C}\) by
\[
h(y) = \sum_{n_1 = 1}^{\infty} a_{n_1} y_1 e^{-n_1 y_1} \cdots \sum_{n_d = 1}^{\infty} a_{n_1 + \cdots + n_d} y_d e^{-n_d y_d}, \quad (4.3)
\]
where \(y = (y_1, \ldots, y_d)\). Further, suppose that for some multi-index \(\alpha\), \(y^\alpha h(y)\) is in \(S(\Delta)\), where
\[
\Delta = \{ y \in (0, \infty)^d : y_2 < y_1, y_3 < y_1 y_2, \ldots, y_d < y_1 \cdots y_{d-1} \}. \quad (4.4)
\]
Then \(L\) meromorphically continues to \(\mathbb{C}^d\). When \(d \geq 2\), there are possible simple poles along the following hyperplanes:

1. \(s_d = 1\) and
2. \(u_k = n\) for all integers \(n \leq \alpha_k + \cdots + \alpha_d\) for \(k = 1, \ldots, d - 1\).
3. \(u_d = n\) for all positive integers \(n \leq \alpha_d\).

When \(d = 1\), there are possible simple poles at \(s_1 = 1, \ldots, 1 + \alpha_1\), with a residue of
\[
\frac{(-1)^{\alpha_1 + 1 - k}}{(k - 1)! (\alpha_1 + 1 - k)!} \lim_{x_1 \to 0^+} \frac{d^{\alpha_1 + 1 - k}}{dx_1^{\alpha_1 + 1 - k}}(x_1^{\alpha_1} h(x_1)) \quad (4.5)
\]
at \(s_1 = k\), for \(k = 1, \ldots, 1 + \alpha_1\).
In one dimension, we have the following corollary to Theorem 4.1:

**Corollary 4.6.** Assume that Equation (1.3) is satisfied (for $d = 1$) and that for some function $A : \mathbb{R} \to \mathbb{C}$, the following conditions hold:

1. The function $g/A$ is in $\mathcal{S}(\mathbb{R})$.
2. For some $\lambda$ in $(0, 1]$ and $h$ in $[-\infty, \infty)$, the function $x \mapsto x^sA(x)\log x$ is in $L^1((0, \lambda))$ for $\Re s > h$.
3. The function $\mu : \mathbb{C} \to \mathbb{C}$ defined by
   \[ \mu(s) = \int_0^\lambda x^sA(x) \, dx. \] (4.6)
   continues to a meromorphic function on $\mathbb{C}$, with poles at $(q_j)_{j=1}^n$, where $n$ might be infinity, with corresponding orders $(r_j)_{j=1}^n$. (Condition (2) insures that $\mu$ is analytic on the right-half plane $\Re s > h$.)

Then $\Psi A$ and $L$ meromorphically continue to $\mathbb{C}$ with a possible pole of order $r_m$ at $s = 2 - j + q_m$ for $m = 1, \ldots, n$, $j = 0, 1, \ldots$, with the possible pole reduced in order by 1 if $j - q_m$ is an integer $\geq 2$.

If condition (1) is replaced by the requirement that $g/A$ be in $S^\beta(\mathbb{R})$, where $\beta$ is a positive integer or infinity, then $\psi A$ and $L$ meromorphically continue to the region
\[ H = \{ s : \Re s > 2 - \beta + h \}, \]
with possible poles being all those described above that lie in $H$. Also, for any $N < \beta$,
\[ L(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^N \frac{(g/A)^{(j)}(0)}{j!} \mu(s + j - 2) + \frac{r_N(s)}{\Gamma(s)} \] (4.7)
on $\Re s > 1 - N + h$, where $r_N$ is analytic on $H$.

In practice, condition (2) of Corollary 4.6 will easily be seen to hold for any choice of the function $A$ that allows $\mu$ to satisfy condition (3); it is a technical condition that is used in the proof of Corollary 4.6 to switch the order of a derivative and an integral.

In one dimension, we also have the following corollary to Corollary 4.5:

**Corollary 4.7.** Let
\[ G(z) = \sum_{n=1}^\infty a_n z^n, \] (4.8)
with $|a_n| \leq Cn^r$ for all positive integers $n$ and some nonnegative real number $r$.

Then $G$ defines a holomorphic function on the open unit disk in $\mathbb{C}$. Suppose that $G$ meromorphically continues to include a neighborhood of $z = 1$ with a pole of order no more than $1 + \alpha$ at $z = 1$. Then we can always choose $\alpha \leq r$, and $L(s) = \sum_{n=1}^\infty a_n z^{-s}$ meromorphically continues to $\mathbb{C}$ with possible simple poles at $s = 1, \ldots, 1 + \alpha$. When $\alpha = 0$, $L(s)$ has one possible simple pole at $s = 1$ whose residue is the same as the residue of $G(z)$ at $z = 1$. 


The natural extension of Corollary 4.7 to higher dimensions would be to define the multi-variable generating function,

\[ G(z) = \sum_{n_1 = 1}^{\infty} a_{n_1,1} z_1^{n_1} \cdots \sum_{n_d = 1}^{\infty} a_{n_1+\ldots+n_d,d} z_d^{n_d}, \]

where \( z = (z_1, \ldots, z_d) \) is in \( \mathbb{C}^d \), and require that \( G \) continue beyond \( D^d \), where \( D \) is the unit disk in \( \mathbb{C} \), with a pole of order no more than \( \alpha_k + 1 \) along the hyperplanes \( z_k = 1, k = 1, \ldots, d \). This condition is insufficient, however, to establish the continuation of \( L \). Nevertheless, as illustrated in Section 9 and Section 10, often the simplest approach is to obtain a closed-form expression for \( G \) and either apply Corollary 4.4 or Corollary 4.6 using \( g(x) = x_1^d \cdots x_d G(e^{-x_1}, \ldots, e^{-x_1 \cdots x_d}) \), or Corollary 4.5 using \( h(y) = y_1 \cdots y_d G(e^{-y_1}, \ldots, e^{-y_d}) \).

In Corollary 4.5 one can allow the powers of \( y \) (the \( \alpha_j \)) to be arbitrary real numbers, though the locations of the poles and the calculations of the residues would differ. A similar statement can be made about Corollary 4.7.

5. Proofs of the main results

5.1. Proof of Theorem 4.1.

Proof. From Equation (1.4) and the identity,

\[ \Gamma(s) = \int_0^\infty w^{s-1} e^{-w} \, dw \]

we have

\[ L(s) \prod_{j=1}^{d} \Gamma(s_j) = \int_0^\infty \cdots \int_0^\infty \sum_{0<n_1<\cdots<n_d} a_{n_1,1} \cdots a_{n_d,d} \]

\[ n_1^{-s_1} \cdots n_d^{-s_d} w_1^{s_1-1} \cdots w_d^{s_d-1} e^{-w_1} \cdots e^{-w_d} \, dw_1 \cdots dw_d. \]

Making the change of variables \( w_k = n_k t_k \),

\[ n_k^{-s_k} w_k^{s_k-1} e^{-w_k} \, dw_k = n_k^{-s_k} (n_k t_k)^{s_k-1} e^{-n_k t_k} \, n_k \, dt_k = t_k^{s_k-1} e^{-n_k t_k} \, dt_k, \]

so

\[ L(s) \prod_{j=1}^{d} \Gamma(s_j) = \int_0^\infty \cdots \int_0^\infty \sum_{0<n_1<\cdots<n_d} a_{n_1,1} \cdots a_{n_d,d} t_1^{s_1-1} e^{-n_1 t_1} \cdots \]

\[ t_d^{s_d-1} e^{-n_d t_d} \, dt_1 \cdots dt_d = \int_0^\infty \cdots \int_0^\infty \sum_{n_1=1}^{d} \cdots \sum_{n_d=1}^{d} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} t_1^{s_1-1} e^{-n_1 t_1} \cdots \]

\[ t_d^{s_d-1} e^{-(n_1+\cdots+n_d) t_d} \, dt_1 \cdots dt_d = \int_0^\infty \cdots \int_0^\infty t_1^{s_1-1} \cdots t_d^{s_d-1} \varphi(t_1, \ldots, t_d) \, dt_1 \cdots dt_d, \]
where
\[
\varphi(t_1, \ldots, t_d) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} e^{-n_1(t_1+\cdots+t_d)} \cdots e^{-n_d t_d}.
\]

As noted in [9], for the multiple-dimensional zeta-function, \( \varphi \) has a singularity of type \( \prod_{k=1}^{d} (t_k + \cdots + t_d) \), and \( \varphi(t) \) would not be in \( S((0, \infty)^d) \) except when \( d = 1 \). We thus proceed precisely as in [9] and [23], except for the presence of the factor \( a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} \), making the change of variables
\[
t_1 = x_1(1-x_2), \ldots, t_{d-1} = x_1 \cdots x_{d-1}(1-x_d), \ t_d = x_1 \cdots x_d
\]
to obtain
\[
L(s) \frac{s_d-1}{\Gamma(u_1) \cdots \Gamma(u_{d-1})} = \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{x_1^{u_1-1} \cdots x_d^{u_d-1}}{\Gamma(u_1) \cdots \Gamma(u_{d-1})} (1-x_k)^{s_k-1} x_k^{u_k-1} g(x_1, \ldots, x_d) \, dx_1 \cdots dx_d
\]
(5.3)
\[
= \int_0^1 \cdots \int_0^1 \int_0^\infty (\psi A)(x_1, \ldots, x_d; s_1, \ldots, s_d) g(A/x)(x_1, \ldots, x_d) \, dx_1 \cdots dx_d
\]
where
\[
g(x_1, \ldots, x_d) = x_1^{d-1} \cdots x_d^{d-1} \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} e^{-n_1 x_1} \cdots e^{-n_d x_d}
\]
(5.4)
is the function of Equation (3.6).

Since \( g/A \) is in \( S(R) \) by assumption, there exists an extension \( f = E(g/A) \) of \( g/A \) to \( S(R^d) \) by Lemma A.2. We can thus solve for \( L \) in Equation (5.3) and write
\[
L(s) = \frac{\Gamma(u_1) \cdots \Gamma(u_{d-1})}{(s_d - 1)} ((\psi A)(\cdot, s_1, \ldots, s_d), f(\cdot)),
\]
which, employing Lemma 5.2, gives an explicit expression for the continuation of \( L \) with the possible poles along the stated hyperplanes.

It remains to prove Equation (4.1); for simplicity, we give the proof for \( d = 2 \). By the growth condition in Equation (1.3),
\[
|g(x)| \leq C_1 C_2 y_1 y_2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{r_1} (m+n)^{r_2} e^{-m y_1} e^{-n y_2},
\]
where \( y \) is defined in Equation (3.5). But,
\[
(m+n)^{r_2} \leq (2 \max \{m, n\})^{r_2} \leq 2^{r_2} (m^{r_2} + n^{r_2}),
\]

Then there exists a constant C in C

\[ \text{Lemma 5.1.} \]

\[ \frac{\text{Equation (5.5).}}{\text{The resulting bound is the same as that in Equation (5.5)}} \]

\[ \text{except that the sum that defines } g \text{ is absolutely convergent.} \]

\[ \text{But } g/A \text{ is in } S(R) \text{ by assumption, so } A \text{ must respect the same bound as in Equation (5.5) (with a different constant). Then from Equation (5.3), for any } \phi \text{ in } S(\mathbb{R}^d), \]

\[ \int_0^1 \int_0^1 \frac{x_1^{u_1-1} (1-x_2)^{s_1-1} y_2^{a_2-1} (1-e^{-y_2})^{r_1+r_2}}{\Gamma(s_1) \Gamma(u_1)} A \phi \, dx_1 \, dx_2, \]

\[ \text{whose integrand is bounded by} \]

\[ C(1-x_2)^{\text{Re } s_1-1} (1+y_1)(1+y_2) \frac{y_1^{r_1+r_2} y_2^{r_2} x_1^{s_1-1} x_2^{a_2-1}}{(1-e^{-y_1})^{r_1+r_2} (1-e^{-y_2})^{r_2}}, \]

\[ \text{where } C \text{ is constant with respect to } x. \]

\[ \text{Using again that } x/(1-e^{-x}) \leq 1 + x \text{ for all } x > 0, \]

\[ \text{we see that the integral in Equation (5.6) is absolutely convergent in the region } \Omega \text{ defined by Re } u_1 - r_1 - 2r_2 > 0, \text{ Re } u_2 - r_2 > 0, \text{ and } \text{Re } s_1 > 0, \text{ which is the same as the region of Equation (4.1) (for } d = 2). \]

Since the sum that defines } g \text{ in Equation (3.6) is absolutely convergent on } R, \text{ we can take any derivative of } g \text{ term-by-term, then bound the derivative as we did in Equation (5.5). The resulting bound is the same as that in Equation (5.5) except that the powers of } 1 - e^{-u/k} \text{ are increased. From this it immediately follows that } g \text{ is in } C^\infty(R) \text{ and in } S((\epsilon, \infty) \times (\epsilon', 1)^{d-1}) \text{ for all } \epsilon > 0 \text{ and } \epsilon' \text{ in } (0, 1). \]

\[ \text{Lemma 5.1. Let } P_r(x) = \sum_{n=1}^\infty n^r x^n \text{ (so } P_r = \text{ Li}_{-r}, \text{ where Li denotes the polylogarithm). Then there exists a constant } c_0 \text{ such that for all } r \geq 0 \text{ and all } x \in [0, 1), \]

\[ 0 \leq P_r(x) \leq c_0 \frac{\Gamma(r+1)x}{(1-x)^{r+1}}, \]  

\[ \text{Proof. Equation (5.7) is equivalent to} \]

\[ \sum_{n=1}^\infty n^r x^n \leq c_0 x \sum_{n=0}^\infty \frac{\Gamma(r+1)}{n!} \left( -\frac{r+1}{n} \right) (-1)^n x^n \]

\[ = c_0 \sum_{n=1}^\infty \frac{\Gamma(r+1)}{(r+n-1)!} x^n = c_0 \sum_{n=1}^\infty \frac{\Gamma(r+n)}{\Gamma(n)} x^n. \]
all \( u \) in \([0,1]\), so \( f(n,u) \geq f(1,u) \). But \( f(1,\cdot) \) is a continuous positive function on the interval \([0,1]\) and so achieves a minimum value, \( R > 0 \). Thus, \( f(n,u) \geq R \), or, 
\[
\Gamma(u+n)/\Gamma(n) \geq Rn^u \text{ for all } u \in [0,1].
\]
Then,
\[
\frac{\Gamma(r+n)}{\Gamma(n)} = (r-1+n)\cdots(r-[r]+n) \frac{\Gamma(r-[r]+n)}{\Gamma(n)} \geq n^{[r]}f(n,r-[r])n^{r-[r]}
\]
\[
\geq Rn^{[r]}n^{r-[r]} = Rn^r.
\]
From this, Equation (5.8) and thus Equation (5.7) follow with \( c_0 = 1/R \). \( \square \)

**Lemma 5.2.** The distribution \( \psi \) of Equation (3.3) is absolutely convergent on \( \text{Re } u_k > 0 \), \( k = 1, \ldots, d \), and \( \text{Re } s_k > 0 \), \( k = 1, \ldots, d - 1 \), and continues to an entire distribution.

**Proof.** The region of absolute convergence follows as in the proof of Theorem 4.1 with \( r_1 = \cdots = r_d = 0 \).

The distribution \( \psi_1 \) is analytic on \( \text{Re } u_1 > 0 \) and continues to an entire function by Lemma 3 of [23] (see Lemma 5.4; this also follows from Corollary 4.6), and \( \psi_k \) for \( k > 1 \) is analytic on \( \text{Re } s_{k-1} > 0 \), \( \text{Re } u_k > 0 \) and continues to an entire function by Lemma 4 of [23]. The entireness of \( \psi \) then follows from Theorem 4.2 with \( A = 1 \). \( \square \)

### 5.2. Proof of Theorem 4.2.

**Proof.** Assume that \( d = 2 \), the proof being entirely analogous for \( d > 2 \). By assumption, \( \overline{\psi}_1 := \psi_1 A_1 \) is a meromorphic distribution on \( u_1 \) and \( \overline{\psi}_2 := \psi_2 A_2 \) is a meromorphic distribution on \( (s_1, u_2) \). Then \( \psi A = \overline{\psi}_1 \otimes \overline{\psi}_2 \) (see Section 2 for the definition of the tensor product of two distributions), and we can write
\[
(\psi A, \varphi) = (\overline{\psi}_1, (\overline{\psi}_2, \varphi)) = (\overline{\psi}_2, (\overline{\psi}_1, \varphi)).
\]
Since \( (\psi A, \varphi) = (\overline{\psi}_1, (\overline{\psi}_2, \varphi)) \), it is meromorphic in \( u_1 \); since \( (\psi A, \varphi) = (\overline{\psi}_2, (\overline{\psi}_1, \varphi)) \) it is meromorphic in \( s_1 \) and \( u_2 \) as well. But a complex-valued function that is meromorphic in each variable separately is meromorphic: this follows from Hartog’s theorem (for instance, see Theorem B.6 p. 15 of [10]). Hence, \( (\psi A, \varphi) \) is meromorphic in \( (u_1, s_1, u_2) \) and so is meromorphic on the subvariety defined by \( s_1 = u_1 - u_2 + 1 \), which, with the change of variables \( s_1 = s_1, s_2 = u_2 + 1 \), means that \( (\psi A, \varphi) \) is meromorphic when viewed as a function of \( (s_1, s_2) \). (These relations come from solving for \( s_1 \) and \( s_2 \) in Equation (3.2).) Since this is true for all \( \varphi \) in \( \mathcal{S}(\mathbb{R}^2) \), the distribution \( \psi A \) is meromorphic. \( \square \)

### 5.3. Proof of Corollary 4.4.

**Proof.** Since \( g \) is in \( \text{Sing } A \), \( g = f A \) for some function \( f \) in \( \mathcal{S}(R) \). Thus, \( L = (\Psi, g) = (\Psi, f A) = (\Psi A, f) \). But \( \Psi A \) continues to a meromorphic distribution by Lemma 5.3, so \( L \) continues to a meromorphic function on \( \mathbb{C}^d \) with the possible simple poles as stated. \( \square \)

**Lemma 5.3.** Let \( A \) be the function defined in Corollary 4.4. Then \( \Psi A \) continues to a meromorphic distribution. When each \( p_j \) and \( q_j \) is a nonnegative integer and \( v_1 = \cdots = v_d = w_2 = \cdots = w_d = 0 \), \( \Psi A \) has the same possible simple poles as those stated for \( L \) in Corollary 4.4.
Proof. Let
\[ A_1(x_1) = x_1^{-p_1}(\log x_1)^{v_1}, \]
\[ A_k(x_k) = x_k^{-p_k}(1 - x_k)^{-q_k}(\log x_k)^{v_k}(\log(1 - x_k))^{w_k}, \quad k = 2, \ldots, d. \]
Then for \( u_1 > p_1 \) and any \( \varphi \) in \( \mathcal{S}(\mathbb{R}) \),
\[
(\psi_1 A_1(x_1), \varphi) = \frac{1}{\Gamma(u_1)} \left( (x_1)^{u_1-p_1-1}(\log x_1)^{v_1}, \varphi \right) = \frac{1}{\Gamma(u_1)} \left( \frac{\partial^{v_1}}{\partial u_1^{v_1}} (x_1)^{u_1-p_1-1}, \varphi \right)
\]
\[ = \frac{1}{\Gamma(u_1)} \frac{\partial^{v_1}}{\partial u_1^{v_1}} ((x_1)^{u_1-p_1-1}, \varphi). \]
The justification for the final equality—which is a matter of bringing the derivative under the integral sign, since for \( u_1 > p_1 \) the distribution \((x_1)^{u_1-p_1-1}\) is regular—is the same as the justification for bringing the derivative under the integral sign in the proof of Corollary 4.6 with the function \( A \) of that corollary being identically 1 here. The function \((\psi_1 A_1(x_1), \varphi)\) thus continues to a meromorphic function of \( u_1 \) since the distribution \((x_1)^{u_1-p_1-1}\) is meromorphic, being the same as the meromorphic distribution \((x_1)^{u_1}\) after a change of variables. This gives a meromorphic continuation of the distribution \(\psi_1 A_1\).

Similarly,
\[
(\psi_k A_k(x_k), \varphi) = \frac{1}{\Gamma(s_k-1)\Gamma(u_k)} \left( \frac{\partial^{v_k}}{\partial s_k^{p_k}} \frac{\partial^{u_k}}{\partial u_k^{v_k}} (1 - x_k)^{s_k-1-q_k-1}(x_k)^{u_k-p_k-1}, \varphi \right),
\]
which we also conclude is meromorphic. Applying Theorem 4.2, we conclude that \(\psi A\) is meromorphic.

The distribution \(x_n^{s-1}\) has simple poles along the hyperplane \( s = n \) for each integer \( n \leq 0 \), and the distribution \((1 - x)^n\) has simple poles along the hyperplanes \( s = n \) and \( t = n \) for each integer \( n \leq 0 \) (see, for instance, [23]). Each derivative in the above expressions for \((\psi_1 A_1(x_1), \varphi)\) and \((\psi_k A_k(x_k), \varphi)\) increases the possible order of the poles by one; the net effect of the poles of the gamma functions in the denominators of these expressions and in the numerator of Equation (3.1) leads to the locations and orders of the possible poles in the statement of the theorem. \( \Box \)

5.4. Proof of Corollary 4.5.

Proof. Let \( A(x) = 1/y(x)^\alpha \), where \( y = (y_1, \ldots, y_d) \) and \( y_k \) is defined in Equation (3.5). The function \( y^\alpha b(y) \) is in \( \mathcal{S}(\Delta) \) by assumption; it follows by Lemma A.1 that \( g(x)^\alpha g(x) = g(x)/A(x) \) is in \( \mathcal{S}(\mathbb{R}) \). But,
\[
A(x) = 1/y(x)^\alpha = x_1^{-(\alpha_1 + \cdots + \alpha_d)} \cdots x_d^{-(\alpha_d)},
\]
is the function defined in Corollary 4.4 with \( p_k = \alpha_k + \cdots + \alpha_d \) and \( q_j = v_j = w_j = 0 \) for each \( j \). It follows that \( g \) is in \( \text{Sing } A \) and we can apply Corollary 4.4 to give the continuation of \( L \) with the possible poles stated in the theorem.
To calculate the residue for \( d = 1 \), we have, for all values of \( s_1 \) in \( \mathbb{C} \),

\[
L(s_1) = (\Psi A, g/A) = \frac{1}{s_1 - 1}(\psi A, g/A)
\]

\[
= \frac{1}{(s_1 - 1)(u_1 - 1) \cdots (u_1 - \alpha_1)}(\overline{\psi}(\cdot, s_1), (g/A)(\cdot)),
\]

where \( u_1 = s_1 - 1 + 1 - 1 = s_1 - 1 \) and where \( \overline{\psi} \) is the entire distribution,

\[
\overline{\psi}(x_1, s_1) = \frac{(x_1)_+^{u_1 - 1 - \alpha_1}}{\Gamma(u_1 - \alpha_1)} = \frac{(x_1)_+^{s_1 - 1 - \alpha_1}}{\Gamma(s_1 - 1 - \alpha_1)}.
\]

Then the residue of \( L(s_1) \) at \( s_1 = 1 \) is given by

\[
R_1 := \text{Res}_{s_1 = 1} L(s_1) = \lim_{s_1 \to 1} (s_1 - 1)L(s_1) = \lim_{s_1 \to 1} \frac{(\overline{\psi}(\cdot, s_1), (g/A)(\cdot))}{(s_1 - 2) \cdots (s_1 - 1 - \alpha_1)}
\]

\[
= \frac{(-1)^{\alpha_1}}{\alpha_1!}(\overline{\psi}(\cdot, 1), (g/A)(\cdot)).
\]

By Lemma 5.4,

\[
\psi(\cdot, 1) = \frac{(x_1)_+^{s_1 - 1 - \alpha_1}}{\Gamma(s_1 - 1 - \alpha_1)} \big|_{s_1 = 1} = \frac{(x_1)_+^{u_1 - 1}}{\Gamma(u_1)} \big|_{u = -\alpha_1} = \delta(\alpha_1)(x_1).
\]

Then,

\[
R_1 = \frac{(-1)^{\alpha_1}}{\alpha_1!}(\delta(\alpha_1), g/A) = \left(\frac{(-1)^{\alpha_1}}{\alpha_1!}\right) (-1)^{\alpha_1}(g/A)^{(\alpha_1)}(0) = \frac{1}{\alpha_1!} \lim_{x_1 \to 0^+} (x_1^{\alpha_1}h(x_1))^{(\alpha_1)}.
\]

More generally, for \( 1 \leq k \leq 1 + \alpha_1 \),

\[
R_k := \text{Res}_{s_1 = k} L(s_1) = \lim_{s_1 \to k} (s_1 - k)L
\]

\[
= \frac{(\overline{\psi}(\cdot, k), (g/A)(\cdot))}{(k - 1)! \cdots (1)(k - 1 - \alpha_1)} = \frac{(-1)^{\alpha_1 + 1 - k}(\overline{\psi}(\cdot, k), (g/A)(\cdot))}{(k - 1)! (\alpha_1 + 1 - k)!}
\]

\[
= \frac{(-1)^{\alpha_1 + 1 - k}}{(k - 1)! (\alpha_1 + 1 - k)!} \lim_{x_1 \to 0^+} (x_1^{\alpha_1}h(x_1))^{(\alpha_1 + 1 - k)}.
\]

Our calculation of the residues in the proof of Corollary 4.5 would also follow from Equation (4.7) of Corollary 4.6.

The following is Lemma 3 of [23], which is from [7]:

**Lemma 5.4.** The regular distribution \( x_+^{s_1-1}/\Gamma(s) \) defined for \( \text{Re } s > 0 \) analytically continues to an entire distribution, with

\[
\frac{x_+^{s_1-1}}{\Gamma(s)} \big|_{s = -n} = \delta^{(n)}(x)
\]

for any nonnegative integer \( n \), where \((\delta^{(n)}, \varphi) = (-1)^{n}\varphi^{(n)}(0)\).
5.5. Proof of Corollary 4.6.

Proof. What we must show is that a function $A$ satisfying conditions (2) and (3) in the statement of Corollary 4.6 always satisfies condition (2) in Theorem 4.1. Since we are working in one dimension,

$$(\Psi A)(x; s) = \frac{x_+^{s-2}A(x)}{(s-1)\Gamma(s-1)} = \frac{x_+^{s-2}A(x)}{\Gamma(s)}$$  \hspace{1cm} (5.9)$$

The function $\phi = g/A$ is in $S^3(\mathbb{R})$ (where $\beta = \infty$ if $g/A$ is assumed to be in $S(\mathbb{R})$), and arguing as in Section I.3.2 p. 48-49 of [7], we have, for any $N < \beta$ and $Re s > 1 - N + h$,

$$(x_+^{s-2}A(x), \phi(x)) = \int_0^\lambda x_+^{s-2}A(x)\phi(x) \, dx = \int_0^\infty x_+^{s-2}A(x)\phi(x) \, dx$$

$$= \int_0^\lambda x_+^{s-2}A(x) \left[ \phi(x) - \sum_{j=0}^N \frac{\phi^{(j)}(0)}{j!} x^j \right] \, dx + \int_\lambda^\infty x_+^{s-2}A(x)\phi(x) \, dx$$

$$+ \sum_{j=0}^N \frac{\phi^{(j)}(0)}{j!} \int_0^\lambda x_+^{s+j-2}A(x) \, dx.$$  \hspace{1cm} (5.10)

Let $\phi_N(x) = \phi(x) - \sum_{j=0}^N \frac{\phi^{(j)}(0)}{j!} x^j$ for $x$ in $[0, \lambda]$. By the remainder theorem for Taylor series and because $\phi$ is in $S^3(\mathbb{R})$, $|\phi_N(x)| \leq Cx^{N+1}$ for some constant $C$, so by condition (2) the first term in the right-hand side of Equation (5.10) is finite as long as $Re s > 1 - N + h$. Also,

$$\frac{d}{ds} \int_0^\lambda x_+^{s-2}A(x)\phi_N(x) \, dx = \int_0^\lambda \frac{\partial}{\partial s} \left( x_+^{s-2}A(x)\phi_N(x) \right) \, dx$$

$$= \int_0^\lambda x_+^{s-2}A(x)\phi_N(x) \, log x \, dx,$$

where we justify the first equality as follows. Let $F(x, s) = x_+^{s-2}A(x)\phi_N(x)$ and $s_0 = (Re s + (1 - N + h))/2$. Then

$$\left| \frac{\partial}{\partial t} F(x,t) \right| = \left| x_+^{t-2}A(x)\phi_N(x) \, log x \right| \leq C \left| x_+^{t+N-1}A(x) \, log x \right|$$

$$\leq C \left| x_+^{s_0+N-1}A(x) \, log x \right|,$$

which is in $L^1((0, \lambda))$ as a function of $x$ for all $Re t > s_0$ by condition (2). Thus, $(\partial/\partial t)F(\cdot, t)$ is bounded by an $L^1((0, \lambda))$–function for all $t$ in an open interval containing $s$—a sufficient condition for bringing the derivative under the integral sign in Equation (5.11) (see, for instance, Theorem 24.5 p.193-194 of [3]). Since the right-hand side of Equation (5.11) is finite by condition (2) and the bound $|\phi_N(x)| \leq Cx^{N+1}$, it follows that the first term in Equation (5.10) is analytic for $Re s > 1 - N + h$.

The second term in the right-hand side of Equation (5.10) is analytic because $A\phi = g$ is in $S((\lambda, \infty))$. This leaves only the third term, which continues by condition (3) to a meromorphic function of $s$ whose possible poles are the same as those of $\sum_{j=0}^N \mu(s + j - 2)$. Thus, $(x_+^{s-2}A(x), \phi(x))$ continues to a meromorphic function on
Re\( s > 1 - N + h \) with poles at \( s = 2 - j + q_m \) of order \( r_m \) for \( m = 1, \ldots, n \) and \( j = 0, \ldots, N \). The function \((\Psi A, \phi(x))\) thus meromorphically continues to all of \( \text{Re}\( s > 1 - N + h \) with the same poles as those of \((x^{s-2} A(x), \phi(x))\), though, as we can see from Equation (5.9), with any poles at \( s = 0, -1, \ldots \) reduced in order by 1 because of the poles of \( \Gamma(s) \), giving the location of the poles in the statement of the theorem. Also, Equation (4.7) follows from Equation (5.9) and Equation (5.10). \( \square \)

Corollary 4.6 can be generalized in a natural way to multiple dimensions, where \( A \) is written in product form with functions similar to Equation (4.6) for the other variables. It follows that for sufficiently small \( \epsilon > 0 \),

\[
\sup_{y \in (0, 2\epsilon)} |y^\beta D^\gamma (y^\alpha h(y))| < \infty
\]

for all nonnegative integers \( \beta \) and \( \gamma \). Combined with the knowledge that \( h \) is in \( S((\epsilon, \infty)) \) for all \( \epsilon > 0 \), this is enough to insure that \( y^\alpha h(y) \) is in \( S((0, \infty)) = S(\Delta) \), and we can apply Corollary 4.5 to complete the proof of the continuation of \( G \).

When \( \alpha = 0 \), the calculation of the residues is particularly simple. Since \( G \) then has a possible simple pole at \( z = 1 \), \( G(z) = c_{-1}/(1 - z) + r(z) \), where \( c_{-1} \) is the residue of \( g \) at \( z = 1 \) and \( r(z) \) is holomorphic in a neighborhood of \( z = 1 \). Then by Equation (4.5),

\[
\text{Res}_{z=1} L = \lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} (x G(e^{-x})) = \lim_{z \to 0^+} (z G(e^{-z})) = \lim_{z \to 0^+} \left( z^{c_{-1}} \frac{e^{-z}}{1 - e^{-z}} + z r(e^{-z}) \right) = c_{-1}.
\]

Finally, from \( |a_n| \leq C n^r \) it follows that for real \( z \) in \([0, 1)\),

\[
| (1 - z)^{\alpha+1} G(z) | \leq (1 - z)^{\alpha+1} C \sum_{n=1}^{\infty} n^r z^n \leq C (1 - z)^{\alpha+1} \frac{z}{(1 - z)^{r+1}} = C z (1 - z)^{\alpha - r},
\]
where the second inequality follows from Lemma 5.1. But $G(z)$ has a pole of order $\alpha + 1$ or less at $z = 1$, so we can always choose $\alpha \leq r$. \hfill \Box

6. More General $L$–functions

We can generalize our definition of $L$ in Equation (1.4) by letting

$$L(s) = \sum_{0<n_1<\cdots<n_d} a_{n_1,1} \cdots a_{n_d,d} \lambda_{n_1,1}^{-s_1} \cdots \lambda_{n_d,1}^{-s_d},$$

where the coefficients $a_{n,k}$ are defined as before and where each $(\lambda_{n,k})_{n=1}^{\infty}$ for $k = 1, \ldots, d$ is a strictly increasing sequence of positive real numbers.

Only small modifications of the proofs in Section 5 are needed to adapt the results of Section 4 to our new definition of $L$. The function $g$ of Equation (3.6) is replaced by

$$g(x) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+n_2,d} y_1(x) e^{-\lambda_{n_1,1} y_1(x)} \cdots y_d(x) e^{-\lambda_{n_d,d} y_d(x)},$$

with similar changes to the functions $h$ and $G$ of Equation (4.3) and Equation (4.9). To obtain the equivalent of Equation (4.1) requires that we place a lower bound on the growth of $\lambda_{n,k}$ with $n$ and incorporate it into the bound on $g$ in Equation (5.5). Also, Corollary 4.7 is no longer valid in general for non-integer $\lambda_{n,1}$, since the function $G$ of Equation (4.9) will not be analytic on the open unit disk.

An important special case of Equation (6.1) is the Hurwitz zeta function, in which $\lambda_{n,k} = n + \theta_k$, so that

$$L(s) = \sum_{0<n_1<\cdots<n_d} a_{n_1,1} \cdots a_{n_d,d} \prod_{k=1}^{d} (n_k + \theta_k)^{-s_k},$$

where $\theta_k$, $k = 1, \ldots, d$ are arbitrary nonnegative real numbers. This causes both the function $g$ of Equation (3.6) and the function $h$ of Equation (4.3) to be multiplied by

$$\nu(x) = e^{-\theta_1 y_1(x)} \cdots e^{-\theta_d y_d(x)}.$$

Because $S(R)$ is closed under multiplication by a bounded $C^\infty$-function all of whose derivatives are bounded, it follows from Lemma A.1 that $\nu$ is in $S(R)$. Thus, the conclusions in Theorem 4.1, Corollary 4.6, Corollary 4.5, and Corollary 4.7 regarding the region of absolute convergence and the location and orders of the possible poles are unchanged. This factor does, however, change the values of the residues and must be incorporated into Equation (4.7).

Because $\nu(0) = 1$, the residue calculation in Equation (4.5) is unchanged at $s = 1 + \alpha_1$, as is the residue calculation in Corollary 4.7.

7. $L$–functions of twisted cusp forms

Let $k$ be a positive integer. The group

$$GL_2^+(\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \ ad - bc > 0 \right\}$$
acts on functions \( \Phi : \mathbb{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \} \to \mathbb{C} \) by the operator
\[
(\Phi_k \gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k}\Phi(\gamma z),
\]
where
\[
\gamma z = \frac{az + b}{cz + d}, \quad \gamma \in GL_2^+(\mathbb{R}).
\]

Let \( N \) be a positive integer. Define the groups
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \text{ mod } N \right\},
\]
\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \text{ mod } N \right\}.
\]

Let \( S_k(\Gamma_1(N)) \) denote the \( \mathbb{C} \)-vector space of cusp forms for \( \Gamma_1(N) \) of weight \( k \). Let \( \epsilon \) be a Dirichlet character modulo \( N \). The \( \mathbb{C} \)-vector space of cusp forms for \( \Gamma_1(N) \) of weight \( k \) and Nebentypus \( \epsilon \) is defined by
\[
S_k(\Gamma_0(N), \epsilon) = \left\{ \Phi \in S_k(\Gamma_1(N)) : \Phi(\gamma z) = \epsilon(d)(cz + d)^k\Phi(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.
\]

Let \( \chi \) be a non-trivial Dirichlet character modulo \( M \). If \( \Phi(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad q = e^{2\pi i nz}, \) is in \( S_k(\Gamma_0(N), \epsilon) \), we define the twist of \( \Phi \) by
\[
(\Phi \otimes \chi)(z) = \sum_{n=1}^{\infty} a(n)\chi(n)q^n.
\]

By Proposition 17 pg. 127 of [12], we know that
\[
\Phi \otimes \chi \in S_k(\Gamma_0(\Gamma_0(N)^+), \epsilon \chi^2).
\]

Further, letting
\[
H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in GL_2^+(\mathbb{Z}),
\]
by [13], pg. 296, we have
\[
H_N : S_k(\Gamma_0(\Gamma_0(N)^+), \epsilon \chi^2) \to S_k\left(\Gamma_0(\Gamma_0(N)^+), \overline{\epsilon \chi^2}\right).
\]

Thus,
\[
(\Phi \otimes \chi)|_k H_N(z) = (NM^2)^{k/2} z^{-k} (\Phi \otimes \chi) \left(-\frac{1}{NM^2 z}\right) = f(z)
\]
for some
\[
f(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_k\left(\Gamma_0(\Gamma_0(N)^+), \overline{\epsilon \chi^2}\right).
\]
That is,

\[ (\Phi \otimes \chi) \left( -\frac{1}{NM^2z} \right) = (NM^2)^{-k/2} z^{k} f(z). \]  

(7.2)

Define the twisted \( L \)-function of \( \Phi \) by

\[ L(s, \Phi \otimes \chi) = \sum_{n=1}^{\infty} a(n) \chi(n) n^{-s}, \quad \text{Re}(s) >> 0. \]

We will now use Corollary 4.5 with \( \alpha = 0 \) (or, equivalently, Theorem 4.1 with \( A \equiv 1 \)) to show that \( L(s, \Phi \otimes \chi) \) has an analytic continuation to an entire function on \( \mathbb{C} \).

Define the twist of the function \( h(y) \) of Equation (4.3) with \( d = 1 \) by

\[ (h \otimes \chi)(y) = \sum_{n=1}^{\infty} a(n) \chi(n) ye^{-ny}. \]

Then by Equation (7.1),

\[ (h \otimes \chi)(y) = \sum_{n=1}^{\infty} a(n) \chi(n) ye^{-ny} = y \left( \Phi \otimes \chi \right) \left( \frac{iy}{2\pi} \right), \quad y \in (0, \infty). \]  

(7.3)

By Equation (7.2),

\[ (\Phi \otimes \chi) \left( \frac{iy}{2\pi} \right) = (-1)^{k/2} (NM^2)^{-3k/2} (2\pi)^k y^{-k} \left( \frac{2\pi i}{NM^2y} \right). \]  

(7.4)

Substituting Equation (7.4) into the RHS of Equation (7.3) yields

\[ (h \otimes \chi)(y) = (-1)^{k} (NM^2)^{-3k/2} (2\pi)^{k} y^{-1-k} \left( \frac{2\pi i}{NM^2y} \right). \]  

(7.5)

Fix some \( \epsilon \) in \((0, 1)\). The growth condition in Equation (1.3) is enough to insure that \( h \otimes \chi \) is in \( S((\epsilon, \infty)) \). We will use Equation (7.5) to show that \( h \otimes \chi \) is also in \( S((0, 1/\epsilon)) \) with \( h \otimes \chi \) and all its derivatives vanishing at the origin. It follows that \( h \otimes \chi \) is in \( S(\Delta) = S((0, \infty)) \), so from Equation (4.7) of Corollary 4.6, \( L \) is entire.

First observe that

\[ f \left( \frac{2\pi i}{NM^2y} \right) = \sum_{n=1}^{\infty} b(n) e^{-(2\pi)^2 n/NM^2y}. \]

Because \( b(n) \) grows polynomially in \( n \),

\[ f \left( \frac{2\pi i}{NM^2y} \right) \in S((0, 1/\epsilon)). \]  

(7.6)

Let \( m \) and \( n \) be nonnegative integers. Then from Equation (7.5) we find that

\[ \frac{d^n}{dy^n} (h \otimes \chi)(y) = \sum_{j=0}^{n} C \left( j, k, n, NM^2 \right) y^{\alpha(j,k,n)} f^{(j)} \left( \frac{2\pi i}{NM^2y} \right), \]
for some constants \( C(j, k, n, NM^2) \) depending on \( j, k, n, NM^2 \), and some integers \( \alpha(j, k, n) \) depending on \( j, k, n \). Then

\[
\sup_{y \in (0, 1/\epsilon)} \left| y^m \frac{d^n}{dy^n} (h \otimes \chi)(y) \right| \leq \sum_{j=0}^{n} \left| C(j, k, n, NM^2) \right| \sup_{y \in (0, 1/\epsilon)} \left| y^{m+\alpha(j, k, n)} f^{(j)} \left( \frac{2\pi i NM^2 y}{N M^2} \right) \right| < \infty
\]

because of Equation (7.6), so \( h \otimes \chi \) is in \( S((0, 1/\epsilon)) \). It follows similarly that \( h \otimes \chi \) and all its derivatives vanish at the origin.

8. \( L \)-functions associated with harmonic numbers

Let \( d = 1 \) and suppose that \( G \) satisfies the properties of Corollary 4.7 with a pole of order \( \alpha \) at 1. Let

\[
b_k = \sum_{j=1}^{k} a_j
\]

and let \( \overline{G}(z) \) be the generating function of Equation (4.9) corresponding to \( (b_k) \). Then by Lemma 8.2, \( \overline{G}(z) = G(z)/(1 - z) \). Thus \( \overline{G} \) also satisfies the properties of Corollary 4.7, with a pole of order \( \alpha + 1 \) at 1.

If, on the other hand, \( G \) does not satisfy the properties of Corollary 4.7, then neither will \( \overline{G} \). An example of this is the sequence \( (a_j) = (1/j) \), whose generating function is \(-\text{log}(1 - z)\), which has an essential singularity at \( z = 1 \). Thus we cannot use Corollary 4.7 to meromorphically continue the single \( L \)-function

\[
L(s) = \sum_{n=1}^{\infty} b_n n^{-s} = \sum_{n=1}^{\infty} \left( \sum_{j=1}^{n} \frac{1}{j} \right) n^{-s}.
\]

We can, however, use Corollary 4.6.

The test function \( g \) of Equation (4.3) corresponding to \( \overline{G} \) is

\[
g(x) = x \overline{G}(e^{-x}) = x \frac{G(e^{-x})}{1 - e^{-x}} = -\frac{x}{1 - e^{-x}} \text{log}(1 - e^{-x}).
\]

Let \( A \) be a non-vanishing function in \( C^\infty(\mathbb{R}) \) defined so that

\[
A(x) = \begin{cases} 
-\text{log}(1 - e^{-x}), & x \text{ in } (0, 1) \\
-\text{log}(1 - e^{-2}), & x \text{ in } (2, \infty).
\end{cases}
\] (8.1)

Letting \( \xi := g/A \), \( \xi(x) = x/(1 - e^{-x}) \) for \( x \) in \((0, 1)\), and we can see, using Lemma 10.3, that \( \xi \) is in \( S(\mathbb{R}) \). Condition (2) of Corollary 4.6 is trivially satisfied, so it remains only to verify that condition (3) of Corollary 4.6 holds.
We have,
\[
\mu(s) = -\int_0^1 t^s \log(1 - e^{-t}) \, dt = -\int_0^1 t^s \log t \, dt + \int_0^1 t^s \log \left( \frac{t}{1 - e^{-t}} \right) \, dt \\
= \frac{1}{(s+1)^2} + \int_0^\infty t^s \phi(t) \, dt - \int_1^\infty t^s \phi(t) \, dt \\
= \frac{1}{(s+1)^2} + \left( x^s_+, \phi(x) \right) + f(s),
\]
where \( f \) is entire and where
\[
\phi(t) := \log \xi(t),
\]
which equals \( \log(x/(1 - e^{-x})) \) for \( x \) in \((0, 1)\). Then \( \phi \) decays at infinity like a Schwartz-class function, and \( \xi(z) \) is an entire function of the complex variable \( z \) that is nonzero at the origin, so \( \phi(z) \) is analytic near the origin. Applying the reasoning in the proof of Corollary 4.7, we conclude that \( \phi \) is in \( S((0, \infty)) \).

This shows that \( \mu(s) \) is analytic on the right-half plane \( \Re s > -1 \) and analytically continues to a meromorphic function on \( \mathbb{C} \). We also know the locations and orders of its possible poles and so can apply Corollary 4.6 to obtain the locations and orders of the possible poles of \( L(s) \). But because we wish to obtain more information about \( L(s) \)—all of its residues and its value at the negative even integers—we instead go directly to Equation (4.7), where we have, for any nonnegative integer \( N \),
\[
L(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^N \frac{B_j'}{j!} \left( \frac{d^j}{dx^j} \right)_{x=0} \left( \frac{1}{1 - e^{-x}} \right) \mu(s+j-2) + \frac{r_N(s)}{\Gamma(s)}
\]
for \( \Re s > -N \), where \( r_N \) is analytic.

Let \( r(x) = x/(1 - e^{-x}) \). Then \( r(-x) = -x/(1 - e^{-(-x)}) = x/(e^x - 1) \) is the exponential generating function for the Bernoulli numbers, so \( r \) is the exponential generating function for \( B'_n := (-1)^n B_n \). Since all odd-index Bernoulli numbers except for \( B_1 \) are zero, \( B'_n = B_n \) for all nonnegative integers except for \( n = 1 \), and \( B'_1 = -B_1 = 1/2 \). Then,
\[
L(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^N \frac{B_j'}{j!} \left[ \frac{1}{(s+j-1)^2} + \left( x^s_+, \phi(x) \right) \right] + \frac{F(s)}{\Gamma(s)},
\]
where
\[
F(s) = r_N(s) + \sum_{j=0}^N f(s+j-2)
\]
is entire. Also,
\[
\eta_j(s) := \left( x^s_+, \phi(x) \right) = \frac{\Gamma(s+j-1)}{\Gamma(s)} \frac{x^{s+j-2}}{\Gamma(s+j-1)} \phi(x)
\]
for \( j \) a nonnegative integer is entire by Lemma 5.4.

We now have sufficient information to determine the orders and residues of all the poles of \( L(s) \) except the pole at \( s = 1 \). Because \( \Gamma(s) \) has a simple pole at \( s = -n \)}
with residue \((-1)^n/n!\), where \(n\) is any nonnegative integer, the pole corresponding to term \(j \geq 1\) in Equation (8.3) corresponds to a simple pole of \(\Gamma(s)\) at \(s = 1 - j\) with a residue of \((-1)^{j-1}/(j-1)!\). Thus, the residue of \(L(s)\) at \(s = 1 - j\) is \((-1)^{j-1}B'_j/j\). Letting \(N = 2k + 1\), since \(B_1 = -1/2\) and \(B_{2k+1} = 0\) for all nonnegative integers \(k\), there are no poles of \(L\) at the even negative integers, there is a simple pole of residue \(1/2\) at \(s = 0\), and there is a simple pole of residue \(-B_{2k}/2k\) at \(s = 1 - 2k\) for \(k = 1, 2, \ldots\).

To determine the residue of \(L\) at \(s = 1\), we switch to a different function \(A\). We choose \(A(x) = -\log x\) for \(x\) in \((0, 1/2)\), \(A(x) = 1\) for \(x\) in \((1, \infty)\), and define \(A(x)\) on \([1/2, 1]\) so that \(A\) is a non-vanishing function in \(C^\infty(\mathbb{R})\). Then \(g/A\) is in \(S^1(R)\), and we can apply Corollary 4.6 with \(\beta_1 = 1\). From Equation (4.7) with \(N = 0\) and using the definition of \(\mu\) in Equation (4.6) with \(\lambda = 1/2\), the principal part of the Laurent expansion of \(L(s)\) about \(s = 1\) is the same as the principal part of

\[-\frac{1}{\Gamma(s)} \int_0^{1/2} x^{s-2} \log x \, dx = -\frac{1}{\Gamma(s)} \frac{2^{1-s}((\log 2)(s-1) + 1)}{(s-1)^2},\]

which is \((s-1)^{-2} + \gamma(s-1)^{-1}\). Thus, the residue of \(L\) at \(s = 1\) is equal to Euler’s constant \(\gamma\).

Actually, we could have used our new version of \(A(x)\) to more simply determine the location of all the poles and calculate their residues by showing that \(\Psi \log x\) and \(\Psi \log(1 - e^{-x})\) have the same poles with the same orders and residues (this follows immediately from Lemma 10.2). This is the approach taken in Section 9.6 p. 293-296 of [5] where the zeta function of the Laplacian of a manifold is analytically continued (though the language of distributions is not used in [5]). Because the poles of \(L\) at the negative even integers vanish, however, we would like to evaluate \(L\) at those points, and this requires our original definition of \(A\) in Equation (8.1), which gave us the exact expression for \(L(s)\) of Equation (8.3), and which we can write as

\[L(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{2k+1} \frac{B'_j}{j!} \frac{1}{(s+j-1)^2} + \sum_{j=0}^{2k+1} \frac{B'_j}{j!} \eta_j(s) + \frac{F(s)}{\Gamma(s)}.\]

We let \(N = 2k + 1\) so that the expression for \(L(s)\) will be valid for \(\text{Re}\, s > -2k - 1\), which includes \(s = -2k\).

Let \(k\) be a positive even integer. Then because \(F(s)/\Gamma(s)\) vanishes at \(s = -2k\) and because \(B_{2k+1} = 0\), we have

\[L(-2k) = \sum_{j=0}^{2k} \frac{B'_j}{j!} \eta_j(-2k). \quad (8.4)\]

Defining \(\xi = g/A\) and \(\phi(t) = \log \xi(t)\) as before, we have, for \(j \geq 2\),

\[\eta_j(-2k) = \left. \Gamma(s+j-1) \right|_{s=-2k} \left. \frac{x^{s+j-2}}{\Gamma(s+j-1)} \phi(x) \right|_{x=-2k}
\]

\[= (-1)^{j-1} \frac{(2k)!}{(2k-j+1)!} \left. \frac{x^{u-1}}{\Gamma(u)} \phi(x) \right|_{u=-(2k-j+1)}
\]

\[= (-1)^{j-1} \frac{(2k)!}{(2k-j+1)!} (-1)^{2k-j+1} \phi^{(2k-j+1)}(0)\]
\[ (2k)! \]
where we used Lemma 5.4 and Lemma 8.1. Equation (8.4) thus becomes
\[ L(-2k) = - \sum_{j=0}^{2k} \frac{B'_j}{j!} (2k)! \frac{B_{2k-j+1}}{(2k-j+1)!} B_{2k-j+1} \]

Only the \( j = 1 \) and \( j = 2k \) terms are nonzero, so
\[ L(-2k) = - \frac{B'_1}{1!} (2k)! \frac{B_{2k-1+1}}{(2k-1+1)!} \frac{B'_{2k}}{(2k)!} (2k)! \frac{B_{2k-2k+1}}{(2k-2k+1)!} 2k - 2k + 1 \]
\[ = - \frac{1}{2} \frac{B_{2k}}{2k} - \frac{B_{2k}}{4k} \frac{B_{2k}}{(2k)!} \frac{B_1}{1!} = - \frac{B_{2k}}{4k} + \frac{B_{2k}}{2} = \frac{1}{2} B_{2k} \left[ 1 - \frac{1}{2k} \right] , \]
in agreement with Corollary 2 of [14].

It might seem that our method of continuing \( L(s) \) using an abstract functional
analytic approach would give insufficient arithmetic information about \( L(s) \) in its
continued domain to evaluate it there. The reason it does give sufficient information,
however, is that we have, in Equation (4.7), the principal part of the continuation
before being divided by the gamma function. The poles of the gamma function move
the residue of the continuation into the constant term of the Laurent series, allowing
us, in principle, to obtain the values of the continued \( L \)-function at the nonpositive
integers (when the function is analytic there).

**Lemma 8.1.** The derivatives of \( \phi \) at the origin are given by
\[ \phi^{(n)}(0) = - \frac{B_n}{n} \]
for all integers \( n \geq 1 \), where \( \phi \) is defined in Equation (8.2).

**Proof.** Observe that
\[ \phi'(x) = \frac{1}{x} - \frac{e^{-x}}{1-e^{-x}} = \frac{1}{x} - \frac{1}{x} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = - \sum_{n=1}^{\infty} \frac{B_n}{n!} x^{n-1} , \]
so, after integrating,
\[ \phi(x) = - \sum_{n=1}^{\infty} \frac{B_n}{nn!} x^n , \]
from which the statement of the lemma follows.

The following well known lemma is a useful tool in dealing with generating functions
(see, for instance, p. 37 of [20]).

**Lemma 8.2.** Suppose that \( P(x) = \sum_{n=0}^{\infty} a_n x^n \) is a formal power series. Then
\[ \frac{P(x)}{1-x} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k \right) x^n \]
as a formal power series.
Proof. This follows from expanding $1/(1 - x)$ as a geometric series and formally multiplying power series.

\[ G(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1 - z}. \]

This is a meromorphic function with a simple pole at $z = 1$. By Corollary 4.7, it follows that the zeta function has a meromorphic continuation with a possible simple pole at $z = 1$. The corresponding function in Equation (4.3), which we call $h_0$, is given by $h_0(y) = y/(e^y - 1)$. By the proof of Corollary 4.7, $h_0$ is in $S((0, \infty))$.

Now let $d \geq 2$. All the coefficients $(a_{n,k})$ are equal to 1, so from Equation (4.9),

\[ G(z) = \sum_{n_1=1}^{\infty} z_{n_1} \cdots \sum_{n_d=1}^{\infty} z_{n_d}^{d} = \frac{z_{1}}{1 - z_{1}} \cdots \frac{z_{d}}{1 - z_{d}}. \]

Then from Equation (4.3),

\[ h(y) = \frac{y_1 e^{-y_1}}{1 - e^{-y_1}} \cdots \frac{y_d e^{-y_d}}{1 - e^{-y_d}} = \frac{y_1}{e^{y_1} - 1} \cdots \frac{y_d}{e^{y_d} - 1} = h_0(y_1) \cdots h_0(y_d). \]

It follows that $h(y)$, being the product of one-dimensional Schwartz-class functions, is in $S((0, \infty)^d)$ and hence in $S(\Delta)$. Applying Corollary 4.5 with $\alpha = 0$ (or, equivalently, Theorem 4.1 with $A \equiv 1$), we conclude that the multiple zeta function has a meromorphic continuation to $\mathbb{C}^d$ with possible simple poles along the hyperplanes, $s_d = 1$ and $s_1 + \cdots + s_d = d - k_1, s_2 + \cdots + s_d = d - 1 - k_2, \ldots, s_{d-1} + s_d = 2 - k_{d-1}$, for all positive integers $k_1, \ldots, k_{d-1}$.

9.2. Periodic coefficients. Assume first that $d = 1$ and that $a_{n+N} = a_n$ for all positive integers $n$. Then

\[ G(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{j=0}^{\infty} \sum_{k=1}^{N} a_{k+jN} z^{k+jN} = \sum_{j=0}^{\infty} z^{jN} \sum_{k=1}^{N} a_k z^k = \frac{zp(z)}{1 - z^N}, \]

where $p(z) = \sum_{k=0}^{N} a_k z^k$ is a degree-$(n - 1)$ polynomial. An argument just like that in Example 1 shows that a meromorphic continuation of $L$ exists with a singularity at $s = 1$. 

9. Simple multi-dimensional applications

9.1. The Riemann zeta function. First we treat the case $d = 1$. We have,

\[ G(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1 - z}. \]
Now let $d = 2$, and let the periods be $N_1$ and $N_2$, so that $a_{n+N_k,k} = a_{n,k}$ for $k = 1, 2$. Letting $M$ be the least common multiple of $N_1$ and $N_2$, we have

$$G(z) = \sum_{n_1=1}^{\infty} a_{n_1,1} z_1^{n_1} \sum_{n_2=1}^{\infty} a_{n_1+n_2,2} z_2^{n_2}$$

$$= \sum_{j_1=0}^{M} \sum_{k_1=1}^{N_1} a_{k_1} z_1^{k_1} \sum_{j_2=0}^{N_2} \sum_{k_2=1}^{N_2} a_{k_1+k_2} z_2^{k_2}$$

$$= \sum_{j_1=0}^{M} z_1^{j_1} \sum_{j_2=0}^{N_2} z_2^{j_2} \sum_{k_1=1}^{M} \sum_{k_2=1}^{N_2} a_{k_1+k_2} z_1^{k_1} z_2^{k_2}$$

$$= \frac{1}{1-z_1^{M-1}} \frac{1}{1-z_2^{N_2-1}} \sum_{k_1=1}^{M} \sum_{k_2=1}^{N_2} a_{k_1+k_2} z_1^{k_1} z_2^{k_2} = \frac{z_1}{1-z_1^{M}} \frac{z_2}{1-z_2^{N_2}} p(z_1, z_2),$$

where $p$ is a polynomial of degree $M - 1$ in $z_1$ and degree $N_2 - 1$ in $z_2$ with, in general, a nonzero constant term. This continues to a meromorphic function on $\mathbb{C}^d$ in the same way as in example 1, and with the same possible poles. The extension of this example to $d > 2$ is clear.

9.3. **Quasi-periodic coefficients.** Let $\alpha_k$, $k = 1, \ldots, d$, be $d$ complex numbers and let $a_{n,k} = e^{i\alpha_k n}$ for $k = 1, \ldots, d$ and $n = 1, 2, \ldots$. The multiple $L$–function with quasi-periodic coefficients is defined by

$$L(s) = \sum_{0 < n_1 < \cdots < n_d} e^{i\alpha_k n_k} n_k^{-s_k}.$$

This $L$–function can be continued very simply by making the change of variables $w_k = e^{i\alpha_k} z_k$, for then $G(w) = G(w_1, \ldots, w_d)$ is the same as the generating function for the multiple zeta function of Example 1. We conclude that quasi-periodic coefficients induce the same possible poles as the multiple zeta functions.

10. **Double $L$–function for harmonic numbers**

In this section we show that the double $L$–function whose coefficients in each single $L$–function are the harmonic numbers,

$$a_{m,1} = a_{m,2} = H_m := \sum_{j=1}^{m} 1/j,$$

meromorphically continues to all of $\mathbb{C}^2$. Specifically, we prove the following:

**Theorem 10.1.** The function

$$L(s_1, s_2) = \sum_{0 < m < n} H_m H_n m^{-s_1} n^{-s_2}$$

continues meromorphically to $\mathbb{C}^2$ with the following possible poles:

1. a simple pole along $s_1 = 1$;
2. a pole of order 3 along $u_1 = n$ for each integer $n \leq 0$;
The proof of Theorem 10.1 occupies the remainder of this section. There are three phases to our continuation.

In phase 1, we determine a closed form for \( G \) and hence for \( g \). In phase 2, we split \( g \) into five functions, \( g_1 \) through \( g_5 \), each of which has a singularity along \( x_2 = 1 \) (see Section 10.8) and various other singularities. In phase 3, we continue \( (\Psi, g_3) \) separately, giving a continuation of \( L = \sum_{j=1}(\Psi, g_j) \).

We will find singularities in \( g_1 \) and \( g_2 \) of types \( 1/(1 - x_2) \), \( \log x_1/(1 - x_2) \), and \( (\log x_1)^2/(1 - x_2) \), and in \( g_2 \) of types \( 1/(x - x_2) \) and \( \log x_1/(1 - x_2) \). Determining the singularities of \( g_4 \) and especially of \( g_5 \) is more involved: it will require us to partition \( R \) into three regions and use a partition of unity to continue portions of \( (\Psi, g_4) \) and \( (\Psi, g_5) \) separately for each region. We will find singularities of the form \( 1/(1 - x_2) \), \( \log x_1/(1 - x_2) \), \( \log x_1 \log x_2/(1 - x_2) \), and \( (\log x_1)^2/(1 - x_2) \) in \( g_4 \), and \( 1/(1 - x_2) \), \( \log x_1/(1 - x_2) \), and \( \log x_2/(1 - x_2) \) in \( g_5 \). Employing Corollary 4.4, these singularities together give the possible poles in the statement of Theorem 10.1.

### 10.1. Calculation of \( G \)

The function \( G \) of Equation (4.9) is given by

\[
G(z) = \sum_{m=1}^{\infty} H_{m} \sum_{n=1}^{\infty} H_{m+n} z_{2}^{n}.
\]

For \( |z_2| < 1 \), where we have absolute convergence,

\[
\sum_{n=1}^{\infty} H_{m+n} z_{2}^{n} = z_{2}^{-m} \sum_{n=1}^{\infty} H_{m} z_{2}^{n} = z_{2}^{-m} \sum_{n=m+1}^{\infty} H_{n} z_{2}^{n}
\]

\[
= z_{2}^{-m} \sum_{n=m+1}^{\infty} \sum_{j=1}^{n} \frac{1}{j} z_{2}^{n} = z_{2}^{-m} \sum_{n=m+1}^{\infty} \frac{1}{n} z_{2}^{n} = z_{2}^{-m} \sum_{n=m+1}^{\infty} \frac{1}{j} z_{2}^{n},
\]

But,

\[
\frac{1}{1 - z_2} \sum_{n=m+1}^{\infty} \frac{1}{n} z_2^n = \sum_{k=0}^{\infty} \frac{1}{k} z_2^k \sum_{n=m+1}^{\infty} \frac{1}{n} z_2^n = \sum_{n=m+1}^{\infty} \frac{1}{n} z_2^n,
\]

so

\[
\sum_{n=1}^{\infty} H_{m+n} z_{2}^{n} = z_{2}^{-m} H_{m} \left[ \sum_{n=0}^{\infty} \frac{z_2^n}{n+1} - \sum_{n=0}^{m} \frac{z_2^n}{n} \right] + z_{2}^{-m} \frac{1}{1 - z_2} \sum_{n=m+1}^{\infty} \frac{1}{n} z_{2}^{n}
\]

\[
= z_{2}^{-m} H_{m} \left[ \frac{1}{1 - z_2} - \frac{1 - z_2^{m+1}}{z_2^{m+1}} \right] + z_{2}^{-m} \frac{1}{1 - z_2} \sum_{n=m+1}^{\infty} \frac{1}{n} z_{2}^{n}
\]

\[
= \frac{z_2}{1 - z_2} H_{m} + \frac{z_2 z_{2}^{-m}}{1 - z_2} \sum_{n=m+1}^{\infty} \frac{1}{n} z_{2}^{n}.
\]
But,
\[
\sum_{n=1}^{m} \frac{z_2^n}{n} = \int_0^{z_2} \sum_{n=0}^{m} t^{n-1} \, dt = \int_0^{z_2} \sum_{n=0}^{m-1} t^n \, dt = \int_0^{z_2} \frac{1 - t^m}{1 - t} \, dt = -\log(1 - z_2) - \int_0^{z_2} \frac{t^m}{1 - t} \, dt
\]
and
\[
\sum_{n=1}^{\infty} \frac{z_2^n}{n} = -\log(1 - z_2), \quad (10.1)
\]
so
\[
\sum_{n=m+1}^{\infty} \frac{z_2^n}{n} = \int_0^{z_2} \frac{t^m}{1 - t} \, dt,
\]
and we conclude that
\[
\sum_{n=1}^{\infty} \frac{H_m z_2^n}{z_1} = \frac{z_2}{1 - z_2} H_m + \frac{z_2^{-m}}{1 - z_2} \int_0^{z_2} \frac{t^m}{1 - t} \, dt,
\]
and therefore that
\[
G(z) = \sum_{m=1}^{\infty} \frac{H_m z_2^m}{z_1} \left[ \frac{z_2}{1 - z_2} H_m + \frac{z_2^{-m}}{1 - z_2} \int_0^{z_2} \frac{t^m}{1 - t} \, dt \right]
\]
\[
= \frac{1}{1 - z_2} \left[ z_2 \sum_{m=1}^{\infty} H_m^2 \frac{z_2^m}{z_1} + \int_0^{z_2} \frac{1}{1 - t} \sum_{m=1}^{\infty} H_m (z_1 / z_2)^m \, dt \right]. \quad (10.2)
\]
The integrals in the above expressions are along any path lying in the open unit disk from the origin to \(z_2\), and a similar statement applies to the integrals in the expressions that follow.

Let
\[
B = \sum_{m=1}^{\infty} H_m^2 \frac{z_2^m}{z_1}.
\]
Then
\[
\sum_{m=1}^{\infty} H_m^2 \frac{z_2^m}{z_1} = \frac{1}{z_1} \sum_{m=1}^{\infty} H_m^2 \frac{z_2^m+1}{z_1} = \frac{1}{z_1} (B - z_1) = \frac{B}{z_1} - 1,
\]
but also,
\[
\sum_{m=1}^{\infty} H_{m+1}^2 z_1^m = \sum_{m=1}^{\infty} \left[H_m + \frac{1}{m+1}\right]^2 z_1^m \\
= \sum_{m=1}^{\infty} H_{m+1}^2 z_1^m + 2 \sum_{m=1}^{\infty} \frac{H_m}{m+1} z_1^m + \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} z_1^m \\
= B + 2 \sum_{m=1}^{\infty} \frac{H_m}{m+1} z_1^m + \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} z_1^m.
\]

But,
\[
\sum_{m=1}^{\infty} \frac{H_m}{m+1} z_1^m = \int_0^{z_1} \frac{1}{z_1} \sum_{m=1}^{\infty} H_m t^m dt = -\frac{1}{z_1} \int_0^{z_1} \frac{\log(1-t)}{1-t} dt \\
= \frac{1}{2z_1} \log(1-z_1)^2,
\]
where we used Lemma 8.2, and
\[
\sum_{m=1}^{\infty} \frac{1}{(m+1)^2} z_1^m = \frac{1}{z_1} \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} z_1^{m+1} = \frac{1}{z_1} (\text{Li}_2(z_1) - 1) = \frac{1}{z_1} \text{dilog}(1-z_1) - 1.
\]

Remark: The function dilog is defined for \(z\) in the open unit disk in \(\mathbb{C}^2\) by
\[
\text{dilog}(z) := \int_0^z \frac{\log t}{1-t} dt = \text{Li}_2(1-z).
\]
It will be more convenient in subsequent sections to use dilog rather than the more conventional dilogarithm function \(\text{Li}_2\).

Thus,
\[
\frac{B}{z_1} - 1 = B + 2 \sum_{m=1}^{\infty} \frac{H_m}{m+1} z_1^m + \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} z_1^m \\
= B + 2 \left[\frac{1}{2z_1} \log(1-z_1)^2\right] + \frac{1}{z_1} \text{dilog}(1-z_1) - 1 \\
= B + \frac{1}{z_1} \log(1-z_1)^2 + \frac{1}{z_1} \text{dilog}(1-z_1) - 1,
\]
which gives
\[
B = \log(1-z_1)^2 + \frac{\text{dilog}(1-z_1)}{1-z_1}.
\] (10.3)

Returning to Equation (10.2) and again applying Lemma 8.2, we have
\[
\int_0^{z_2} \frac{1}{1-t} \sum_{m=1}^{\infty} H_m(z_1 t/z_2)^m dt = -\int_0^{z_2} \frac{1}{1-t} \frac{\log(1-z_1 t/z_2)}{1-z_1 t/z_2} dt.
\]
Using a partial fraction decomposition of the integrand, we have

\[
\int_0^{z_2} \frac{1}{1-t} \log(1 - z_1 t/z_2) \, dt = \frac{1}{1 - \xi} \int_0^{z_2} \left[ \frac{1}{1-t} - \frac{\xi}{1 - \xi t} \right] \log(1 - \xi t) \, dt \\
= \frac{1}{1 - \xi} \left[ \int_0^{z_2} \frac{\log(1 - \xi t)}{1 - t} \, dt - \xi \int_0^{z_2} \frac{\log(1 - \xi t)}{1 - \xi t} \, dt \right] \\
= \frac{1}{1 - \xi} \left[ \int_0^{z_2} \frac{\log(1 - \xi t)}{1 - t} \, dt + \frac{1}{2} \log(1 - (\xi z_2)^2) \right],
\]

where \( \xi = z_1/z_2 \). The integral above can be expressed using dilogarithms as

\[
\int_0^{z_2} \frac{\log(1 - \xi t)}{1 - t} \, dt = -\frac{1}{2} \log(1 - z_2)^2 + \text{dilog}(1 - \xi z_2) \\
- \text{dilog}(1 - z_2) - \text{dilog}\left(\frac{1 - \xi z_2}{1 - z_2}\right).
\]

From Equation (10.2), we then have

\[
G(z) = \frac{1}{1 - z_2} \left[ \frac{z_2 \log(1 - z_1)^2}{1 - z_1} + \frac{z_2 \text{dilog}(1 - z_1)}{1 - z_1} - \int_0^{z_2} \frac{1}{1-t} \log(1 - z_1 t/z_2) \, dt \right] \\
= \frac{1}{1 - z_2} \left[ \frac{z_2 \log(1 - z_1)^2}{1 - z_1} + \frac{z_2 \text{dilog}(1 - z_1)}{1 - z_1} - \frac{1}{1 - z_1 z_2} \left( -\frac{1}{2} \log(1 - z_2)^2 \right) \\
+ \text{dilog}(1 - z_1) - \text{dilog}(1 - z_2) - \text{dilog}\left(\frac{1 - z_1}{1 - z_2}\right) + \frac{1}{2} \log(1 - z_1)^2 \right] \\
= \frac{1}{1 - z_2} \left[ \left( \frac{z_2}{1 - z_1} - \frac{z_2}{2(z_2 - z_1)} \right) \log(1 - z_1)^2 + \left( \frac{z_2}{1 - z_1} - \frac{z_2}{z_2 - z_1} \right) \text{dilog}(1 - z_1) \\
- \frac{z_2 - z_1}{z_2 - z_1} \left( -\frac{1}{2} \log(1 - z_2)^2 - \text{dilog}(1 - z_2) - \text{dilog}\left(\frac{1 - z_1}{1 - z_2}\right) \right) \right] \\
= \frac{1}{1 - z_2} \left[ \frac{z_2(2z_2 - z_1 - 1)}{2(1 - z_1)(z_2 - z_1)} \log(1 - z_1)^2 - \frac{z_2(1 - z_2)}{1 - z_1} \text{dilog}(1 - z_1) \\
- \frac{z_2 - z_1}{z_2 - z_1} \left( -\frac{1}{2} \log(1 - z_2)^2 - \text{dilog}(1 - z_2) - \text{dilog}\left(\frac{1 - z_1}{1 - z_2}\right) \right) \right] \\
= \frac{1}{(1 - z_2)(1 - z_1/z_2)} \left[ \frac{(2z_2 - z_1 - 1)}{2(1 - z_1)} \log(1 - z_1)^2 - \frac{1 - z_2}{1 - z_1} \text{dilog}(1 - z_1) \right. \\
\left. + \frac{1}{2} \log(1 - z_2)^2 + \text{dilog}(1 - z_2) + \text{dilog}\left(\frac{1 - z_1}{1 - z_2}\right) \right].
\]

The singularity along \( z_1 = z_2 \) is removable (though there is a singularity at \( (1, 1) \)), as can be seen from the first expression for \( G \) above. This is as must be, since we know that \( G \) is analytic on \( D \times D \), where \( D \) is the open unit disk in \( \mathbb{C} \).

10.2. **Calculation of** \( g \). **Using** the identity,

\[
\text{dilog}(x) = -\text{dilog}(1/x) - (1/2)(\log x)^2, \quad (10.4)
\]
which holds for all nonzero $x$, we have,

$$\frac{1}{2} \log(1 - z_2)^2 + \dilog \left( \frac{1 - z_1}{1 - z_2} \right)$$

$$= \frac{1}{2} \log(1 - z_2)^2 - \dilog \left( \frac{1 - z_2}{1 - z_1} \right) - \frac{1}{2} \log \left( \frac{1 - z_1}{1 - z_2} \right)^2$$

$$= - \dilog \left( \frac{1 - z_2}{1 - z_1} \right) - \frac{1}{2} \log(1 - z_1)^2 + \log(1 - z_2) \log(1 - z_1).$$

We then have,

$$g(x_1, x_2) = x_1^2 x_2 G(e^{-x_1}, e^{-x_1 x_2})$$

$$= x_1^2 x_2 \frac{(2e^{-x_1 x_2} - e^{-x_1} - 1) \log(1 - e^{-x_1})^2}{2(1 - e^{-x_1})(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})}$$

$$+ \frac{x_1^2 x_2}{(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})} \left[ -\frac{1}{1 - e^{-x_1}} \dilog(1 - e^{-x_1})$$

$$- \frac{1}{2} \log(1 - e^{-x_1})^2 + \log(1 - e^{-x_1 x_2}) \log(1 - e^{-x_1})$$

$$+ \dilog(1 - e^{-x_1 x_2}) - \dilog \left( \frac{1 - e^{-x_1 x_2}}{1 - e^{-x_1}} \right) \right].$$

We can write this as $g = g_1 + \cdots + g_5$, where

$$g_1(x) = k_1(x) \log(1 - e^{-x_1})^2,$$

$$g_2(x) = -k_2(x) \dilog(1 - e^{-x_1}),$$

$$g_3(x) = -k_3(x) \frac{1}{2} \log(1 - e^{-x_1})^2,$$

$$g_4(x) = k_3(x) \log(1 - e^{-x_1 x_2}) \log(1 - e^{-x_1}),$$

$$g_5(x) = k_3(x) \left( \dilog(1 - e^{-x_1 x_2}) - \dilog \left( \frac{1 - e^{-x_1 x_2}}{1 - e^{-x_1}} \right) \right),$$

with

$$k_1(x) = x_1^2 x_2 \frac{(2e^{-x_1 x_2} - e^{-x_1} - 1)}{2(1 - e^{-x_1})(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})},$$

$$k_2(x) = \frac{x_1^2 x_2}{(1 - e^{-x_1})(1 - e^{-x_1(1-x_2)})},$$

$$k_3(x) = \frac{x_1^2 x_2}{(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})}.$$
Since \((\Psi, g) = (\Psi, g_1) + \cdots + (\Psi, g_5)\), we can continue \(L\) by applying \(\Psi\) to each of the test functions \(g_1\) through \(g_5\) separately.

**10.3. Continuation of \((\Psi, g_1)\).** We define a partition of unity \(\{\varphi_1, \varphi_2\}\) of \(R\) so that \(\varphi_1 \equiv 1\) on \((0, 1/4) \times (0, 1)\) and \(\varphi_1 \equiv 0\) on \((1/2, \infty) \times (0, 1)\). By Lemma 10.2,

\[\varphi_1 g_1 = k_1(x) \left( (\log x_1)^2 + 2f(x_1) \log x_1 + f(x_1)^2 \right) \varphi_1.\]

Since \((1 - x_2)k_1(x)\) is in \(C^\infty(R)\), we can split \(\varphi_1 g_1\) into three parts, applying Corollary 4.4 three times with \(A(x) = \log(x_1)^2/(1 - x_2)\), \(\log(x_1)/ (1 - x_2)\), and \(1/(1 - x_2)\)
to continue \((\Psi, \varphi_1 g_1)\).

Because of Lemma 10.3, we can continue \((\Psi, \varphi_2 g_1)\) using \(A(x) = 1/(1 - x_2)\).

**Lemma 10.2.** There exists an entire function \(f\) such that

\[\log(1 - e^{-z}) = \log z + f(z).\] (10.5)

**Proof.** As in Lemma 8.1,

\[\frac{d}{dz} \left( \log \left( \frac{z}{(1 - e^{-z})} \right) \right) = \frac{1}{z} - \frac{e^{-z}}{1 - e^{-z}} = \frac{1}{z} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = -\sum_{n=1}^{\infty} \frac{B_n}{n!} z^{n-1},\]

so, after integrating,

\[\log(z/(1 - e^{-z})) = -\sum_{n=1}^{\infty} \frac{B_n}{nn!} z^{n},\]

from which the statement of the lemma follows. \(\square\)

**Lemma 10.3.** The functions \(\log(1 - e^{-x})\) and \(\text{dilog}(1 - e^{-x})\) are in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\) for any \(\epsilon > 0\).

**Proof.** Fix \(\epsilon > 0\). Then for any \(n > 0\),

\[\frac{d^n}{dx^n} \log(1 - e^{-x}) = \sum_{j=1}^{n} C_j(r(x))^j,\]

where \(r(x) = 1/(e^x - 1)\) is in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\) and \(C_j\) are integer constants. Thus all the derivatives of \(\log(1 - e^{-x})\) are in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\). It only remains to show that \(\log(1 - e^{-x})\)

Itself decays faster than any polynomial, which it does, since for any \(m > 0\), by L’Hopital’s rule,

\[\lim_{x \to \infty} x^m \log(1 - e^{-x}) = \lim_{m \to \infty} \frac{\log(1 - e^{-x})}{x^m} = \lim_{m \to \infty} \frac{r(x)}{-m x^{m-1}} = 0.\]

Thus, we conclude that \(\log(1 - e^{-x})\) is in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\)

Then \((d/dx) \text{dilog}(1 - e^{-x}) = \log(1 - e^{-x})\), which is in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\). Since the derivative of \(\text{dilog}(1 - e^{-x})\) is in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\), we need only verify that \(\text{dilog}(1 - e^{-x})\) itself decays faster than any polynomial, which it does, since for any \(m > 0\), by L’Hopital’s rule (since \(\text{dilog}(1) = 0\),

\[\lim_{x \to \infty} x^m \text{dilog}(1 - e^{-x}) = \lim_{m \to \infty} \frac{\text{dilog}(1 - e^{-x})}{x^m} = \lim_{m \to \infty} \frac{\log(1 - e^{-x})}{-m x^{m-1}} = 0.\]

Thus, we conclude that \(\text{dilog}(1 - e^{-x})\) is also in \(\mathcal{S}(\langle \epsilon, \infty \rangle)\). \(\square\)
10.4. Continuation of \((\Psi, g_2)\). Let \(\{\varphi_1, \varphi_2\}\) be the partition of unity of Section 10.3. Using Lemma 10.3 and the identity,
\[
dilog(1 - x) = -dilog(x) + \pi^2/6 - \log x \log(1 - x),
\]
we have
\[
\varphi_{1g_2}(x) = k_2(x)(\text{dilog}(e^{-x_1}) - \pi^2/6 + \log e^{-x_1} \log(1 - e^{-x_1}))\varphi_1(x)
= k_2(x)(\text{dilog}(e^{-x_1}) - \pi^2/6 - x_1 \log x_1 + x_1 f(x_1))\varphi_1(x).
\]
Because \(\text{dilog}(z)\) is analytic on the disk of radius 1 centered at \(z = 1\), \(\text{dilog}(e^{-x_1})\varphi_1(x)\) is in \(S(R)\). We can therefore split \(\varphi_{1g_2}\) into three parts, applying Corollary 4.4 twice with \(A(x) = \log x_1/(1 - x_2)\) and \(1/(1 - x_2)\) to continue \((\Psi, \varphi_{1g_2})\). (The factor of \(x_1\) could be used to remove one of the possible poles that results from Corollary 4.4; however, these possible poles are already present, to higher order, in the continuation of \((\Psi, g_1)\).)

Because of Lemma 10.3, we can continue \((\Psi, \varphi_{2g_2})\) using \(A(x) = 1/(1 - x_2)\).

10.5. Continuation of \((\Psi, g_3)\). The continuation of \((\Psi, g_3)\) is virtually identical to that of \((\Psi, g_1)\), so we suppress the details.

10.6. Continuation of \((\Psi, g_4)\). We define a partition of unity \(\{\varphi_1, \varphi_2, \varphi_3\}\) of \(R\) as follows. First, define \(\varphi_1\) so that \(\varphi_1 \equiv 1\) on \((0, 3) \times (0, 1)\) and \(\varphi_1 \equiv 0\) on \((4, \infty) \times (0, 1)\). Next, let
\[
S_1 = \{(x_1, x_2) \in R : x_1 > 3 \text{ and } x_2 < 1/x_1\},
S_2 = \{(x_1, x_2) \in R : x_1 > 2 \text{ and } x_2 < 2/x_1\},
\]
and let \(\{\overline{\varphi}_2, \overline{\varphi}_3\}\) be a partition of unity defined so that \(\varphi_1 \equiv 1\) on \(S_1\) and \(\varphi_2 \equiv 0\) on \(R \setminus S_2\). Then let \(\varphi_2 = (1 - \varphi_1)\overline{\varphi}_2\) and \(\varphi_3 = (1 - \varphi_1)\overline{\varphi}_3\).

The relevant facts about this partition are the following:
1. on the support of \(\varphi_1\), both \(x_1\) and \(x_1x_2\) are bounded (by 4);
2. on the support of \(\varphi_2\), \(x_1\) is bounded away from 0 (by 3) and \(x_1x_2\) is bounded (by 2);
3. on the support of \(\varphi_3\), \(x_1\) is bounded away from 0 (by 3), and \(x_1x_2\) is bounded away from zero (by 1).

We will use \((\Psi, g_4) = (\Psi, \varphi_{1g_4}) + (\Psi, \varphi_{2g_4}) + (\Psi, \varphi_{3g_4})\) and continue the three components of \((\Psi, g_4)\) separately. The proof of Corollary 4.4 uses only the properties of \(g\) stated in the theorem, so even though no \(\varphi_jg_4\) is the function \(g\) of Equation (3.6) for any \(L\)-function (it cannot be because it is singular along \(x_2 = 1\), which \(g\) never is), we can still apply Corollary 4.4 to continue each \((\Psi, \varphi_jg_4)\).

First, \(\varphi_3(1 - x_2)g_4\) is in \(S(R)\) by an application of Lemma 10.4, so we can continue \((\Psi, \varphi_3g_4)\) using Corollary 4.4 with \(A(x) = 1/(1 - x_2)\).

To continue \((\Psi, \varphi_2g_4)\), we have,
\[
\varphi_2g_4(x) = \frac{\log(1 - e^{-x_1x_2})}{1 - x_2}\varphi_2(x)\xi(x),
\]
where $\xi(x) = -k_3(x)(1-x_2)\log(1-e^{-x_1})$. Then $\varphi_2\xi$ is in $S(R)$ by Lemma 10.3, and by Equation (10.5),

$$
\varphi_2 g_4(x) = \frac{\log x_1 + \log x_2 + f(x_1 x_2)}{1-x_2} \varphi(x) \xi(x).
$$

Because $|x_1 x_2| \leq 2$ on the support of $\varphi_2$, the partial derivatives of $f(x_1 x_2)$ grow no faster than polynomially in $x_1$ on the support of $\varphi_2$, so $\varphi_2(x) f(x_1 x_2) \xi(x)$ is in $S(R)$. Splitting $\varphi_2 g_4$ into three parts and applying Corollary 4.4 three times with $A(x) = (\log x_1)/(1-x_2)$, $A(x) = (\log x_2)/(1-x_2)$, and $A(x) = 1/(1-x_2)$ establishes the continuation of $(\Psi, \varphi_2 g_4)$ to a function meromorphic on $\mathbb{C}^2$.

To continue $(\Psi, \varphi_1 g_4)$, we have,

$$
\varphi_1 g_4(x) = \frac{\log(1-e^{-x_1 x_2})}{1-x_2} \log(1-e^{-x_1}) \varphi(x) \xi(x),
$$

where $\xi(x) = -k_3(x)(1-x_2)$, and $\varphi_1 \xi$ is in $S(R)$. Using Equation (10.5) twice gives

$$
\varphi_1 g_4(x) = \frac{(\log(x_1 x_2) + f(x_1 x_2))(\log x_1 + f(x_1))}{1-x_2} \varphi(x) \xi(x)
$$

$$
= \frac{(\log x_1)^2}{1-x_2} \varphi(x) \xi(x) + \frac{\log x_1 \log x_2}{1-x_2} \varphi(x) \xi(x) + \frac{\log x_1 f(x_1) \varphi(x) \xi(x)}{1-x_2} + \frac{\log x_2}{1-x_2} f(x_1) \varphi(x) \xi(x)
$$

$$
+ \frac{1}{1-x_2} f(x_1) f(x_1 x_2) \varphi(x) \xi(x).
$$

Because both $x_1$ and $x_1 x_2$ are bounded on the support of $\varphi_1$, each of $f(x_1) \varphi(x) \xi(x)$, $f(x_1 x_2) \varphi(x) \xi(x)$, and $f(x_1) f(x_1 x_2) \varphi(x) \xi(x)$ are in $S(R)$. Hence, we can continue $(\Psi, \varphi_1 g_4)$ by splitting $\varphi_1 g_4$ into six parts and using $A(x) = (\log x_1)^2/(1-x_2)$, $A(x) = \log x_1 \log x_2/(1-x_2)$, $A(x) = \log x_1/(1-x_2)$ (twice), $A(x) = \log x_2/(1-x_2)$, and $A(x) = 1/(1-x_2)$.

**Lemma 10.4.** Define $F : R \rightarrow \mathbb{C}$ by $F(x) = \log(1-e^{-x_1 x_2}) \theta(x)$ where $\theta$ is in $S(R)$ and is supported on a region of $R$ for which $x_1 x_2$ is bounded away from zero. Then $F$ is in $S(R)$.

**Proof.** Because $x_1 x_2$ is bounded away from zero on the support of $\theta$, the function $\log(1-e^{-x_1 x_2})$ is bounded on the support of $\theta$. By a direct calculation, it can also be shown that each partial derivative of $\log(1-e^{-x_1 x_2})$ is bounded by a polynomial in $x_1$ on the support of $\theta$. By repeated use of the Leibniz rule it follows that $F$ is in $S(R)$.

**10.7. Continuation of $(\Psi, g_5)$.** We use the same partition of unity as in Section 10.6. Let

$$
H(x) = \text{dilog}(1-e^{-x_1 x_2}) - \text{dilog}(\alpha(x)),
$$

where

$$
\alpha(x) = (1-e^{-x_1 x_2})/(1-e^{-x_1})
$$
assumes values in $(0, 1)$ on $R$. Then by Lemma 10.8, $\varphi_3(x)(1 - x_2)g_5(x) = \varphi_3(x)(1 - x_2)k_3(x)H(x)$ is in $S(R)$, and $(\Psi, \varphi_2g_5)$ can be continued to a function meromorphic on $\mathbb{C}^2$ by applying Corollary 4.4 with $A(x) = 1/(1 - x_2)$.

Now consider the continuation of $(\Psi, \varphi_2g_5)$. Using Equation (10.6), we have,

$$
\text{dilog}(1 - e^{-x_1x_2}) = \frac{\pi^2}{6} - \text{dilog}(e^{-x_1x_2}) - \log(e^{-x_1x_2}) \log(1 - e^{-x_1x_2})
$$

(10.8)

and

$$
\text{dilog}(\alpha(x)) = \frac{\pi^2}{6} - \text{dilog}(1 - \alpha(x)) - \log(\alpha(x)) \log(1 - \alpha(x)),
$$

(10.9)

so

$$
H(x) = \text{dilog}(1 - \alpha(x)) - \text{dilog}(e^{-x_1x_2}) + x_1x_2 \log(1 - e^{-x_1x_2})
$$

$$
+ \log(\alpha(x)) \log(1 - \alpha(x))
$$

$$
= F_1(x_1, x_2) + x_1x_2 \log(1 - e^{-x_1x_2}) + (\log(1 - e^{-x_1x_2}) - \log(1 - e^{-x_1}))
$$

$$
(\log(e^{-x_1x_2} - e^{-x_1}) - \log(1 - e^{-x_1}))
$$

$$
= F_1(x_1, x_2) + x_1x_2 \log(1 - e^{-x_1x_2}) + (\log(1 - e^{-x_1x_2}) - \log(1 - e^{-x_1}))
$$

$$
(\log(e^{-x_1x_2} - e^{-x_1} - x_1x_2) - \log(1 - e^{-x_1}))
$$

$$
= F_1(x_1, x_2) + (\log(1 - e^{-x_1x_2}) - \log(1 - e^{-x_1}))
$$

$$
(\log(1 - e^{-x_1} - x_1x_2) - \log(1 - e^{-x_1})) + x_1x_2 \log(1 - e^{-x_1})
$$

$$
= F_1(x_1, x_2) + \log(1 - e^{-x_1x_2}) (\log(1 - e^{-x_1x_2} - 1) - \log(1 - e^{-x_1}))
$$

$$
- \log(1 - e^{-x_1}) (\log(1 - e^{-x_1x_2} - 1) - \log(1 - e^{-x_1}) - x_1x_2)
$$

$$
= F_1(x_1, x_2) + F_2(x_1, x_2) + F_3(x_1, x_2),
$$

where

$$
F_1(x_1, x_2) = \text{dilog}(1 - \alpha(x)) - \text{dilog}(e^{-x_1x_2}),
$$

(10.10)

$$
F_2(x_1, x_2) = - \log(1 - e^{-x_1}) (\log(1 - e^{-x_1x_2}) - \log(1 - e^{-x_1}) - x_1x_2),
$$

and

$$
F_3(x_1, x_2) = \log(1 - e^{-x_1x_2}) (\log(1 - e^{-x_1x_2}) - \log(1 - e^{-x_1})).
$$

The function $\varphi_2F_1$ is in $S(R)$ by Lemma 10.6 and $\varphi_2F_2$ is in $S(R)$ by Lemma 10.3 and Lemma 10.5. Then $\varphi_2(x)(1 - x_2)k_3(x)(F_1 + F_2)(x)$ is in $S(R)$; therefore, the function $(\Psi, \varphi_2(x)k_3(x)(F_1 + F_2)(x))$ can be continued to a meromorphic function on $\mathbb{C}^2$ by applying Corollary 4.4 with $A(x) = 1/(1 - x_2)$.

Also, $\varphi_2(x)(\log(1 - e^{-x_1(1-x_2)}) - \log(1 - e^{-x_1}))$ is in $S(R)$ by Lemma 10.3 and Lemma 10.5, so $(\Psi, \varphi_2(x)k_3(x)F_3(x))$ can be continued to a meromorphic function
on \(\mathbb{C}^2\) using Corollary 4.4 as in the continuation of \(\varphi_2 g_4\) in Section 10.6. Since \(g_5 = k_3(F_1 + F_2 + F_3)\), it follows that \((\Psi, \varphi_2 g_5)\) can be continued to a meromorphic function on \(\mathbb{C}^2\).

To continue \((\Psi, \varphi_1 g_5)\), the final piece of \((\Psi, g_3)\), we have

\[
(\Psi, \varphi_1 g_5) = (\Psi, \varphi_1 k_3(x) \dilog(1 - e^{-x_1 x_2})) - (\Psi, \varphi_1 k_3(x) \dilog(\alpha(x))).
\]

Using Equation (10.8) and Equation (10.5), we have

\[
dilog(1 - e^{-x_1 x_2}) = \frac{\pi^2}{6} - \dilog(e^{-x_1 x_2}) + x_1 x_2(\log x_1 + \log x_2 + f(x_1 x_2)),
\]

where \(f\) is entire as a function of one complex variable. Then, since \(\dilog\) is real analytic on \((0, 2)\) and hence on any domain of \((0, 1]\) that is bounded away from zero, \(\varphi_1(\pi^2/6 - \dilog(e^{-x_1 x_2}) + x_1 x_2 f(x_1 x_2))\) is in \(S(R)\), and we continue \((\Psi, \varphi_1 k_3(x) \dilog(1 - e^{-x_1 x_2}))\) using Corollary 4.4 as in the continuation of \(\varphi_2 g_4\) in Section 10.6.

Now let \(\varphi_{1,1}\) and \(\varphi_{1,2}\) be functions in \(C^\infty(R)\) such that \(\varphi_1 = \varphi_{1,1} + \varphi_{1,2}\) with \(\varphi_{1,1} \equiv 0\) on \((0, \infty) \times (0, 1/4)\) and \(\varphi_{1,2} \equiv 0\) on \((0, \infty) \times (3/4, 1)\), and observe that \(\alpha\) is in \(C^\infty(R)\), \(1 - \alpha\) is bounded away from zero on the support of \(\varphi_{1,1}\), and \(1 - \alpha\) is bounded away from zero on the support of \(\varphi_{1,2}\). Thus, \((\Psi, \varphi_{1,1} k_3(x) \dilog(\alpha(x)))\) can be continued using Corollary 4.4 with \(A(x) = 1/(1 - x_2)\).

To continue \((\Psi, \varphi_{1,2} k_3(x) \dilog(\alpha(x)))\), we can take advantage of Equation (10.6). Because \(1 - \alpha\) is bounded away from zero on the support of \(\varphi_{1,2}\), the function \(\varphi_{1,2}(\pi^2/6 - \dilog(1 - \alpha(x)))\) is in \(S(R)\), so \((\Psi, \varphi_{1,2} k_3(x)(\pi^2/6 - \dilog(1 - \alpha(x)))\) is meromorphic, again using Corollary 4.4 with \(A(x) = 1/(1 - x_2)\).

Since \(\varphi_{1,2} \log(1 - \alpha(x))\) is in \(C^\infty(R)\), we have,

\[
\varphi_{1,2} k_3(x) \log(\alpha(x))(1 - \alpha(x)) = \frac{\log(\alpha(x))}{1 - x_2} \eta(x),
\]

where \(\eta(x) = \varphi_{1,2}(1 - x_2) k_3(x) \log(1 - \alpha(x))\) is in \(S(R)\). Then,

\[
\log(\alpha(x)) = \log(1 - e^{-x_1 x_2}) - \log(1 - e^{-x_1}),
\]

giving singularities like those for \(\varphi_1 g_4\) in Section 10.6 and \(g_1\) in Section 10.3, so we can continue \((\Psi, \varphi_{1,2} k_3(x) \log(\alpha(x)) \log(1 - \alpha(x)))\) using the approaches in those sections.

Using Equation (10.6) and adding together the three functions we have continued gives the continuation of \((\Psi, \varphi_1 k_3(x) \dilog(\alpha(x)))\), and hence of \((\Psi, \varphi_1 g_5)\) and, finally, of \(L\).

**Lemma 10.5.** The function \(\log(1 - e^{-x_1(1-x_2)})\) is in \(S(\Omega)\) for any domain \(\Omega\) in \(R\) for which \(x_1\) is bounded away from 0 and \(x_2\) is bounded away from 1.

**Proof.** First observe that

\[
|\log(1 - e^{-x_1(1-x_2)})| \leq |\log(1 - e^{-c_0 x_1})|
\]

where \(c_0 = \inf_{x \in \Omega}(1 - x_2)\). Then for any multi-index \(\beta\),

\[
\sup_{x \in \Omega} |x^\beta \log(1 - e^{-x_1(1-x_2)})| \leq \sup_{x \in \Omega} |x^\beta \log(1 - e^{-c_0 x_1})| \leq \sup_{x \in \Omega} |x_1^\beta \log(1 - e^{-c_0 x_1})|.
\]

This is finite, as we can see by using L’Hospital’s rule as in the proof of Lemma 10.3 (where \(c_0\) was equal to 1), since also \(x_1\) is bounded away from 0.
One can show inductively that for any multi-index $\alpha$ for which $|\alpha| \geq 1$,
\[
D^\alpha \log(1 - e^{-x_1(1-x_2)}) = \sum_{j=1}^{|\alpha|} P_j(x_1, 1 - x_2)(r(x))^j,
\]
where $r(x) = 1/(e^{x_1(1-x_2)} - 1)$ and $P_j$ are polynomials. But $|r(x)| \leq 1/(e^{c_\alpha x_1} - 1)$, so for any multi-index $\beta$,
\[
\sup_{x \in \Omega} |x^\beta (r(x))^j| \leq \sup_{x \in \Omega} |x^\beta|/(e^{c_\alpha x_1} - 1)^j \leq \sup_{x \in \Omega} x_1^\beta|/(e^{c_\alpha x_1} - 1)^j
\]
is finite, since also $x_1$ is bounded away from 0. It follows that $\log(1 - e^{-x_1(1-x_2)})$ is in $S(\Omega)$.

**Lemma 10.6.** The function $F_1$ of Equation (10.10) is in $S(\Omega)$, where $\Omega$ is the interior of the support of $\varphi_2$.

**Proof.** Because
\[
1 - \alpha(x) = e^{-x_1 x_2} - e^{-x_1} \alpha(x),
\]
as $x_1$ grows large, the two terms in $F_1$ become exponentially close, and so cancel to produce exponential decay. Our proof of this lemma is a formal verification of this observation.

Let $\beta$ be a multi-index. Then for integer constants $C_{\gamma,\delta}$,
\[
D^\beta F_1(x_1, x_2) = \sum_{\gamma: \gamma \leq \beta} C_{\gamma,\delta} \left[ \dilog(\alpha(x)) \prod_{k=1}^{n(\gamma)} D^{\delta(k)}(1 - \alpha(x)) \right.
\]
\[
- \dilog(\alpha(x)) \prod_{k=1}^{n(\gamma)} D^{\delta(k)}(e^{-x_1 x_2})
\]
\[
= \sum_{\gamma: \gamma \leq \beta} C_{\gamma,\delta} \left[ \dilog(\alpha(x)) - \dilog(\alpha(x)) \right]
\]
\[
+ \sum_{\gamma: \gamma \leq \beta} C_{\gamma,\delta} \dilog(\alpha(x)) \prod_{k=1}^{n(\gamma)} D^{\delta(k)}(1 - \alpha(x)) - \prod_{k=1}^{n(\gamma)} D^{\delta(k)}(e^{-x_1 x_2}),
\]
where $n(\gamma) \leq |\gamma|$ and the $\delta(\gamma, k)$ are multi-indices. (For many values of $\gamma$, there will be distinct values of $k_1$ and $k_2$ such that $\delta(\gamma, k_1) = \delta(\gamma, k_2)$).

Using Equation (10.11) and applying the mean value theorem, we have
\[
|\dilog(\alpha(x)) - \dilog(\alpha(x))| = |\dilog(\alpha(x)) - \dilog(\alpha(x)) - \dilog(\alpha(x)) - \dilog(\alpha(x))| = |\alpha(x)| \dilog(\alpha(x)) + |\alpha(x)| \dilog(\alpha(x)) + 1(\xi)|
\]
for some $\xi$ in $(1 - \alpha(x), e^{-x_1x_2})$.

Because $|\text{dilog}^{(k)}(x)|$ is decreasing on $(0,1)$ for all nonnegative integers $k$, by Lemma 10.9, $|\text{dilog}^{(|\beta|-|\gamma|)+1}(\xi)| \leq |\text{dilog}^{(|\beta|-|\gamma|)}(1 - \alpha(x))| \leq |\text{dilog}^{(|\beta|-|\gamma|)}(e^{-2}/2)|$, which is a constant that depends only on $|\beta| - |\gamma|$. Since also $0 < \alpha(x) < 1$, we have

$$|\text{dilog}^{(|\beta|-|\gamma|)}(e^{-x_1x_2}) - \text{dilog}^{(|\beta|-|\gamma|)}(1 - \alpha(x))| \leq C_{|\beta|-|\gamma|} e^{-x_1}.$$ 

By the same reasoning,

$$\left|\text{dilog}^{(|\beta|-|\gamma|)}(1 - \alpha(x))\right| \leq C'_{|\beta|-|\gamma|}. $$

Also,

$$\left|D^{\delta(1)}(e^{-x_1x_2})\right| = x_1^{\delta(1)} e^{-x_1x_2} \leq P_\delta(x),$$

(10.12)

where $P_\delta$ is a polynomial.

Combining these bounds with the bound in Lemma 10.7 and the bound on $D^\delta F_1(x_1, x_2)$ above, it follows that $F_1$ is in $\mathcal{S}(\Omega)$.

**Lemma 10.7.** Let $n$ be a nonnegative integer, and let $\delta(k), k = 1, \ldots, n$ be any multi-index. Then for some polynomial $P_\delta$,

$$\left|\prod_{k=1}^n D^{\delta(k)}(1 - \alpha(x)) - \prod_{k=1}^n D^{\delta(k)}(e^{-x_1x_2})\right| \leq P_\delta(x) e^{-x_1}$$

for all $x$ in the interior of the support of $\varphi_2$.

**Proof.** We proceed by induction on $n$. For $n = 1$, we have, using Equation (10.11),

$$D^{\delta(1)}(1 - \alpha(x)) - D^{\delta(1)}(e^{-x_1x_2}) = -D^{\delta(1)}(e^{-x_1}\alpha(x)) = -\partial_1^{\delta(1)}(e^{-x_1}\partial_2^{\delta(1)}\alpha(x))$$

$$= \sum_{k=0}^{\delta(1)} C_{k,\delta(1)} e^{-x_1}\partial_1^{k}\partial_2^{\delta(1)}\alpha(x) = e^{-x_1}\sum_{k=0}^{\delta(1)} C_{k,\delta(1)} D^{(k,\delta(1))}\alpha(x),$$

where $C_{k,\delta(1)}$ are integer constants.

One can show by induction that $D^{(k,\delta(1))}\alpha(x)$ is always a finite sum of terms of the form

$$C e^{-k_1x_1x_2} x_1^{e_1} x_2^{e_2}$$

for integer constants $k_1, k_2, k_3$, and $C$, and nonnegative integer constants $e_1$ and $e_2$, with $e_1 + e_2 \leq k + \delta(1)_2$. Thus,

$$\left|D^{\delta(1)}(1 - \alpha(x)) - D^{\delta(1)}(e^{-x_1x_2})\right| \leq P_\delta(x) e^{-x_1}$$

on the support of $\varphi_2$ for some polynomial $P_\delta$, establishing the induction hypothesis for $n = 1$. It also follows from Equation (10.12), which holds for any multi-index, that

$$\left|D^{\delta(1)}(1 - \alpha(x))\right| \leq P_\delta(x).$$

(10.13)
Now assume that the induction hypothesis is true for \( n = m - 1 \). Then
\[
\left| \prod_{k=1}^{m} D^{\delta(k)}(1 - \alpha(x)) - \prod_{k=1}^{m} D^{\delta(k)}(e^{-x_1 x_2}) \right|
\leq \left| D^{\delta(m)}(1 - \alpha(x)) \right| \left| \prod_{k=1}^{m-1} D^{\delta(k)}(1 - \alpha(x)) - \prod_{k=1}^{m-1} D^{\delta(k)}(e^{-x_1 x_2}) \right|
+ \left| D^{\delta(m)}(1 - \alpha(x)) - D^{\delta(m)}(e^{-x_1 x_2}) \right| \left| \prod_{k=1}^{m} D^{\delta(k)}(e^{-x_1 x_2}) \right|.
\]

The induction hypothesis for \( n = m \) now follows by applying Equation (10.12) and Equation (10.13), which hold for any multi-indices, along with the induction hypothesis for \( n = 1 \) and the induction hypothesis for \( n = m - 1 \).

\[ \square \]

**Lemma 10.8.** The function \( H \) of Equation (10.7) is in \( S(\Omega) \), where \( \Omega \) is the interior of the support of \( \varphi_3 \).

**Proof.** The proof is almost identical to that of Lemma 10.6. Equation (10.11) is still the starting point for the proof, though instead of having \( 1 - \alpha(x) \) bounded away from 0 by \( e^{-2}/2 \), we now have \( 1 - e^{-x_1 x_2} \) bounded away from 0 by \( 1 - e^{-1} \). In all other respects, the proof parallels that of Lemma 10.6 so closely, that we suppress the details.

\[ \square \]

**Lemma 10.9.** On the support of \( \varphi_2 \), \( 1 - \alpha(x) > e^{-x_1 x_2}/2 > e^{-2}/2 \).

**Proof.** On the support of \( \varphi_2 \),
\[
x_1 > 3 > 2 + \log 2 \text{ and } x_1 x_2 < 2
\Rightarrow x_1(1 - x_2) = x_1 - x_1 x_2 > x_1 - 2 > \log 2
\Rightarrow e^{x_1(1-x_2)} > 2 \Rightarrow \frac{1}{2} e^{-x_1 x_2} > e^{-x_1} > e^{-x_1} \alpha(x)
\Rightarrow e^{-x_1 x_2} > e^{-x_1} \alpha(x) + \frac{1}{2} e^{-x_1 x_2}
\Rightarrow 1 - \alpha(x) = e^{-x_1 x_2} - e^{-x_1} \alpha(x) > \frac{1}{2} e^{-x_1 x_2} > e^{-2}/2,
\]

where we used Equation (10.11).

\[ \square \]

10.8. **The singularity along** \( x_2 = 1 \). Each of the functions \( g_1 \) through \( g_5 \) has a singularity along \( x_2 = 1 \) because of the factors \( k_1, k_2, \) and \( k_3 \). In removing this singularity, we introduced a simple pole at \( s_1 = 1 \) into the continuation of \( L \). The test function \( g \) itself, however, we know has singularities only along \( x_1 = 0 \) and \( x_2 = 0 \); when added the singularities of \( g_1 \) through \( g_5 \) cancel along \( x_2 = 1 \) (though \( (0, 1) \) is still a singular point). It does not appear feasible to avoid splitting \( g \) into components to continue \( L \); however, a careful analysis of our argument would give information about the residues at \( s = 1 \). It is possible that we would find that these residues cancel.
Appendix A. An extension operator on $S(R)$

In this section we prove two analytic results used in the proofs of section 2. The second of these results consists of the construction of an extension operator on $S(R)$.

Lemma A.1. Let $y = y(x)$ be the transformation $y_k = x_1 \cdots x_k$, $k = 1, \ldots, d$. If $f$ is in $S(\Delta)$ then $f \circ y$ is in $S(R)$, where $R$ is defined in Equation (3.4) and $\Delta$ is defined in Equation (4.4).

Proof. First observe that $\Delta = y^{-1}(R)$.

For $d = 1$ there is nothing to prove since $y_1 = x_1$. Assume that $d = 2$, and let $\tilde{f} = f \circ y$. Then since $y_1 = x_1$ and $y_2 = x_1 x_2$,

$$\frac{\partial^k \tilde{f}}{\partial x_1^k} = \frac{\partial^k f}{\partial y_1^k},$$

$$\frac{\partial \tilde{f}}{\partial x_2} = \frac{\partial f}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial f}{\partial y_2} \frac{\partial y_2}{\partial x_2} = x_1 \frac{\partial f}{\partial y_2},$$

$$\frac{\partial^2 \tilde{f}}{\partial x_1^2} = x_1 \frac{\partial}{\partial y_2} \left( \frac{\partial f}{\partial y_2} \right) = x_1 \frac{\partial f}{\partial y_2} .\ldots,$$

$$\frac{\partial^k \tilde{f}}{\partial x_2^k} = x_1 \frac{\partial^k f}{\partial y_2^k}.$$

Thus, for any multi-index $\alpha$,

$$D^\alpha \tilde{f} = x_1^{\alpha_1} D^\alpha f = y_1^{\alpha_2} D^\alpha f.$$

If $f$ is in $S(\Delta)$ then for any multi-indices $\alpha$ and $\beta$,

$$\sup_{x \in R} \left| x^\alpha D^\beta \tilde{f} \right| \leq \sup_{x \in R} \left| x_1^{\alpha_1} D^\beta \tilde{f} \right| = \sup_{y \in \Delta} \left| y_1^{\alpha_1} y_1^{\beta_1} D^\beta f \right| < \infty.$$

(The inequality above follows from observing that $|x_k| < 1$ for all $k = 1, \ldots, d$.) We conclude that $\tilde{f}$ is in $S(R)$.

For $d \geq 3$, though the derivatives will involve a sum of terms, each term can be bounded as above, and the proof is very similar. $\square$

Lemma A.2. There exists a continuous linear extension operator $E$ that maps $S(R)$ to $S(\mathbb{R}^d)$ and $S^\gamma(R)$ to $S^\gamma(\mathbb{R}^d)$ for any multi-index $\gamma$, where $R$ is defined as in Equation (3.4).

Proof. For simplicity of notation, we give the proof for $d = 2$; nothing significant changes for $d > 2$. Also, the proof for $d = 1$ is an obvious simplification of the argument for $d = 2$.

Let $f$ be in $S(R)$. We prove the existence of the extension $\tilde{f} := Ef$ in three steps, as follows:

**Step 1:** We extend $f$ to a function $u$ in $C^\infty(\mathbb{R}^2)$ much as in the proof of Theorem 5’ of Chapter VI of [17], though we do so explicitly so we can more easily make the calculations required to establish Schwartz decay.


Because $f$ is in $C^\infty(R)$, we can extend $f$ continuously to the boundary of $R$. We then define $u$ on $(-\infty,0] \times (0,1)$ as in Equation (24) p. 182 of [17] by

$$u(x,y) = \int_1^\infty f(x - c_0\lambda x, y)\psi(\lambda)\,d\lambda$$

(A.1)

for $x \leq 0$, and $u(x,y) = f(x,y)$ for $x$ in $R$. Here, $\psi$ is as in Lemma 1 p. 182 of [17], and we use $c_0$ in place of $c_0$ of [17]. Also, in the notation of [17], $\Delta(x,y) = -x$.

Because $f$ is in $S(R)$, $u$ and all its derivatives are continuous, as we can verify directly from Equation (A.1); hence, $u$ is in $C^\infty((-\infty,\infty) \times (0,1))$.

Next we extend $u$ to $\mathbb{R}^2$ as follows. Let $\{\phi_-, \phi_+\}$ be a partition of unity of $R$ defined so that $\phi_+$ is equal to 1 on the set $\{(x,y) \in R : 3/4 \leq y < 1\}$, $\phi_-$ is equal to 1 on the set $\{(x,y) \in R : 0 < y \leq 1/4\}$, and both are constant along horizontal lines.

Then define $u_-$ and $u_+$ in $C^\infty(\mathbb{R}^2)$ by

$$u_-(x,y) = \begin{cases} \int_1^\infty (u\phi_-(x,y) - c_0\lambda y)\psi(\lambda)\,d\lambda, & y \leq 0, \\ u(x,y)\phi_-(x,y), & 0 < y < 1, \\ 0, & y \geq 1, \end{cases}$$

$$u_+(x,y) = \begin{cases} \int_1^\infty (u\phi_+(x,y) + c_0\lambda(y - 1))\psi(\lambda)\,d\lambda, & y \geq 1, \\ u(x,y)\phi_+(x,y), & 0 < y < 1, \\ 0, & y \leq 0. \end{cases}$$

In both integrals above we treat $u$ as being zero whenever $\phi_-$ or $\phi_+$ is zero (the value we choose for $u$ does not matter).

Finally, define $u$ in $C^\infty(\mathbb{R}^2)$ by

$$u(x,y) = u_-(x,y) + u_+(x,y),$$

and observe that $u$ is an extension of $f$ to all of $\mathbb{R}^2$, and $u$ is in $C^\infty(\mathbb{R}^2)$ by the same reasoning as before.

**Step 2:** Let $\varphi_h$ and $\varphi_v$ in $C^\infty(\mathbb{R}^2)$ assume values in $[0,1]$ and be such that $\varphi_h \equiv 1$ on $[0,\infty)$, $\varphi_h \equiv 0$ on $(-\infty,-1]$, $\varphi_v \equiv 1$ on $[0,1]$, and $\varphi_v \equiv 0$ on $[2,\infty)$ and on $(-\infty,-1]$. Then $\varphi := \varphi_h\varphi_v$ is in $C^\infty(\mathbb{R}^2)$ and assumes values in $[0,1]$, is identically 1 on $R$, and is identically 0 on the complement in $\mathbb{R}^2$ of $(-1,\infty) \times (-1,2)$.

Define $\tilde{f}$ in $C^\infty(\mathbb{R}^2)$ by

$$\tilde{f} = \varphi u.$$

**Step 3:** The function $\tilde{f}$ has Schwartz decay in all directions except possibly along the positive $x$-axis when $y$ is in $[1,2]$ or in $(-1,0]$, because in all other directions, $\tilde{f}$ either equals $f$, which has Schwartz decay, or becomes zero after a finite distance. So we need only show that $|x^n y^m \partial^j x \partial^k y f(x,y)|$ is bounded for all nonnegative integers $m$, $n$, $j$, and $k$ on two subsets of $\mathbb{R}^2$: $R_1 = (0,\infty) \times (-1,0)$ and $R_2 = (0,\infty) \times (1,2)$.

First we consider only partial derivatives of $x$. Assume that $(x,y)$ is in $R_1$, and that $m$, $n$, and $j$ are nonnegative integers. Then, since $\varphi$ is constant along horizontal
rays in $R_1$,
\[
|x^my^n\partial_x^j\tilde{f}(x, y)| = |\varphi(x, y)x^my^n\partial_x^j u(x, y)| \leq |x^my^n\partial_x^j u(x, y)| = |x^my^n\partial_x^j u_-(x, y)|
\]
\[
= |x^my^n\partial_x^j \int_1^{\infty} (f\phi_-)(x, y - c_0\lambda y)\psi(\lambda) \, d\lambda|
\]
\[
= \int_1^{\infty} (x^my^n\partial_x^j f(x, y - c_0\lambda y))\phi_-\psi(\lambda) \, d\lambda
\]
\[
\leq \sup |\psi| \sup_{y' \in (0,1)} |x^my^n\partial_x^j f(x, y')| \left| \int_1^{\infty} \phi_-(x, y - c_0\lambda y) \, d\lambda \right|
\]
\[
\leq \frac{1}{c_0} \sup |\psi| \sup_{y' \in (0,1)} |x^my^n\partial_x^j f(x, y')|.
\]

The second and third equalities follow from the definitions of $u$ and $u_-$ (and $u$ becomes $f$ in the integral because $x > 0$). The fourth equality uses the constancy of $\phi_-$ along horizontal lines. The last inequality follows by a change of variables and the observation that $\phi_-$ is supported in a strip of vertical width less than 1.

Thus,
\[
\sup_{(x, y) \in R_1} |x^my^n\partial_x^j\tilde{f}(x, y)| \leq \frac{1}{c_0} \sup |\psi| \sup_{x > 0} \sup_{y' \in (0,1)} |x^my^n\partial_x^j f(x, y')|
\]
\[
= \frac{1}{c_0} \sup |\psi| \sup_{(x, y') \in R} |x^my^n\partial_x^j f(x, y')|,
\]
which is finite by the assumption that $f$ is in $S(R)$. The bound on $R_2$ is obtained similarly.

Bounding $|x^my^n\partial_x^j\partial_y^k\tilde{f}(x, y)|$ is more tedious, because both $\varphi$ and $\phi_-$ have nonzero partial derivatives in the $y$-direction. If we write this as $|x^my^n\partial_x^j\partial_y^k\tilde{f}(x, y)|$, we can start with the calculation above then perform the partial derivatives in $y$. This will result in a sum of terms including partial derivatives of $\varphi$, $\phi_-$, and $f$. Each term, however, will be just as above, with $\varphi$ and $\phi_-$ replaced by partial derivatives of these functions, and with partial derivatives in both $x$ and $y$. Since all the partial derivatives of $\varphi$ and $\phi_-$ are bounded, this does not change the argument for each term, and we see that $|x^my^n\partial_x^j\partial_y^k\tilde{f}(x, y)|$ is bounded as well.

The linearity of the extension operator $\mathcal{E}f = \tilde{f}$ is clear from the definition of $\tilde{f}$, and its continuity follows from the bounds we established above.

\[\square\]

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