ANALYTIC CONTINUATION OF MULTIPLE DIRICHLET SERIES USING DISTRIBUTIONS

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Abstract. We use Gelfand’s theory of analytic continuation of tempered distributions and Stein’s theory of bounded extension operators to prove a sufficient condition for the meromorphic continuation to \( \mathbb{C}^d \) of a class of \( d \)-dimensional multiple Dirichlet series of Euler-Zagier type.

1. Introduction and statements of the main results

A \( d \)-dimensional multiple Dirichlet series, \( d \geq 1 \), is a function of the form

\[
L(s) = \int_X a(n_1, \ldots, n_d) n_1^{-s_1} \cdots n_d^{-s_d} d\mu(n_1, \ldots, n_d)
\]

where \( s = (s_1, \ldots, s_d) \), \((X, \mu)\) is a measure space, \( a \) is \( \mu \)-measurable, and the integral is over \( X \). For certain choices of \( a \), \( L \) will map a domain of \( \mathbb{C}^d \) into \( \mathbb{C} \). Special cases of Equation (1) include classical one-dimensional Dirichlet series, where \( \mu \) is a one-dimensional discrete measure, and the multiple Dirichlet series for the Riemann zeta function studied by Diaconu, Goldfeld, and Hoffstein in [DGH], where \( \mu \) is a product measure containing both discrete and continuous (Lebesgue) measures on \( \mathbb{R} \). Another special case of Equation (1) is a multiple Dirichlet series of Euler-Zagier type, which we study in this paper.

Let \( (a_{n,k})_{n=1}^{\infty}, k = 1, \ldots, d \), be \( d \) sequences of complex numbers, and assume that each sequence satisfies the growth condition

\[
|a_{n,k}| \leq C_k n^{r_k}, \quad k = 1, \ldots, d,
\]

for some nonnegative real constants \( C_k \) and \( r_k \). We define a multiple Dirichlet series of Euler-Zagier type by

\[
L(s) = \sum_{0 < n_1 < \cdots < n_d} a_{n_1,1} \cdots a_{n_d,d} n_1^{-s_1} \cdots n_d^{-s_d},
\]

which is absolutely convergent and analytic in the region

\[
\text{Re}(s_k + \cdots + s_d) > d - k + 1 + r_k + \cdots + r_d \quad \text{for} \quad k = 1, \ldots, d.
\]

Our primary objective in this paper is to use Gelfand’s theory of analytic continuation of tempered distributions and Stein’s theory of bounded extension operators.
for prove a sufficient condition for the meromorphic continuation of Equation (3) to \( \mathbb{C}^d \) (see Theorem 1.1).

In the important special case \( a_{n,k} = 1 \) for all \( n \) and \( k \), Equation (3) is the Euler-Zagier multiple zeta function of depth \( d \),
\[
\zeta_d(s) = \sum_{0 < n_1 < \ldots < n_d} n_1^{-s_1} \cdots n_d^{-s_d},
\]
which was defined by D. Zagier in [Za2]. The function \( \zeta_d(s) \) has been the focus of intense study in recent years, appearing in connection with arithmetic and hyperbolic geometry, moduli spaces, number theory, and quantum physics (see for example [BD, Go1, Ko1, Ko2, T, Za3]). Other examples of Equation (3) include the multiple Dirichlet \( L \)–function [Go2], the multiple Dedekind zeta function [Ma2, Z2], and multiple Dirichlet series [M] related to iterated Shimura integrals and noncommutative modular symbols.

In [Go2, Z1], it is shown that the multiple zeta function \( \zeta_d(s) \) continues to a meromorphic function on \( \mathbb{C}^d \), and the locations of the poles, all of which are simple, are identified. The authors employ a \( d \)-dimensional Mellin transform to show that the zeta function equals, in its region of absolute convergence, a meromorphic distribution applied to a test function. This gives an explicit continuation of \( \zeta_d(s) \) to all of \( \mathbb{C}^d \), the poles being the same as those of the distribution.

The meromorphic distribution in [Go2, Z1] is
\[
\Psi(x_1, \ldots, x_d; s_1, \ldots, s_d) = \frac{\Gamma(u_1) \cdots \Gamma(u_{d-1})}{(s_d - 1)!} \psi(x_1, \ldots, x_d; s_1, \ldots, s_d),
\]
where
\[
u_k = s_k + \cdots + s_d - d + k - 1
\]
and
\[
\psi(x_1, \ldots, x_d; s_1, \ldots, s_d) = \frac{(x_1)^{u_1-1} \cdots (x_d)^{u_d-1}}{\Gamma(u_1) \cdots \Gamma(u_d)} \prod_{k=2}^{d} \frac{(1 - x_k)^{s_k-1} - (x_k)^{s_k-1}}{\Gamma(s_k-1) \Gamma(u_k)}
\]
where \( t_+ := t(0, \infty) \). This is a regular distribution in the domain of \( \mathbb{C}^d \) defined by \( s_1 > 0, \ldots, s_{d-1} > 0 \) and \( u_1 > 0, \ldots, u_d > 0 \). Because, \( \psi \) continues to an entire distribution as we show in Lemma 4.2, the poles of \( \Psi \) arise from the other factors in Equation (5).

The test function of [Go2, Z1] is defined on
\[
R = (0, \infty) \times (0, 1)^{d-1}
\]
by
\[
g(x) = g(x_1, \ldots, x_d) = \frac{y_1(x) \cdots y_d(x)}{y_n(x) - 1} \cdots (y_n(x) - 1),
\]
and
\[
y_k = y_k(x) = x_1 \cdots x_k.
\]
Let $S(Ω)$ be the space of Schwartz class functions on an open subset $Ω$ of $\mathbb{R}^d$ (see Section 2). Since $Ψ$ is zero for $x$ outside of $R$, the value of the pairing $(Ψ, g)$ does not depend upon the value of $g$ outside of $R$; nonetheless, it is essential that $g$ extend to a test function on all of $\mathbb{R}^d$. One way to see that $g$ extends for the multiple zeta function $ζ_d(s)$ is to view $g$ as a function of $y = y(x)$. The function $g(y)$ then factors into a product, each factor of which is in $S((0, ∞))$. Hence $g(y)$ is in $S((0, ∞)^d) ∩ S(g(R))$.

By Lemma 3.1, $g(x)$ is thus in $S(R)$, which we show in Theorem 3.2 is a sufficient (and, of course, necessary) condition to insure that $g(x)$ extends to $S(\mathbb{R}^d)$.

If we apply the continuation argument of [Go2, Z1] to the multiple Dirichlet series Equation (3), as we show in the proof of Theorem 1.1, the only thing that changes is the nature of the test function $g$, which is now equal to

$$g(x) = \sum_{n_1=1}^{∞} \cdots \sum_{n_d=1}^{∞} a_{n_1, \ldots, n_d} y_1(x) e^{-n_1 y_1(x)} \cdots y_d(x) e^{-n_d y_d(x)}.$$  

(10)

Although $g$ is always in $C^∞(R)$ and in $S((ε, ∞) \times (ε′, 1)^{d-1})$ for all $ε > 0$ and $ε′$ in $(0, 1)$ (see the comment following the proof of Theorem 1.1), it will no longer necessarily extend to $S(\mathbb{R}^d)$ because it need no longer lie in $S(R)$, its behavior on the boundary (specifically, on $\{x \in \mathbb{R}^d : x_1 \cdots x_d = 0\}$) not being controllable a priori. Hence, we are abusing terminology when we refer to $g$ as a test function when it is not in $S(R)$ and so does not extend to a true test function in $S(\mathbb{R}^d)$.

Our main innovation is to circumvent this problem, in many cases, by shifting the poor behavior of the test function $g$ near the boundary of $R$ to the distribution $Ψ$ by dividing $g$ by a function $A$ and multiplying $Ψ$ by the same function $A$, giving a new test function and distribution. Since $\int_R (ΨA)(g/A) = \int_R Ψg$, if $A$ can be chosen so that $g/A$ is in $S(R)$ and $ΨA$ is meromorphic (though with more poles than $Ψ$), then the continuation argument can be completed as in [Go2, Z1], though now there will be more poles not all of which need be simple. This approach leads to the following theorem.

**Theorem 1.1.** Assume that Equation (2) is satisfied, and let $A : R → \mathbb{C}$ be such that the following two conditions hold:

1. $g/A$ is in $S(R)$, and
2. $ψA$ is a meromorphic distribution on $\mathbb{C}^d$.

Then by necessity, the absolute region of convergence of $ψA$ (that is, the region on which $ψA$ is a regular distribution) contains the region defined by

$$\text{Re } u_k > r_k + \cdots + (d - k + 1)r_d, \quad k = 1, \ldots, d$$

and $L$ meromorphically continues to all of $\mathbb{C}^d$ with the following possible poles:

1. a simple pole along the hyperplane $s_d = 1$;
2. a simple pole along the hyperplane $u_k = n$ for all integers $n ≤ 0$ for $k = 1, \ldots, d - 1$;
3. the same poles and same orders as those of $ψA$.  


If, however, a possible pole in (3) matches one in (1) or (2), then the possible order of the pole increases by 1.

**Remark 1.2.** The term pole has a dual meaning. In Theorem 1.1 and the corollaries that follow, we are using the interpretation given on p. 168 of [Gu2] of a pole as a holomorphic subvariety of dimension \(d - 1\), the order of the pole being defined on p. 168 of [Gu2] as well. The subvariety may contain both points of indeterminacy and poles in the sense of the definition on p. 164 of [Gu2]. In dimension 1, the two terms can be identified, and it is only in this case that we will use the second meaning of the term pole.

A pole is either empty or is a subvariety of dimension \(d - 1\). The points of indeterminacy form a subvariety that is either empty or of dimension \(d - 2\) (in dimension 1, then, there are no points of indeterminacy).

When we say that \(S\) is the set of possible poles of a function, we mean that the set of all poles of the function is a subset of \(S\). When we say that the possible order of a pole is \(n\), it means that the pole has an order greater-than-or-equal-to \(n\) (and so might not be a pole at all).

**Remark 1.3.** The region in Equation (11) is smaller than the region of absolute convergence of \(L\) in Equation (4), which means that it is not, in general, maximal.

There are, of course, practical difficulties in applying Theorem 1.1. The first difficulty one faces is in obtaining a closed form for the original test function \(g\), or at least sufficient information about \(g\) to understand its behavior on the boundary of \(R\). A functional relationship on the coefficients or on their generating function is usually needed. If \(g\) can be found, a function \(A\) can be determined by the asymptotic behavior of \(g\) near the boundary of \(R\). The second difficulty is in determining whether \(A\) meets the second criterion of Theorem 1.1, and if so, the location and order of the poles of \(\psi A\) that result. The conditions on \(A\) in Theorem 1.4 and Corollary 1.6 give fairly general and practicable sets of conditions to address the second difficulty. Both of these difficulties are considerably lessened in one dimension, where it is often possible to obtain a closed form expression for the test function \(g\), and where we need only understand the behavior of \(g\) near zero.

**Theorem 1.4.** Condition (2) of Theorem 1.1 that \(\psi A\) is meromorphic is met if \(A\) can be written in product form, \(A(x) = A_1(x_1) \cdots A_d(x_d)\), where each \(\psi_k A_k\) is a meromorphic distribution on \(u_1\) in \(\mathbb{C}\) for \(k = 1\) and on \((s_{k-1}, u_k)\) in \(\mathbb{C}^2\) for \(k > 1\), where

\[
\psi_1 := \frac{(x_1)^{u_1-1}}{\Gamma(u_1)} \quad \text{and} \quad \psi_k := \frac{(1 - x_k)^{s_k-1} (x_k)^{u_k-1}}{\Gamma(s_{k-1}) \Gamma(u_k)}, \quad k = 2, \ldots, d.
\]

(Here, we are treating \(u_1, \ldots, u_d\) and \(s_1, \ldots, s_{d-1}\) as independent complex variables.)

Furthermore, the poles of \(\psi A\) are precisely those of each \(\psi_k A_k\), \(k = 1, \ldots, d\), though now as codimension-one subvarieties of \(\mathbb{C}^d\) rather than of \(\mathbb{C}\) or \(\mathbb{C}^2\). (For instance, if \(d = 4\) and \(\psi_2 A_2\) has a pole along the one-dimensional hyperplane \(s_1 = 2\) of \(\mathbb{C}^2\), then \(\psi A\) has a pole along the three-dimensional hyperplane \(s_1 = 2\).)
Definition 1.5. The function \( g : \mathbb{R} \to \mathbb{R} \) has the singularity type of the function \( A : \mathbb{R} \to \mathbb{R} \) if \( g = fA \) for some \( f \) in \( S(\mathbb{R}) \), where \( R \) is defined in Equation (8). We denote the set of all functions with the singularity type of \( A \) by \( \text{Sing} A \). (Observe that \( A \) is not in \( \text{Sing} A \), since the constant function 1 is not in \( S(\mathbb{R}) \).)

Corollary 1.6 is a corollary of Theorem 1.1 and Theorem 1.4.

Corollary 1.6. Let \( p_1, \ldots, p_d, q_2, \ldots, q_d \) be real numbers, let \( v_1, \ldots, v_d, w_2, \ldots, w_d \) be nonnegative integers, and let

\[
A(x) = x^{-p_1} \cdots x_d^{-p_d} (1 - x_2)^{-q_2} \cdots (1 - x_d)^{-q_d} (\log x_1)^{v_1} \cdots (\log x_d)^{v_d} \\
(\log(1 - x_2))^{w_2} \cdots (\log(1 - x_d))^{w_d}.
\]

If \( g \) is in \( \text{Sing} A \), then \( L = (\Psi, g) \) can be continued to a function meromorphic on \( \mathbb{C}^d \). When \( p_1, \ldots, p_d \) and \( q_2, \ldots, q_d \) are nonnegative integers, \( L \) has possible poles along the following hyperplanes:

1. For \( k = 1, \ldots, d - 1 \), poles of order \( w_k + 1 \) along \( s_k = 1, \ldots, q_k \) and poles of order \( w_k \) along \( s_k = n \) for each integer \( n \leq 0 \);
2. A simple pole along \( s_d = 1 \);
3. For \( k = 1, \ldots, d - 1 \), poles of order \( v_k + 1 \) along \( u_k = 1, \ldots, p_k \) and poles of order \( v_k \) along \( u_k = n \) for each integer \( n \leq 0 \);
4. Poles of order \( v_d + 1 \) along \( u_d = 1, \ldots, p_d \) and poles of order \( v_d \) along \( u_d = n \) for each integer \( n \leq 0 \).

As we will see (in Equation (15)), \( g \) is always bounded by a function like \( A \) of Corollary 1.6. It is a much stronger statement, however, to say that \( g \) is in \( \text{Sing} A \).

We will use the techniques developed in this paper to prove that the double Dirichlet series whose coefficients are the harmonic numbers,

\[
a_{m,1} = a_{m,2} = H_m := \sum_{j=1}^{m} 1/j,
\]

meromorphically continues to all of \( \mathbb{C}^2 \). Specifically, we prove the following:

Theorem 1.7. The function

\[
L(s_1, s_2) = \sum_{0 < m < n} H_m H_n m^{-s_1} n^{-s_2}
\]

continues meromorphically to \( \mathbb{C}^2 \) with the following possible poles:

1. A simple pole along \( s_1 = 1 \);
2. A pole of order 3 along \( u_1 = n \) for each integer \( n \leq 0 \);
3. A pole of order 2 along \( u_2 = n \) for each integer \( n \leq 0 \).

Although the function \( L(s_1, s_2) \) is of some arithmetic interest (its special values are related to Zagier’s Kronecker limit formula for real quadratic fields [Ma1, Za1]), this is not our primary focus. Rather, we believe that the proof of Theorem 1.7 is important in that it illustrates the type of analytic techniques required to successfully apply our method.
Indeed, there has been a great deal of interest in the meromorphic continuation of a different type of multiple Dirichlet series of the form Equation (1) with interesting arithmetic applications (see e.g., [BFH, DGH]). Of particular interest is the multiple Dirichlet series for the Riemann zeta function studied in [DGH], whose meromorphic continuation to a certain tube domain in \( \mathbb{C}^d \) would imply a conjectured formula for the \( 2k \)-th moments of the Riemann zeta function. We believe that the techniques developed in this paper can be applied to the meromorphic continuation of this type of multiple Dirichlet series, and that the proof of Theorem 1.7 will be instructive in this regard.

This paper is organized as follows. In Section 2 we review the necessary background on analytic continuation of tempered distributions. In Section 3, which we believe is of some independent interest, we construct a continuous linear extension operator on the Schwartz class \( S(R) \). In Section 4 we prove the main results. In Section 5 we explain how the main results can be adapted to a more general multiple Dirichlet series. Finally, in Section 6 we apply our method to prove Theorem 1.7.

2. ANALYTIC CONTINUATION OF TEMPERED DISTRIBUTIONS

We give here a brief overview of the analytic continuation of tempered distributions, referring the reader to [GS], especially Sections 3.1-3.3 of Chapter 1, for a more in depth discussion. Section 2 of [Z1] is also helpful in this regard.

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \). Then \( S(\Omega) \), the Schwartz-class functions on \( \Omega \), are defined to be the set of all complex-valued \( C^\infty \)-functions \( f \) on \( \Omega \) such that

\[
\rho_{\alpha,\beta}(f) := \sup_{x \in \Omega} |x^\alpha D^\beta f(x)| < \infty
\]

for all \((d\text{-dimensional})\) multi-indices \( \alpha \) and \( \beta \). A multi-index \( \alpha \) is an ordered pair of \( d \) nonnegative integers \((\alpha_1, \ldots, \alpha_d)\), \( x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d} \), and

\[
D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}
\]

where \( |\alpha| = \alpha_1 + \cdots + \alpha_d \). This definition is equivalent to that of [GS] and [Z1] when \( \Omega = \mathbb{R}^d \), and it is only in this case that we use \( S(\Omega) \) as a space of test functions.

Endowed with the sufficient family of semi-norms, \( \{\rho_{\alpha,\beta}\} \), \( S(\mathbb{R}^d) \) is a Fréchet space.

A (tempered) distribution is an element of \( \mathcal{S}'(\mathbb{R}^d) \), the dual space of \( \mathcal{S}(\mathbb{R}^d) \); that is, the set of all continuous linear functionals on \( \mathcal{S}(\mathbb{R}^d) \), continuity being with respect to all the semi-norms \( \rho_{\alpha,\beta} \) separately. A distribution \( \psi \) applied to a test function \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \) is written as \((\psi, \varphi)\), the operation \((\cdot, \cdot)\) or, more explicitly, \((\cdot, \cdot)_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)}\), defining a pairing of \( \mathcal{S}'(\mathbb{R}^d) \) and \( \mathcal{S}(\mathbb{R}^d) \). If, for some locally integrable function \( \overline{\psi} \), \((\psi, \varphi) = \int_{\mathbb{R}^d} \overline{\psi} \varphi \) for all \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \), then the distribution is called a regular distribution, and \( \psi \) and \( \overline{\psi} \) are normally identified.

A distribution \( \psi \) is analytic (meromorphic) if for any test function \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \), \((\psi, \varphi)\) is analytic (meromorphic) in some domain in \( \mathbb{C}^d \). If \( \psi \) is regular and analytic on some domain of \( \mathbb{C}^d \) and, for any test function \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \), \((\psi, \varphi) \) analytically continues to an analytic or meromorphic function, then \( \psi \) is said to analytically continue to an
analytic or meromorphic distribution. (For short, we will sometimes say that the
distribution meromorphically continues.) A region on which \( \psi \) is regular and analytic
is called a region of absolute convergence of \( \psi \).

We will also have a need for the tensor product of distributions, which we define
as follows. Let \( \psi_1 \) and \( \psi_2 \) be distributions in \( S'(\mathbb{R}^d_1) \) and \( S'(\mathbb{R}^d_2) \), and let \( \varphi \) be in
\( S(\mathbb{R}^{d_1+d_2}) \). Then the functions

\[
\varphi_1(x_1) = (\psi_2(\cdot), \varphi(x_1, \cdot))_{S'(\mathbb{R}^d_2), S(\mathbb{R}^d_1)}, \quad \varphi_2(x_2) = (\psi_1(\cdot), \varphi(\cdot, x_2))_{S'(\mathbb{R}^d_1), S(\mathbb{R}^d_1)}
\]

are in \( S(\mathbb{R}^d_1) \), \( S(\mathbb{R}^d_2) \), respectively: this follows from the general fact that for any fixed \( f \) in \( S'(\mathbb{R}^d_2) \) and \( \varphi \) in \( S(\mathbb{R}^{d_1+d_2}) \), \( x \mapsto (f(\cdot), \varphi(x, \cdot)) \) lies in \( S(\mathbb{R}^d_1) \), along with
the symmetric relation with the order of the variables transposed. Then we define
\( \psi_1 \otimes \psi_2 \) by

\[
(\psi_1 \otimes \psi_2, \varphi)_{S'(\mathbb{R}^{d_1+d_2}), S(\mathbb{R}^{d_1+d_2})} := (\psi_1, \varphi_1)_{S'(\mathbb{R}^d_1), S(\mathbb{R}^d_1)} = (\psi_2, \varphi_2)_{S'(\mathbb{R}^d_2), S(\mathbb{R}^d_2)},
\]

which we can write more concisely as

\[
(\psi_1 \otimes \psi_2, \varphi) = (\psi_1, (\psi_2, \varphi)) = (\psi_2, (\psi_1, \varphi)). \tag{12}
\]

To show that this definition is consistent, we must show that equality holds in the
last two expressions in Equation (12). So suppose first that \( \varphi = \varphi_1 \otimes \varphi_2 \). Then
\[
(\psi_1, (\psi_2, \varphi_2)) = (\psi_1, \varphi_1(\psi_2, \varphi_2)) = (\psi_1, \varphi_1(\psi_2, \varphi_2)) = (\psi_2, \varphi_2)(\psi_1, \varphi_1) = (\psi_2, (\psi_1, \varphi)),
\]
where in the second and third equalities we used the linearity of the pairings involved,
and in the final equality we used the symmetric equality with the order of \( \psi_1 \) and \( \psi_2 \)
transposed. This shows that Equation (12) is well-defined for test functions that are
product-form and hence by linearity for all test functions in \( S(\mathbb{R}^{d_1}) \otimes S(\mathbb{R}^{d_2}) \). But
\( S(\mathbb{R}^{d_1}) \otimes S(\mathbb{R}^{d_2}) \) is dense in \( S(\mathbb{R}^{d_1+d_2}) \) (a nontrivial fact) so the definition is, in fact,
well-defined for all distributions in \( S(\mathbb{R}^{d_1+d_2}) \).

Equation (12) can also be seen as the analog of Fubini’s theorem for tempered
distributions. In fact, it follows for regular distributions by an application of Fubini’s
theorem, and hence is a natural definition of the tensor product of two distributions.

3. A CONTINUOUS LINEAR EXTENSION OPERATOR ON \( S(R) \)

In this section we prove two analytic results used in the proofs of Section 4. In the
second result, which may be of some independent interest, we construct a continuous
linear extension operator on the Schwartz class \( S(R) \).

**Lemma 3.1.** Let \( y = y(x) \) be the transformation \( y_k = x_1 \cdots x_k, k = 1, \ldots, d \), and let \( R \) be defined as in Equation (8). If \( f \) is in \( S(y(R)) \) then \( f \circ y \) is in \( S(R) \).

**Proof.** Let \( f \) be in \( S(y(R)) \) and let \( \widetilde{f} = f \circ y \). Applying the chain rule, we can see
that for any multi-index \( \beta \),

\[
D^\beta \widetilde{f}(x) = \sum_{j=1}^{N} C_j x^{\gamma^j}(D^{\gamma^j} f)(y(x))
\]
for some positive integers $N$ and $(C_j)$ and multi-indices $(\gamma^j)$ with each $\gamma^j \leq \beta$. It follows that for any multi-indices $\alpha$ and $\beta$,

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta \overline{f}(x)| \leq \sum_{j=1}^{N} C_j \sup_{x \in \mathbb{R}} |x^{\alpha + \gamma^j} (D^{\gamma^j} f)(y(x))|$$

$$\leq \sum_{j=1}^{N} C_j \sup_{x \in \mathbb{R}} |x_1^{\alpha_1 + \gamma_1^j} (D^{\gamma_1^j} f)(y(x))| = \sum_{j=1}^{N} C_j \sup_{y \in y(R)} |y_1^{\alpha_1 + \gamma_1^j} D^{\gamma_1^j} f(y)|,$$

where we used the fact that $|x_k| < 1$ for all $k = 2, \ldots, d$. But this is finite because $f$ is in $\mathcal{S}(y(R))$ and we conclude that $\overline{f}$ is in $\mathcal{S}(R)$.

\textbf{Theorem 3.2.} There exists a continuous linear extension operator $\mathcal{E}$ that maps $\mathcal{S}(R)$ to $\mathcal{S}(\mathbb{R}^d)$, where $R$ is defined as in Equation (8).

\textbf{Proof.} For simplicity of notation, we give the proof for $d = 2$; nothing significant changes for $d > 2$. Also, the proof for $d = 1$ is an obvious simplification of the argument for $d = 2$.

Let $f$ be in $\mathcal{S}(R)$. We prove the existence of the extension $\tilde{f} := \mathcal{E}f$ in three steps, as follows:

\textbf{Step 1:} We extend $f$ to a function $u$ in $C^\infty(\mathbb{R}^2)$ much as in the proof of Theorem 5′ of Chapter VI of [St], though we do so explicitly so we can more easily make the calculations required to establish Schwartz decay.

Because $f$ is in $C^\infty(R)$, we can extend $f$ continuously to the boundary of $R$. We then define $u$ on $(-\infty, 0] \times (0, 1)$ as in Equation (24) p. 182 of [St] by

$$u(x, y) = \int_{1}^{\infty} f(x - c_0 \lambda x, y) \psi(\lambda) \, d\lambda$$

(13)

for $x \leq 0$, and $u(x, y) = f(x, y)$ for $x$ in $R$. Here, $\psi$ is as in Lemma 1 p. 182 of [St], and we use $c_0$ in place of $2c$ of [St]. Also, in the notation of [St], $\Delta(x, y) = -x$.

Because $f$ is in $\mathcal{S}(R)$, $u$ and all its derivatives are continuous, as we can verify directly from Equation (13); hence, $u$ is in $C^\infty((\infty, \infty) \times (0, 1))$.

Next we extend $u$ to $\mathbb{R}^2$ as follows. Let $\{\phi_-, \phi_+\}$ be a partition of unity of $R$ defined so that $\phi_+$ equals 1 on the set $\{(x, y) \in R : 3/4 \leq y < 1\}$, $\phi_-$ equals 1 on the set $\{(x, y) \in R : 0 < y \leq 1/4\}$, and both are constant along horizontal lines. Then define $u_-$ and $u_+$ in $C^\infty(\mathbb{R}^2)$ by

$$u_-(x, y) = \begin{cases} \int_{1}^{\infty} (u\phi_-(x, y - c_0 \lambda y) \psi(\lambda) \, d\lambda, & y \leq 0, \\ u(x, y)\phi_-(x, y), & 0 < y < 1, \\ 0, & y \geq 1, \end{cases}$$

$$u_+(x, y) = \begin{cases} \int_{1}^{\infty} (u\phi_+(x, y + c_0 \lambda(y - 1)) \psi(\lambda) \, d\lambda, & y \geq 1, \\ u(x, y)\phi_+(x, y), & 0 < y < 1, \\ 0, & y \leq 0. \end{cases}$$

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In both integrals above we treat \( u \) as being zero whenever \( \phi_- \) or \( \phi_+ \) is zero (the value we choose for \( u \) does not matter).

Finally, define \( u \) in \( C^\infty(\mathbb{R}^2) \) by
\[
u(x, y) = u_-(x, y) + u_+(x, y),
\]
and observe that \( u \) is an extension of \( f \) to all of \( \mathbb{R}^2 \), and \( u \) is in \( C^\infty(\mathbb{R}^2) \) by the same reasoning as before.

**Step 2:** Let \( \varphi_h \) and \( \varphi_v \) in \( C^\infty(\mathbb{R}^2) \) assume values in \([0, 1]\) and be such that \( \varphi_h \equiv 1 \) on \([0, \infty)\), \( \varphi_h \equiv 0 \) on \((-\infty, -1)\), \( \varphi_v \equiv 1 \) on \([0, 1]\), and \( \varphi_v \equiv 0 \) on \([2, \infty)\) and on \((-\infty, -1)\). Then \( \varphi \equiv \varphi_h \varphi_v \) is in \( C^\infty(\mathbb{R}^2) \) and assumes values in \([0, 1]\), is identically 1 on \( R \), and is identically 0 on the complement in \( \mathbb{R}^2 \) of \((-1, \infty) \times (-1, 2)\).

Define \( \tilde{f} \) in \( C^\infty(\mathbb{R}^2) \) by
\[
\tilde{f} = \varphi u.
\]

**Step 3:** The function \( \tilde{f} \) has Schwartz decay in all directions except possibly along the positive \( x \)-axis when \( y \) is in \([1, 2]\) or in \((-1, 0]\), because in all other directions, \( f \) either equals \( f \), which has Schwartz decay, or becomes zero after a finite distance. So we need only show that \( |x^m y^n \partial_x^j \partial_y^k \tilde{f}(x, y)| \) is bounded for all nonnegative integers \( m, n, j, \) and \( k \) on two subsets of \( \mathbb{R}^d \): \( R_1 = (0, \infty) \times (-1, 0) \) and \( R_2 = (0, \infty) \times (1, 2)\).

First we consider only partial derivatives of \( x \). Assume that \( (x, y) \) is in \( R_1 \), and that \( m, n, j \) are nonnegative integers. Then, since \( \varphi \) is constant along horizontal rays in \( R_1 \),
\[
|x^m y^n \partial_x^j \tilde{f}(x, y)| = |\varphi(x, y)x^m y^n \partial_x^j u(x, y)| \leq |x^m y^n \partial_x^j u_+(x, y)|.
\]

The second and third equalities follow from the definitions of \( u \) and \( u_- \) (and \( u \) becomes \( f \) in the integral because \( x > 0 \)). The fourth equality uses the constancy of \( \phi_- \) along horizontal lines. The last inequality follows by a change of variables and the observation that \( \phi_- \) is supported in a strip of vertical width less than 1.

Thus,
\[
\sup_{(x, y) \in R_1} |x^m y^n \partial_x^j \tilde{f}(x, y)| \leq \frac{1}{c_0} \sup_{\psi} \sup_{(x, y) \in \mathbb{R}^2} |x^m y^n \partial_x^j f(x, y')|,
\]

and observe that \( u \) is an extension of \( f \) to all of \( \mathbb{R}^2 \), and \( u \) is in \( C^\infty(\mathbb{R}^2) \) by the same reasoning as before.
which is finite by the assumption that \( f \) is in \( S(R) \). The bound on \( R_2 \) is obtained similarly.

Bounding \(|x^m y^n \partial_y^k \tilde{f}(x, y)|\) is more tedious, because both \( \varphi \) and \( \phi_- \) have nonzero partial derivatives in the \( y \)-direction. If we write this as \(|x^m y^n \partial_y^k \partial_y^j \tilde{f}(x, y)|\), we can start with the calculation above then perform the partial derivatives in \( y \). This will result in a sum of terms including partial derivatives of \( \varphi \), \( \phi_- \), and \( f \). Each term, however, will be just as above, with \( \varphi \) and \( \phi_- \) replaced by partial derivatives of these functions, and with partial derivatives in both \( x \) and \( y \). Since all the partial derivatives of \( \varphi \) and \( \phi_- \) are bounded, this does not change the argument for each term, and we see that \(|x^m y^n \partial_y^k \partial_y^j \tilde{f}(x, y)|\) is bounded as well.

The linearity of the extension operator \( E \tilde{f} = \tilde{f} \) is clear from the definition of \( \tilde{f} \), and its continuity follows from the bounds we established above. \( \square \)

4. Proofs of the main results

4.1. Proof of Theorem 1.1.

Proof. From Equation (3) and the identity,

\[ \Gamma(s) = \int_0^\infty w^{s-1} e^{-w} \, dw \]

we have

\[
L(s) \prod_{j=1}^d \Gamma(s_j) = \int_0^\infty \cdots \int_0^\infty \sum_{0 < n_1 < \cdots < n_d} a_{n_1,1} \cdots a_{n_d,d} \prod_{k=1}^d n_k^{-s_k} \prod_{k=1}^d w_k^{s_k-1} e^{-w_k} \, dw_k \cdots dw_d.
\]

Making the change of variables \( w_k = n_k t_k \),

\[
n_k^{-s_k} w_k^{s_k-1} e^{-w_k} \, dw_k = n_k^{-s_k} (n_k t_k)^{s_k-1} e^{-n_k t_k} n_k \, dt_k = n_k^{-s_k} e^{-n_k t_k} \, dt_k,
\]

so

\[
L(s) \prod_{j=1}^d \Gamma(s_j) = \int_0^\infty \cdots \int_0^\infty \sum_{0 < n_1 < \cdots < n_d} a_{n_1,1} \cdots a_{n_d,d} \prod_{k=1}^d t_k^{s_k-1} e^{-n_k t_k} \, dt_k \cdots dt_d
\]

\[
= \int_0^\infty \cdots \int_0^\infty \sum_{n_1=1}^d \sum_{n_d=1}^d a_{n_1,1} \cdots a_{n_d,d} t_1^{s_1-1} e^{-n_1 t_1} \cdots t_d^{s_d-1} e^{-n_d t_d} \, dt_1 \cdots dt_d
\]

\[
= \int_0^\infty \cdots \int_0^\infty \phi(t_1, \ldots, t_d) \, dt_1 \cdots dt_d,
\]
where
\[ \varphi(t_1, \ldots, t_d) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} e^{-n_1(t_1+\cdots+t_d)} \cdots e^{-n_d t_d}. \]

As noted in [Go2], for the multiple-dimensional zeta-function, \( \varphi \) has a singularity of type \( \prod_{k=1}^{d} (t_k + \cdots + t_d) \), and \( \varphi(t) \) would not be in \( \mathcal{S}((0, \infty)^d) \) except when \( d = 1 \). We thus proceed precisely as in [Go2, Z1], except for the presence of the factor \( a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} \), making the change of variables
\[ t_1 = x_1(1 - x_2), \ldots, t_{d-1} = x_1 \cdots x_{d-1}(1 - x_d), \quad t_d = x_1 \cdots x_d \]
to obtain
\[
L(s) = \frac{s_d - 1}{\Gamma(u_1) \cdots \Gamma(u_{d-1})}
= \int_{0}^{1} \cdots \int_{0}^{1} \frac{x_1^{u_1-1} \cdots x_{d}^{u_d-1}}{\Gamma(u_1) \cdots \Gamma(u_{d})} g(x_1, \ldots, x_d) \, dx_1 \cdots dx_d
= \int_{0}^{1} \cdots \int_{0}^{1} (\psi A)(x_1, \ldots, x_d; s_1, \ldots, s_d)(g/A)(x_1, \ldots, x_d) \, dx_1 \cdots dx_d,
\]
where
\[
g(x_1, \ldots, x_d) = x_1^{d_d-1} \cdots x_d \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} e^{-n_1x_1} \cdots e^{-n_dx_d}
= \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\cdots+n_d,d} (x_1 e^{-n_1x_1}) \cdots (x_d e^{-n_dx_d})
\]
is the function of Equation (10).

Since \( g/A \) is in \( \mathcal{S}(R) \) by assumption, there exists an extension \( f = \mathcal{E}(g/A) \) of \( g/A \) to \( \mathcal{S}(\mathbb{R}^d) \) by Theorem 3.2. We can thus solve for \( L \) in Equation (14) and write
\[
L(s) = \frac{\Gamma(u_1) \cdots \Gamma(u_{d-1})}{(s_d - 1)} ((\psi A)(\cdot, s_1, \ldots, s_d), f(\cdot)),
\]
which, employing Lemma 4.2, gives an explicit expression for the continuation of \( L \) with the possible poles along the stated hyperplanes.

It remains to prove Equation (11); for simplicity, we give the proof for \( d = 2 \). By the growth condition in Equation (2),
\[
|g(x)| \leq C_1 C_2 y_1 y_2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{r_1}(m + n)^{r_2} e^{-my_1} e^{-ny_2},
\]
where \( y \) is defined in Equation (9). But,
\[
m^{r_1}(m + n)^{r_2} \leq m^{r_1}(2 \max \{m, n\})^{r_2} \leq m^{r_1}2^{2r_2}(m^{r_2} + n^{r_2}) = 2^{r_2}(m^{r_1+r_2} + m^{r_1} n^{r_2}) \leq 2^{r_2}(m^{r_1+r_2} n^{r_2} + m^{r_1+r_2} n^{r_2}) \leq 2^{r_2+1} m^{r_1+r_2} n^{r_2},
\]
where
\[ |g(x)| \leq C y_1 y_2 \sum_{m=1}^{\infty} m^{r_1+r_2} e^{-my_1} \sum_{n=1}^{\infty} n^{r_2} e^{-ny_2} = C y_1 y_2 \sum_{m=1}^{\infty} m^{r_1+r_2} (e^{-y_1})^m \sum_{n=1}^{\infty} n^{r_2} (e^{-y_2})^n \]
\[ \leq C \frac{y_1 y_2 e^{-y_1}}{(1 - e^{-y_1})^{r_1+r_2+1}} \leq \frac{(1 + y_1)(1 + y_2)e^{-y_1}e^{-y_2}}{(1 - e^{-y_1})^{r_1+r_2}(1 - e^{-y_2})^{r_2}}, \]

(15)

where we used Lemma 4.1 and the observation that \( x/(1 - e^{-x}) \leq 1 + x \) for all \( x > 0 \). This also shows that the sum that defines \( g \) is absolutely convergent.

But \( g/A \) is in \( S(R) \) by assumption, so \( A \) must respect the same bound as in Equation (15) (with a different constant). Then from Equation (7), for any \( \phi \) in \( S(\mathbb{R}^d) \),
\[ (\psi A, \phi) = \int_0^1 \int_0^\infty \frac{x_1^{s_1-1} (1 - x_2)^{s_2-1} x_2^{r_1+r_2} e^{-y_1} e^{-y_2}}{\Gamma(s_1) \Gamma(s_2)} A\phi \, dx_1 \, dx_2, \]

(16)

whose integrand is bounded by
\[ C(1 - x_2)^{Re s_1-1} (1 + y_1)(1 + y_2) (1 - e^{-y_1})^{r_1+r_2+1} \]
\[ \leq C \frac{y_1 y_2 e^{-y_1}}{(1 - e^{-y_1})^{r_1+r_2+1}} \leq \frac{(1 + y_1)(1 + y_2)e^{-y_1}e^{-y_2}}{(1 - e^{-y_1})^{r_1+r_2}(1 - e^{-y_2})^{r_2}}, \]

where \( C \) is constant with respect to \( x \). Using again that \( x/(1 - e^{-x}) \leq 1 + x \) for all \( x > 0 \), we see that the integral in Equation (16) is absolutely convergent in the region \( \Omega \) defined by \( Re u_1 - r_1 - 2r_2 > 0 \), \( Re u_2 - r_2 > 0 \), and \( Re s_1 > 0 \), which is the same as the region of Equation (11) (for \( d = 2 \)).

Since the sum that defines \( g \) in Equation (10) is absolutely convergent on \( R \), we can take any derivative of \( g \) term-by-term, then bound the derivative as we did in Equation (15). The resulting bound is the same as that in Equation (15) except that the powers of \( 1 - e^{-y_k} \) are increased. From this it immediately follows that \( g \) is in \( C^\infty(R) \) and in \( S((\epsilon, \infty) \times (\epsilon', 1)^{d-1}) \) for all \( \epsilon > 0 \) and \( \epsilon' \) in \( (0, 1) \).

Lemma 4.1. Let \( P_r(x) = \sum_{n=1}^{\infty} n^r x^n \) (so \( P_r = Li_{-r} \), where \( Li \) denotes the polylogarithm). Then there exists a constant \( c_0 \) such that for all \( r \geq 0 \) and all \( x \) in \([0, 1)\),
\[ 0 \leq P_r(x) \leq c_0 \frac{\Gamma(r+1)x}{(1-x)^{r+1}}. \]

(17)

Proof. Equation (17) is equivalent to
\[ \sum_{n=1}^{\infty} n^r x^n \leq c_0 x \sum_{n=0}^{\infty} \Gamma(r+1) \left( \frac{-r+1}{n} \right) (-1)^n x^n \]
\[ = c_0 \sum_{n=1}^{\infty} \Gamma(r+1) \left( \frac{r+n-1}{n-1} \right) x^n = c_0 \sum_{n=1}^{\infty} \frac{\Gamma(r+n)}{\Gamma(n)} x^n. \]

(18)

Define \( f : \mathbb{Z}^{\geq 1} \times [0, 1) \to \mathbb{R} \) by \( f(n, u) = (\Gamma(u+n)/\Gamma(n))/n^u \). Then \( f(\cdot, u) \) is a nondecreasing function since \( f(n+1, u)/f(n, u) = (1+u/n)/(1+1/n)^u \geq 1 \) for all \( u \) in \([0, 1] \), so \( f(n, u) \geq f(1, u) \). But \( f(1, \cdot) \) is a continuous positive function on
the interval $[0, 1]$ and so achieves a minimum value, $R > 0$. Thus, $f(n, u) \geq R$, or, \( \Gamma(u + n)/\Gamma(n) \geq Rn^u \) for all $u$ in $[0, 1]$. Then,

\[
\frac{\Gamma(r + n)}{\Gamma(n)} = (r - 1 + n) \cdots (r - [r] + n) \frac{\Gamma(r - [r] + n)}{\Gamma(n)} \\
\geq n^{[r]} f(n, r - [r]) n^{r - [r]} \geq Rn^{[r]} n^{r - [r]} = Rn^r.
\]

From this, Equation (18) and thus Equation (17) follow with $c_0 = 1/R$. \(\square\)

**Lemma 4.2.** The distribution $\psi$ of Equation (7) is absolutely convergent on $Re u_k > 0$, $k = 1, \ldots, d$, and $Re s_k > 0$, $k = 1, \ldots, d - 1$, and continues to an entire distribution.

**Proof.** The region of absolute convergence follows as in the proof of Theorem 1.1 with $r_1 = \cdots = r_d = 0$.

The distribution $\psi_1$ is analytic on $Re u_1 > 0$ and continues to an entire function by Lemma 3 of [Z1], and $\psi_k$ for $k > 1$ is analytic on $Re s_{k-1} > 0$, $Re u_k > 0$ and continues to an entire function by Lemma 4 of [Z1]. The entireness of $\psi$ then follows from Theorem 1.4 with $A = 1$. \(\square\)

4.2. **Proof of Theorem 1.4.**

**Proof.** Assume that $d = 2$, the proof being entirely analogous for $d > 2$. By assumption, $\overline{\psi}_1 := \psi_1 A_1$ is a meromorphic distribution on $\mathbb{R}^2$ and $\overline{\psi}_2 := \psi_2 A_2$ is a meromorphic distribution on $(s_1, u_2)$. Then $\psi A = \overline{\psi}_1 \otimes \overline{\psi}_2$ (see Section 2 for the definition of the tensor product of two distributions), and we can write, for any $\varphi$ in $S(\mathbb{R}^2)$,

\[
(\psi A, \varphi) = (\overline{\psi}_1, \overline{\psi}_2, \varphi)) = (\overline{\psi}_2, (\overline{\psi}_1, \varphi)).
\]

Since $(\psi A, \varphi) = (\overline{\psi}_1, (\overline{\psi}_2, \varphi))$, it is meromorphic in $u_1$; since $(\psi A, \varphi) = (\overline{\psi}_2, (\overline{\psi}_1, \varphi))$ it is meromorphic in $s_1$ and $u_2$ as well. But a complex-valued function that is meromorphic in each variable separately is meromorphic: this follows from Hartog’s theorem (for instance, see Theorem B.6 p. 15 of [Gu1]). Hence, $(\psi A, \varphi)$ is meromorphic in $(u_1, s_1, u_2)$ and so is meromorphic on the subvariety defined by $s_1 = u_1 - u_2 + 1$, which, with the change of variables $s_1 = s_1, s_2 = u_2 + 1$, means that $(\psi A, \varphi)$ is meromorphic when viewed as a function of $(s_1, s_2)$. (These relations come from solving for $s_1$ and $s_2$ in Equation (6).) Since this is true for all $\varphi$ in $S(\mathbb{R}^2)$, the distribution $\psi A$ is meromorphic. \(\square\)

4.3. **Proof of Corollary 1.6.**

**Proof.** Since $g$ is in $Sing A$, $g = f A$ for some function $f$ in $S(\mathbb{R})$. Thus, $L = (\Psi, g) = (\Psi, f A) = (\Psi A, f)$. But $\Psi A$ continues to a meromorphic distribution by Lemma 4.3, so $L$ continues to a meromorphic function on $\mathbb{C}^d$ with the possible simple poles as stated. \(\square\)

**Lemma 4.3.** Let $A$ be the function defined in Corollary 1.6. Then $\Psi A$ continues to a meromorphic distribution. When each $p_j$ and $q_j$ is a nonnegative integer, $\Psi A$ has the same possible simple poles as those stated for $L$ in Corollary 1.6.
Proof. Let
\[ A_1(x_1) = x_1^{-p_1}(\log x_1)^{v_1}, \]
\[ A_k(x_k) = x_k^{-p_k} (1 - x_k)^{-q_k}(\log x_k)^{v_k} (\log(1 - x_k))^{w_k}, \quad k = 2, \ldots, d. \]
Then for \( u_1 > p_1 \) and any \( \varphi \) in \( \mathcal{S}(\mathbb{R}) \),
\[ (\psi_1 A_1(x_1), \varphi) = \frac{1}{\Gamma(u_1)} ((x_1)^{u_1-p_1-1}(\log x_1)^{v_1}, \varphi) \]
\[ = \frac{1}{\Gamma(u_1)} \left( \frac{\partial^{p_1}}{\partial u_1^{v_1}} (x_1)^{u_1-p_1-1}, \varphi \right) = \frac{1}{\Gamma(u_1)} \frac{\partial^{v_1}}{\partial u_1^{v_1}} ((x_1)^{u_1-p_1-1}, \varphi). \]
In the final equality, we brought the derivative under the integral sign (for \( u_1 > p_1 \) the distribution \( (x_1)^{u_1-p_1-1} \) is regular). This is justified since for any \( u_1 > p_1 \), there is an open interval containing \( u_1 \) on which \( (\partial^{v_1}/\partial u_1^{v_1}) (x_1)^{u_1-p_1-1} \) is bounded by an \( L^1 \)-function. This is a sufficient condition for bringing the derivative under the integral sign (see, for instance, Theorem 24.5 p.193-194 of [AB]). The function \( (\psi_1 A_1(x_1), \varphi) \) thus continues to a meromorphic function of \( u_1 \) since the distribution \( (x_1)^{u_1-p_1-1} \) is meromorphic, being the same as the meromorphic distribution \( (x_1)^{u_1} \) after a change of variables. This gives a meromorphic continuation of the distribution \( \psi A_1 \).

Similarly,
\[ (\psi_k A_k(x_k), \varphi) = \frac{1}{\Gamma(s_k-1) \Gamma(u_k)} \left( \frac{\partial^{v_k}}{\partial s_k^{v_k}} \frac{\partial^{p_k}}{\partial u_k^{v_k}} (1 - x_k)^{s_k-1-q_k-1} (x_k)^{u_k-p_k-1}, \varphi \right), \]
which we also conclude is meromorphic. Applying Theorem 1.4, we conclude that \( \psi A \) is meromorphic.

The distribution \( x_s^{-1} \) has simple poles along the hyperplane \( s = n \) for each integer \( n \leq 0 \), and the distribution \( x_t^t (1 - x)^t \) has simple poles along the hyperplanes \( s = n \) and \( t = n \) for each integer \( n \leq 0 \) (see, for instance, [Z1]). Each derivative in the above expressions for \( (\psi_1 A_1(x_1), \varphi) \) and \( (\psi_k A_k(x_k), \varphi) \) increases the possible order of the poles by one; the net effect of the poles of the gamma functions in the denominators of these expressions and in the numerator of Equation (5) leads to the locations and orders of the possible poles in the statement of the theorem. \( \square \)

5. More General Multiple Dirichlet Series

We can generalize our definition of \( L \) in Equation (3) by letting
\[ L(s) = \sum_{0<n_1<\cdots<n_d} a_{n_1,1} \cdots a_{n_d,d} \lambda_{n_1,1}^{-s_1} \cdots \lambda_{n_d,1}^{-s_d}, \quad (19) \]
where the coefficients \( a_{n,k} \) are defined as before and where each \( (\lambda_{n,k})_{n=1}^{\infty} \) for \( k = 1, \ldots, d \) is a strictly increasing sequence of positive real numbers.

Only small modifications of the proofs in Section 4 are needed to adapt the results of Section 1 to our new definition of \( L \). The function \( g \) of Equation (10) is replaced
by

$$ g(x) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{n_1,1} \cdots a_{n_1+\ldots+n_d,d} y_1(x) e^{-\lambda_{n_1,1} y_1(x)} \cdots y_d(x) e^{-\lambda_{n_d,d} y_d(x)}.$$ 

To obtain the equivalent of Equation (11) requires that we place a lower bound on the growth of $\lambda_{n,k}$ with $n$ and incorporate it into the bound on $g$ in Equation (15).

An important special case of Equation (19) is the Hurwitz zeta function, in which $\lambda_{n,k} = n + \theta_k$, so that

$$ L(s) = \sum_{0<n_1<\ldots<n_d} a_{n_1,1} \cdots a_{n_d,d} \prod_{k=1}^{d} (n_k + \theta_k)^{-s_k},$$

where $\theta_k$, $k = 1, \ldots, d$ are arbitrary nonnegative real numbers. This causes the function $g$ of Equation (10) to be multiplied by

$$ \nu(x) = e^{-\theta_1 y_1(x)} \cdots e^{-\theta_1+\ldots+\theta_d y_d(x)}.$$ 

Because $\mathcal{S}(R)$ is closed under multiplication by a bounded $C^\infty$-function all of whose derivatives are bounded, it follows from Lemma 3.1 that $\nu$ is in $\mathcal{S}(R)$. Thus, the conclusions in Theorem 1.1 regarding the region of absolute convergence and the location and orders of the possible poles are unchanged.

6. Proof of Theorem 1.7

In this section we prove Theorem 1.7. We should remark that Theorem 1.7 can also be proved using a variant of Euler-Maclaurin summation formula and an unfolding procedure similar to that used to meromorphically continue $\zeta_d(s)$ in [Ma2]. However, as discussed earlier, the proof we give is important in that it illustrates the analytic techniques required to successfully apply our method.

Our approach is to split $g$ into five functions, $g_1$ through $g_5$, each of which has a singularity along $x_2 = 1$ (see Section 6.7) and various other singularities. We then continue each $(\Psi, g_j)$ separately, giving a continuation of $L = \sum_{j=1}^{5} (\Psi, g_j)$.

We will find singularities in $g_1$ and $g_3$ of types $1/(1-x_2)$, $\log x_1/(1-x_2)$, and $(\log x_1)^2/(1-x_2)$, and in $g_2$ of types $1/(x-x_2)$ and $\log x_1/(1-x_2)$. Determining the singularities of $g_4$ and especially of $g_5$ is more involved: it will require us to partition $R$ into three regions and use a partition of unity to continue portions of $(\Psi, g_4)$ and $(\Psi, g_5)$ separately for each region. We will find singularities of the form $1/(1-x_2)$, $\log x_1/(1-x_2)$, $\log x_2/(1-x_2)$, $\log x_1 \log x_2/(1-x_2)$, and $(\log x_1)^2/(1-x_2)$ in $g_4$, and $1/(1-x_2)$, $\log x_1/(1-x_2)$, and $\log x_2/(1-x_2)$ in $g_5$. Employing Corollary 1.6, these singularities together give the possible poles in the statement of Theorem 1.7.

6.1. Decomposition of $g$. We have $g(x_1, x_2) = x_1^2 x_2 G(e^{-x_1}, e^{-x_1 x_2})$, where

$$ G(z) = \sum_{m=1}^{\infty} H_{m+z_1} \sum_{n=1}^{\infty} H_{m+n+z_2}.$$
We can write this as
\[ G(z) = \frac{1}{(1 - z_2)(1 - z_1/z_2)} \left[ \frac{2z_2 - z_1 - 1}{2(1 - z_1)} \log(1 - z_1)^2 - \frac{1 - z_2}{1 - z_1} \dilog(1 - z_1) \\
+ \frac{1}{2} \log(1 - z_2)^2 + \dilog(1 - z_2) + \log \left( \frac{1 - z_1}{1 - z_2} \right) \right]. \]

The singularity of \( G \) along \( z_1 = z_2 \) is removable (though there is a singularity at \((1, 1)\)), as can be seen from the first expression for \( G \) above. This is as must be, since we know that \( G \) is analytic on \( D \times D \), where \( D \) is the open unit disk in \( \mathbb{C} \).

Using the identity,
\[ \dilog(x) = -\dilog(1/x) - (1/2)(\log x)^2, \]
which holds for all nonzero \( x \), we have,
\[
\frac{1}{2} \log(1 - z_2)^2 + \dilog \left( \frac{1 - z_1}{1 - z_2} \right) = \frac{1}{2} \log(1 - z_2)^2 - \dilog \left( \frac{1 - z_2}{1 - z_1} \right) - \frac{1}{2} \log \left( \frac{1 - z_1}{1 - z_2} \right)^2
\]
\[ = -\dilog \left( \frac{1 - z_2}{1 - z_1} \right) - \frac{1}{2} \log(1 - z_1)^2 + \log(1 - z_2) \log(1 - z_1). \]

We then have,
\[
g(x_1, x_2) = x_1^2 x_2 G(e^{-x_1}, e^{-x_1 x_2})
\]
\[ = x_1^2 x_2 \frac{2(e^{-x_1 x_2} - e^{-x_1} - 1) \log(1 - e^{-x_1})^2}{(1 - e^{-x_1})(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})} + \frac{x_1^2 x_2}{(1 - e^{-x_1 x_2})(1 - e^{-x_1(1-x_2)})} \left[ - \frac{1 - e^{-x_1 x_2}}{1 - e^{-x_1}} \dilog(1 - e^{-x_1}) - \frac{1}{2} \log(1 - e^{-x_1})^2 \\
+ \log(1 - e^{-x_1 x_2}) \log(1 - e^{-x_1}) + \dilog(1 - e^{-x_1 x_2}) - \dilog \left( \frac{1 - e^{-x_1 x_2}}{1 - e^{-x_1}} \right) \right]. \]

We can write this as \( g = g_1 + \cdots + g_5 \), where
\[
g_1(x) = k_1(x) \log(1 - e^{-x_1})^2,
\]
\[
g_2(x) = -k_2(x) \dilog(1 - e^{-x_1}),
\]
\[
g_3(x) = -k_3(x) \frac{1}{2} \log(1 - e^{-x_1})^2,
\]
\[
g_4(x) = k_3(x) \log(1 - e^{-x_1 x_2}) \log(1 - e^{-x_1}),
\]
\[
g_5(x) = k_3(x) \left( \dilog(1 - e^{-x_1 x_2}) - \dilog \left( \frac{1 - e^{-x_1 x_2}}{1 - e^{-x_1}} \right) \right),
\]

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with
\[ k_1(x) = x_1^2x_2^2 \frac{(2e^{-x_1x_2} - e^{-x_1} - 1)}{2(1 - e^{-x_1})(1 - e^{-x_1x_2})(1 - e^{-x_1(1-x_2)})}, \]
\[ k_2(x) = \frac{x_2^2x_2}{(1 - e^{-x_1})(1 - e^{-x_1(1-x_2)})}, \]
\[ k_3(x) = \frac{x_1^2x_2}{(1 - e^{-x_1x_2})(1 - e^{-x_1(1-x_2)})}. \]

For \( j = 1, 2, 3 \), \((1 - x_2)k_j\) is a meromorphic function on \( \mathbb{C}^2 \) whose singularities are removable; hence \((1 - x_2)k_j\) is entire on \( \mathbb{C}^2 \) and restricts to a function in \( C^\infty(R) \). Further, a simple induction argument shows that each of the derivatives of each \( k_j \) is bounded on \( R \) by some polynomial in \( x_1 \) (a different polynomial for each derivative).

Since \((\Psi, g) = (\Psi, g_1) + \cdots + (\Psi, g_5)\), we can continue \( L \) by applying \( \Psi \) to each of the test functions \( g_1 \) through \( g_5 \) separately.

6.2. **Continuation of** \((\Psi, g_1)\). We define a partition of unity \:\{\varphi_1, \varphi_2\} of \( R \) so that \( \varphi_1 \equiv 1 \) on \((0, 1/4) \times (0, 1)\) and \( \varphi_1 \equiv 0 \) on \((1/2, \infty) \times (0, 1)\). By Lemma 6.1,
\[ \varphi_1 g_1 = k_1(x) ((\log x_1)^2 + 2f(x_1)\log x_1 + f(x_1)^2) \varphi_1. \]

Since \((1 - x_2)k_1(x)\) is in \( C^\infty(R) \), we can split \( \varphi_1 g_1 \) into three parts, applying Corollary 1.6 three times with \( A(x) = (\log x_1)^2/(1 - x_2), \log(x_1)/(1 - x_2), \) and \( 1/(1 - x_2) \) to continue \((\Psi, \varphi_1 g_1)\).

Because of Lemma 6.2, we can continue \((\Psi, \varphi_2 g_1)\) using \( A(x) = 1/(1 - x_2) \).

**Lemma 6.1.** There exists an entire function \( f \) such that
\[ \log(1 - e^{-z}) = \log z + f(z). \tag{20} \]

**Proof.** Observe that
\[ \frac{d}{dz} \left( \log \left( \frac{z}{1 - e^{-z}} \right) \right) = \frac{1}{z} - \frac{e^{-z}}{1 - e^{-z}} = \frac{1}{z} - \frac{1}{z} \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = - \sum_{n=1}^{\infty} B_n \frac{z^{n-1}}{n!}, \]
so, after integrating,
\[ \log(z/(1 - e^{-z})) = - \sum_{n=1}^{\infty} \frac{B_n}{nn!} z^n, \]
from which the statement of the lemma follows. \( \square \)

**Lemma 6.2.** The functions \( \log(1 - e^{-x}) \) and \( \text{dilog}(1 - e^{-x}) \) are in \( \mathcal{S}((\epsilon, \infty)) \) for any \( \epsilon > 0 \).

**Proof.** Fix \( \epsilon > 0 \). Then for any \( n > 0 \),
\[ \frac{d^n}{dx^n} \log(1 - e^{-x}) = \sum_{j=1}^{n} C_j (r(x))^j, \]
where \( r(x) = 1/(e^x - 1) \) is in \( \mathcal{S}((\epsilon, \infty)) \) and \( C_j \) are integer constants. Thus all the derivatives of \( \log(1 - e^{-x}) \) are in \( \mathcal{S}((\epsilon, \infty)) \). It only remains to show that \( \log(1 - e^{-x}) \)
itself decays faster than any polynomial, which it does, since for any \( m > 0 \), by L’Hôpital’s rule,

\[
\lim_{x \to -\infty} x^m \log(1 - e^{-x}) = \lim_{m \to -\infty} \frac{\log(1 - e^{-x})}{x^{-m}} = \lim_{m \to -\infty} \frac{r(x)}{-m x^{-m-1}} = 0.
\]

Thus, we conclude that \( \log(1 - e^{-x}) \) is in \( S((\epsilon, \infty)) \).

Then \( (d/dx) \log(1 - e^{-x}) = \log(1 - e^{-x}) \), which is in \( S((\epsilon, \infty)) \). Since the derivative of \( \log(1 - e^{-x}) \) is in \( S((\epsilon, \infty)) \), we need only verify that \( \log(1 - e^{-x}) \) itself decays faster than any polynomial, which it does, since for any \( m > 0 \), by L’Hôpital’s rule (since \( \log(1) = 0 \)),

\[
\lim_{x \to -\infty} x^m \log(1 - e^{-x}) = \lim_{m \to -\infty} \frac{\log(1 - e^{-x})}{x^{-m}} = \lim_{m \to -\infty} \frac{r(x)}{-m x^{-m-1}} = 0.
\]

Thus, we conclude that \( \log(1 - e^{-x}) \) is also in \( S((\epsilon, \infty)) \). \( \square \)

6.3. **Continuation of \((\Psi, g_2)\).** Let \( \{\varphi_1, \varphi_2\} \) be the partition of unity of Section 6.2. Using Lemma 6.2 and the identity,

\[
dilog(1 - x) = -dilog(x) + \pi^2/6 - \log x \log(1 - x), \tag{21}
\]

we have

\[
\varphi_1 g_2(x) = k_2(x)(dilog(e^{-x_1}) - \pi^2/6 + \log e^{-x_1} \log(1 - e^{-x_1}))(\varphi_1(x)
\]

\[
= k_2(x)(dilog(e^{-x_1}) - \pi^2/6 - x_1 \log x_1 + x_1 f(x_1)) \varphi_1(x).
\]

Because \( \text{dilog}(z) \) is analytic on the disk of radius 1 centered at \( z = 1 \), \( \text{dilog}(e^{-x_1}) \varphi_1(x) \) is in \( S(R) \). We can therefore split \( \varphi_1 g_2 \) into three parts, applying Corollary 1.6 twice with \( A(x) = \log x_1/(1 - x_2) \) and \( 1/(1 - x_2) \) to continue \((\Psi, \varphi_1 g_2)\). (The factor of \( x_1 \) could be used to remove one of the possible poles that results from Corollary 1.6; however, these possible poles are already present, to higher order, in the continuation of \((\Psi, g_1)\).)

Because of Lemma 6.2, we can continue \((\Psi, \varphi_2 g_2)\) using \( A(x) = 1/(1 - x_2) \).

6.4. **Continuation of \((\Psi, g_3)\).** The continuation of \((\Psi, g_3)\) is virtually identical to that of \((\Psi, g_1)\), so we suppress the details.

6.5. **Continuation of \((\Psi, g_4)\).** We define a partition of unity \( \{\varphi_1, \varphi_2, \varphi_3\} \) of \( R \) as follows. First, define \( \varphi_1 \) so that \( \varphi_1 \equiv 1 \) on \((0, 3) \times (0, 1)\) and \( \varphi_1 \equiv 0 \) on \((4, \infty) \times (0, 1)\). Next, let

\[
S_1 = \{(x_1, x_2) \in R : x_1 > 3 \text{ and } x_2 < 1/x_1\},
\]

\[
S_2 = \{(x_1, x_2) \in R : x_1 > 2 \text{ and } x_2 < 2/x_1\},
\]

and let \( \{\varphi_2, \varphi_3\} \) be a partition of unity defined so that \( \varphi_2 \equiv 1 \) on \( S_1 \) and \( \varphi_2 \equiv 0 \) on \( R \setminus S_2 \). Then let \( \varphi_2 = (1 - \varphi_1)\varphi_2 \) and \( \varphi_3 = (1 - \varphi_1)\varphi_3 \).

The relevant facts about this partition are the following:

1. on the support of \( \varphi_1 \), both \( x_1 \) and \( x_1 x_2 \) are bounded (by 4);
2. on the support of \( \varphi_2 \), \( x_1 \) is bounded away from 0 (by 3) and \( x_1 x_2 \) is bounded (by 2);

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(3) on the support of $\varphi_3$, $x_1$ is bounded away from 0 (by 3), and $x_1x_2$ is bounded away from zero (by 1).

We will use $(\Psi, g_4) = (\Psi, \varphi_1 g_4) + (\Psi, \varphi_2 g_4) + (\Psi, \varphi_3 g_4)$ and continue the three components of $(\Psi, g_4)$ separately. The proof of Corollary 1.6 uses only the properties of $g$ stated in the theorem, so even though no $\varphi_j g_4$ is the function $g$ of Equation (10) for any $L$–function (it cannot be because it is singular along $x_2 = 1$, which $g$ never is), we can still apply Corollary 1.6 to continue each $(\Psi, \varphi_j g_4)$.

First, $\varphi_3(1 - x_2) g_4$ is in $\mathcal{S}(R)$ by an application of Lemma 6.3, so we can continue $(\Psi, \varphi_3 g_4)$ using Corollary 1.6 with $A(x) = 1/(1 - x_2)$.

To continue $(\Psi, \varphi_2 g_4)$, we have,

$$
\varphi_2 g_4(x) = \frac{\log(1 - e^{-x_1x_2})}{1 - x_2} \varphi_2(x) \xi(x),
$$

where $\xi(x) = \frac{k_3(x)(1 - x_2) \log(1 - e^{-x_1})}{1 - x_2}$. Then $\varphi_2 \xi$ is in $\mathcal{S}(R)$ by Lemma 6.2, and by Equation (20),

$$
\varphi_2 g_4(x) = \frac{\log x_1 + \log x_2 + f(x_1 x_2)}{1 - x_2} \varphi_2(x) \xi(x).
$$

Because $|x_1x_2| \leq 2$ on the support of $\varphi_2$, the partial derivatives of $f(x_1x_2)$ grow no faster than polynomially in $x_1$ on the support of $\varphi_2$, so $\varphi_2(x) f(x_1x_2) \xi(x)$ is in $\mathcal{S}(R)$. Splitting $\varphi_2 g_4$ into three parts and applying Corollary 1.6 three times with $A(x) = (\log x_1)/(1 - x_2)$, $A(x) = (\log x_2)/(1 - x_2)$, and $A(x) = 1/(1 - x_2)$ establishes the continuation of $(\Psi, \varphi_2 g_4)$ to a function meromorphic on $\mathbb{C}^2$.

To continue $(\Psi, \varphi_1 g_4)$, we have,

$$
\varphi_1 g_4(x) = \frac{\log(1 - e^{-x_1x_2})}{1 - x_2} \log(1 - e^{-x_1}) \varphi_1(x) \xi(x),
$$

where $\xi(x) = k_3(x)(1 - x_2)$, and $\varphi_1 \xi$ is in $\mathcal{S}(R)$. Using Equation (20) twice gives

$$
\varphi_1 g_4(x) = \frac{(\log(x_1x_2) + f(x_1x_2))(\log x_1 + f(x_1))}{1 - x_2} \varphi_1(x) \xi(x)
$$

$$
= \frac{(\log x_1)^2}{1 - x_2} \varphi_1(x) \xi(x) + \frac{\log x_1 \log x_2}{1 - x_2} \varphi_1(x) \xi(x) + \frac{\log x_1}{1 - x_2} f(x_1) \varphi_1(x) \xi(x)
$$

$$
+ \frac{\log x_2}{1 - x_2} f(x_1) \varphi_1(x) \xi(x) + \frac{\log x_1}{1 - x_2} f(x_1x_2) \varphi_1(x) \xi(x)
$$

$$
+ \frac{1}{1 - x_2} f(x_1) f(x_1x_2) \varphi_1(x) \xi(x).
$$

Because on the support of $\varphi_1$ both $x_1$ and $x_1x_2$ are bounded, $f(x_1) \varphi_1(x) \xi(x)$, $f(x_1x_2) \varphi_1(x) \xi(x)$, and $f(x_1) f(x_1x_2) \varphi_1(x) \xi(x)$ are in $\mathcal{S}(R)$. Hence, we can continue $(\Psi, \varphi_1 g_4)$ by splitting $\varphi_1 g_4$ into six parts and applying Corollary 1.6 with $A(x) = (\log x_1)^2/(1 - x_2)$, $A(x) = \log x_1 \log x_2/(1 - x_2)$, $A(x) = \log x_1/(1 - x_2)$ (twice), $A(x) = \log x_2/(1 - x_2)$, and $A(x) = 1/(1 - x_2)$.
Lemma 6.3. Define $F : R \to \mathbb{C}$ by $F(x) = \log(1 - e^{-x_1 x_2})\theta(x)$ where $\theta$ is in $\mathcal{S}(R)$ and is supported on a region of $R$ for which $x_1 x_2$ is bounded away from zero. Then $F$ is in $\mathcal{S}(R)$.

Proof. Because $x_1 x_2$ is bounded away from zero on the support of $\theta$, the function $\log(1 - e^{-x_1 x_2})$ is bounded on the support of $\theta$. By a direct calculation, it can also be shown that each partial derivative of $\log(1 - e^{-x_1 x_2})$ is bounded by a polynomial in $x_1$ on the support of $\theta$. By repeated use of the Leibnitz rule it follows that $F$ is in $\mathcal{S}(R)$. □

6.6. Continuation of $(\Psi, g_5)$. We use the same partition of unity as in Section 6.5. Let

$$H(x) = \text{dilog}(1 - e^{-x_1 x_2}) - \text{dilog}(\alpha(x)), \quad (22)$$

where

$$\alpha(x) = (1 - e^{-x_1 x_2})/(1 - e^{-x_1})$$

assumes values in $(0, 1)$ on $R$. Then by Lemma 6.7, $\varphi_3(x)(1 - x_2)g_5(x) = \varphi_3(x)(1 - x_2)k_3(x)H(x)$ is in $\mathcal{S}(R)$, and $(\Psi, \varphi_3 g_5)$ can be continued to a function meromorphic on $\mathbb{C}^2$ by applying Corollary 1.6 with $A(x) = 1/(1 - x_2)$.

Now consider the continuation of $(\Psi, \varphi_2 g_5)$. Using Equation (21), we have,

$$\text{dilog}(1 - e^{-x_1 x_2}) = \frac{\pi^2}{6} - \text{dilog}(e^{-x_1 x_2}) - \log(1 - e^{-x_1 x_2})\log(1 - e^{-x_1 x_2})$$

$$= \frac{\pi^2}{6} - \text{dilog}(e^{-x_1 x_2}) + x_1 x_2 \log(1 - e^{-x_1 x_2}), \quad (23)$$

and

$$\text{dilog}(\alpha(x)) = \frac{\pi^2}{6} - \text{dilog}(1 - \alpha(x)) - \log(\alpha(x))\log(1 - \alpha(x)).$$

We can use these relations to show that

$$H(x) = F_1(x_1, x_2) + F_2(x_1, x_2) + F_3(x_1, x_2),$$

where

$$F_1(x_1, x_2) = \text{dilog}(1 - \alpha(x)) - \text{dilog}(e^{-x_1 x_2}), \quad (24)$$

$$F_2(x_1, x_2) = -\log(1 - e^{-x_1})(\log(1 - e^{-x_1(1 - x_2)}) - \log(1 - e^{-x_1}) - x_1 x_2),$$

and

$$F_3(x_1, x_2) = \log(1 - e^{-x_1 x_2})(\log(1 - e^{-x_1(1 - x_2)}) - \log(1 - e^{-x_1})).$$

The function $\varphi_2 F_1$ is in $\mathcal{S}(R)$ by Lemma 6.5 and $\varphi_2 F_2$ is in $\mathcal{S}(R)$ by Lemma 6.2 and Lemma 6.4. Then $\varphi_2(x)(1 - x_2)k_3(x)(F_1 + F_2)(x)$ is in $\mathcal{S}(R)$; therefore, the function $(\Psi, \varphi_2(x)k_3(x)(F_1 + F_2)(x))$ can be continued to a meromorphic function on $\mathbb{C}^2$ by applying Corollary 1.6 with $A(x) = 1/(1 - x_2)$.

Also, $\varphi_2(x)(\log(1 - e^{-x_1(1 - x_2)}) - \log(1 - e^{-x_1}))$ is in $\mathcal{S}(R)$ by Lemma 6.2 and Lemma 6.4, so $(\Psi, \varphi_2(x)k_3(x)F_3(x))$ can be continued to a meromorphic function.
on \( \mathbb{C}^2 \) using Corollary 1.6 as in the continuation of \( \varphi_{2}g_{4} \) in Section 6.5. Since \( g_{5} = k_{3}(F_{1} + F_{2} + F_{3}) \), it follows that \((\Psi, \varphi_{2}g_{3})\) can be continued to a meromorphic function on \( \mathbb{C}^2 \).

To continue \((\Psi, \varphi_{1}g_{5})\), the final piece of \((\Psi, g_{5})\), we have
\[
(\Psi, \varphi_{1}g_{5}) = (\Psi, \varphi_{1}k_{3}(x) \text{dilog}(1 - e^{-x_{1}x_{2}})) - (\Psi, \varphi_{1}k_{3}(x) \text{dilog}(\alpha(x))).
\]

Using Equation (23) and Equation (20), we have
\[
dilog(1 - e^{-x_{1}x_{2}}) = \frac{\pi^2}{6} - \text{dilog}(e^{-x_{1}x_{2}}) + x_{1}x_{2}(\log x_{1} + \log x_{2} + f(x_{1}x_{2})),
\]
where \( f \) is entire as a function of one complex variable. Then, since dilog is real analytic on \((0, 2)\) and hence on any domain of \((0, 1]\) that is bounded away from zero, \( \varphi_{1}(\pi^2/6 - \text{dilog}(e^{-x_{1}x_{2}}) + x_{1}x_{2}f(x_{1}x_{2})) \) is in \( S(R) \), and we continue \((\Psi, \varphi_{1}k_{3}(x) \text{dilog}(1 - e^{-x_{1}x_{2}}))\) using Corollary 1.6 as in the continuation of \( \varphi_{2}g_{4} \) in Section 6.5.

Now let \( \varphi_{1,1} \) and \( \varphi_{1,2} \) be functions in \( C^{\infty}(R) \) such that \( \varphi_{1} = \varphi_{1,1} + \varphi_{1,2} \) with \( \varphi_{1,1} \equiv 0 \) on \((0, \infty) \times (0, 1/4)\) and \( \varphi_{1,2} \equiv 0 \) on \((0, \infty) \times (3/4, 1)\), and observe that \( \alpha \) is in \( C^{\infty}(R) \), \( \alpha \) is bounded away from zero on the support of \( \varphi_{1,1} \), and \( 1 - \alpha \) is bounded away from zero on the support of \( \varphi_{1,2} \). Thus, \((\Psi, \varphi_{1,1}k_{3}(x) \text{dilog}(\alpha(x)))\) can be continued using Corollary 1.6 with \( A(x) = 1/(1 - x_{2}) \).

To continue \((\Psi, \varphi_{1,2}k_{3}(x) \text{dilog}(\alpha(x)))\), we can take advantage of Equation (21). Because \( 1 - \alpha \) is bounded away from zero on the support of \( \varphi_{1,2} \), the function \( \varphi_{1,2}(\pi^2/6 - \text{dilog}(1 - \alpha(x))) \) is in \( S(R) \), so \((\Psi, \varphi_{1,2}k_{3}(x)(\pi^2/6 - \text{dilog}(1 - \alpha(x)))) \) is meromorphic, again using Corollary 1.6 with \( A(x) = 1/(1 - x_{2}) \).

Since \( \varphi_{1,2} \log(1 - \alpha(x)) \) is in \( C^{\infty}(R) \), we have
\[
\varphi_{1,2}k_{3}(x) \log(\alpha(x)) \log(1 - \alpha(x)) = \frac{\log(\alpha(x))}{1 - x_{2}} \eta(x),
\]
where \( \eta(x) = \varphi_{1,2}(1 - x_{2})k_{3}(x) \log(1 - \alpha(x)) \) is in \( S(R) \). Then,
\[
\log(\alpha(x)) = \log(1 - e^{-x_{1}x_{2}}) - \log(1 - e^{-x_{1}}),
\]
giving singularities like those for \( \varphi_{1}g_{4} \) in Section 6.5 and \( g_{1} \) in Section 6.2, so we can continue \((\Psi, \varphi_{1,2}k_{3}(x) \log(\alpha(x)) \log(1 - \alpha(x)))\) using the approaches in those sections.

Using Equation (21) and adding together the three functions we have continued gives the continuation of \((\Psi, \varphi_{1}k_{3}(x) \text{dilog}(\alpha(x)))\), and hence of \((\Psi, \varphi_{1}g_{3})\) and, finally, of \( L \).

**Lemma 6.4.** The function \( \log(1 - e^{-x_{1}(1 - x_{2})}) \) is in \( S(\Omega) \) for any domain \( \Omega \) in \( R \) for which \( x_{1} \) is bounded away from \( 0 \) and \( x_{2} \) is bounded away from 1.

**Proof.** First observe that on any \( \Omega \) with the properties described in the statement of the lemma,
\[
| \log(1 - e^{-x_{1}(1 - x_{2})}) | \leq | \log(1 - e^{-c_{0}x_{1}}) |,
\]
where \( c_{0} = \inf_{x_{2} \in \Omega} (1 - x_{2}) \). Then for any multi-index \( \beta \),
\[
\sup_{x \in \Omega} | x^{\beta} \log(1 - e^{-x_{1}(1 - x_{2})}) | \leq \sup_{x \in \Omega} | x^{\beta} \log(1 - e^{-c_{0}x_{1}}) | \leq \sup_{x \in \Omega} | x_{1}^{\beta_{1}} \log(1 - e^{-c_{0}x_{1}}) |.
\]
This is finite, as we can see by using L’Hopital’s rule as in the proof of Lemma 6.2 (where \( c_0 \) was equal to 1), since also \( x_1 \) is bounded away from 0.

One can show inductively that for any multi-index \( \alpha \) for which \(|\alpha| \geq 1\),

\[
D^\alpha \log(1 - e^{-x_1(1 - x_2)}) = \sum_{j=1}^{|\alpha|} P_j(x_1, 1 - x_2)(r(x))^j,
\]

where \( r(x) = 1/(e^{x_1(1 - x_2)} - 1) \) and \( P_j \) are polynomials. But \(|r(x)| \leq 1/(e^{c_0x_1} - 1)\), so for any multi-index \( \beta \),

\[
\sup_{x \in \Omega} |x^\beta(r(x))^j| \leq \sup_{x \in \Omega} |x^\beta|/(e^{c_0x_1} - 1)^j \leq \sup_{x \in \Omega} |x^\beta|/(e^{c_0x_1} - 1)^j
\]

is finite, since also \( x_1 \) is bounded away from 0. It follows that \( \log(1 - e^{-x_1(1 - x_2)}) \) is in \( S(\Omega) \).

\[\text{Lemma 6.5.} \quad \text{The function } F_1 \text{ of Equation (24) is in } S(\Omega), \text{ where } \Omega \text{ is the interior of the support of } \varphi_2.\]

\[\text{Proof.} \quad \text{Because}
\]

\[
1 - \alpha(x) = e^{-x_1x_2} - e^{-x_1\alpha(x)}, \quad (25)
\]

as \( x_1 \) grows large, the two terms in \( F_1 \) become exponentially close, and so cancel to produce exponential decay. Our proof of this lemma is a formal verification of this observation.

Let \( \beta \) be a multi-index. Then for integer constants \( C_{\gamma, \beta} \),

\[
D^\beta F_1(x_1, x_2) = \sum_{\{\gamma: |\gamma| \leq |\beta|\}} C_{\gamma, \beta} \left[ \begin{array}{c}
\text{dilog}(|\beta| - |\gamma|)(1 - \alpha(x)) \prod_{k=1}^{n(\gamma)} D^{\delta(\gamma, k)}(1 - \alpha(x)) \\
- \text{dilog}(|\beta| - |\gamma|)(e^{-x_1x_2}) \prod_{k=1}^{n(\gamma)} D^{\delta(\gamma, k)}(e^{-x_1x_2})
\end{array} \right] 
\]

\[
= \sum_{\{\gamma: |\gamma| \leq |\beta|\}} C_{\gamma, \beta} \left[ \text{dilog}(|\beta| - |\gamma|)(1 - \alpha(x)) - \text{dilog}(|\beta| - |\gamma|)(e^{-x_1x_2}) \prod_{k=1}^{n(\gamma)} D^{\delta(\gamma, k)}(e^{-x_1x_2}) \\
+ \sum_{\{\gamma: |\gamma| \leq |\beta|\}} C_{\gamma, \beta} \text{dilog}(|\beta| - |\gamma|)(1 - \alpha(x)) \prod_{k=1}^{n(\gamma)} D^{\delta(\gamma, k)}(1 - \alpha(x)) - \prod_{k=1}^{n(\gamma)} D^{\delta(\gamma, k)}(e^{-x_1x_2}) \right],
\]

where \( n(\gamma) \leq |\gamma| \) and the \( \delta(\gamma, k) \) are multi-indices. (For many values of \( \gamma \), there will be distinct values of \( k_1 \) and \( k_2 \) such that \( \delta(\gamma, k_1) = \delta(\gamma, k_2) \).)
Using Equation (25) and applying the mean value theorem, we have
\[
|\text{dilog}(|\beta|) (e^{-x_1 x_2}) - \text{dilog}(|\beta|) (1 - \alpha(x))| \\
= |\text{dilog}(|\beta|) (e^{-x_1 x_2}) - \text{dilog}(|\beta|) (e^{-x_1} - e^{-x_1} \alpha(x))| \\
= |e^{-x_1} \alpha(x) \text{dilog}(|\beta|)| + 1 (\xi)|
\]
for some \( \xi \) in \((1 - \alpha(x), e^{-x_1} x_2)\).

Because \(|\text{dilog}(k)(x)|\) is decreasing on \((0, 1)\) for all \(k \geq 0\), we have, by Lemma 6.8,
\[
|\text{dilog}(|\beta|)(\xi)| \leq |\text{dilog}(|\beta|)(1 - \alpha(x))| \leq |\text{dilog}(|\beta|)(e^{-2}/2)|,
\]
which is a constant that depends only on \(|\beta| - |\gamma|\). Since also \(0 < \alpha(x) < 1\), we have
\[
|\text{dilog}(|\beta|)(e^{-x_1} x_2) - \text{dilog}(|\beta|)(1 - \alpha(x))| \leq C_{|\beta| - |\gamma|} e^{-x_1}.
\]

By the same reasoning,
\[
|\text{dilog}(|\beta|)(1 - \alpha(x))| \leq C_{|\beta| - |\gamma|}.
\]

Also,
\[
|D^{(\gamma,k)}(e^{-x_1} x_2)| = x_1^{(\gamma,k)} x_2^{(\gamma,k)} e^{-x_1} x_2 \leq P_\delta(x),
\]
where \(P_\delta\) is a polynomial.

Combining these bounds with the bound in Lemma 6.6 and the bound on \(D^\beta F_1(x_1, x_2)\) above, it follows that \(F_1\) is in \(\mathcal{S}(\Omega)\). \(\square\)

**Lemma 6.6.** Let \(n\) be a nonnegative integer, and let \(\delta = (\delta(1), \ldots, \delta(n))\) be any multi-index. Then for some polynomial \(P_\delta\),
\[
\prod_{k=1}^n D^{(k)}(1 - \alpha(x)) - \prod_{k=1}^n D^{\delta}(e^{-x_1} x_2) \leq P_\delta(x) e^{-x_1}
\]
for all \(x\) in the interior of the support of \(\varphi_2\).

**Proof.** We proceed by induction on \(n\). For \(n = 1\), we have, using Equation (25),
\[
D^{(1)}(1 - \alpha(x)) - D^{(1)}(e^{-x_1} x_2) = -D^{(1)}(e^{-x_1} \alpha(x)) \\
= -\partial_1^{(1)}(e^{-x_1} \partial_2^{(1)} \alpha(x)) = \sum_{k=0}^{\delta(1)} C_{k,\delta(1)} e^{-x_1} \partial_1^{k} \partial_2^{(1)} \alpha(x) \\
= e^{-x_1} \sum_{k=0}^{\delta(1)} C_{k,\delta(1)} D^{(k,\delta(1))} \alpha(x),
\]
where \(C_{k,\delta(1)}\) are integer constants.

One can show by induction that \(D^{(k,\delta(1))} \alpha(x)\) is always a finite sum of terms of the form
\[
C e^{-k_1 x_1 x_2} e^{-k_2 x_1} (1 - e^{-x_1})^{k_3} x_1^{\alpha_1} x_2^{\alpha_2}
\]
for integer constants \( k_1, k_2, k_3 \), and \( C \), and nonnegative integer constants \( e_1 \) and \( e_2 \), with \( e_1 + e_2 \leq k + \delta(1,2) \). Thus,

\[
|D^{(1)}(1 - \alpha(x)) - D^{(1)}(e^{-x_1x_2})| \leq P_\delta(x)e^{-x_1}
\]
on the support of \( \varphi_2 \) for some polynomial \( P_\delta \), establishing the induction hypothesis for \( n = 1 \). It also follows from Equation (26), which holds for any multi-index, that

\[
|D^{(1)}(1 - \alpha(x))| \leq P_\delta(x).
\]

(27)

Now assume that the induction hypothesis is true for \( n = m - 1 \). Then

\[
\begin{align*}
|\prod_{k=1}^{m} D^{(k)}(1 - \alpha(x)) - \prod_{k=1}^{m} D^{(k)}(e^{-x_1x_2})| \\
\leq |D^{(m)}(1 - \alpha(x))| \left| \prod_{k=1}^{m-1} D^{(k)}(1 - \alpha(x)) - \prod_{k=1}^{m-1} D^{(k)}(e^{-x_1x_2}) \right| \\
+ |D^{(m)}(1 - \alpha(x)) - D^{(m)}(e^{-x_1x_2})| \left| \prod_{k=1}^{m-1} D^{(k)}(e^{-x_1x_2}) \right|.
\end{align*}
\]

The induction hypothesis for \( n = m \) now follows by applying Equation (26) and Equation (27), which hold for any multi-indices, along with the induction hypothesis for \( n = 1 \) and the induction hypothesis for \( n = m - 1 \).

\[\square\]

Lemma 6.7. The function \( H \) of Equation (22) is in \( S(\Omega) \), where \( \Omega \) is the interior of the support of \( \varphi_3 \).

Proof. The proof is almost identical to that of Lemma 6.5. Equation (25) is still the starting point for the proof, though instead of having \( 1 - \alpha(x) \) bounded away from 0 by \( e^{-2}/2 \), we now have \( 1 - e^{-x_1x_2} \) bounded away from 0 by \( 1 - e^{-1} \). In all other respects, the proof parallels that of Lemma 6.5 so closely, that we suppress the details.

\[\square\]

Lemma 6.8. On the support of \( \varphi_2 \), \( 1 - \alpha(x) > e^{-x_1x_2}/2 > e^{-2}/2 \).

Proof. On the support of \( \varphi_2 \),

\[
\begin{align*}
x_1 > 3 > 2 + \log 2 \text{ and } x_1x_2 < 2 & \implies x_1(1 - x_2) = x_1 - x_1x_2 > x_1 - 2 > \log 2 \\
& \implies e^{x_1(1 - x_2)} > 2 \implies \frac{1}{2}e^{-x_1x_2} > e^{-x_1} > e^{-x_1}\alpha(x) \\
& \implies e^{-x_1x_2} > e^{-x_1}\alpha(x) + \frac{1}{2}e^{-x_1x_2} \\
& \implies 1 - \alpha(x) = e^{-x_1x_2} - e^{-x_1}\alpha(x) > \frac{1}{2}e^{-x_1x_2} > e^{-2}/2,
\end{align*}
\]

where we used Equation (25).
6.7. Remark on the singularity along $x_2 = 1$. Each of the functions $g_1$ through $g_5$ has a singularity along $x_2 = 1$ because of the factors $k_1$, $k_2$, and $k_3$. In removing this singularity, we introduced a simple pole at $s_1 = 1$ into the continuation of $L$. The test function $g$ itself, however, we know has singularities only along $x_1 = 0$ and $x_2 = 0$: when added, the singularities of $g_1$ through $g_5$ cancel along $x_2 = 1$ (though $(0, 1)$ is still a singular point). It does not appear feasible to avoid splitting $g$ into components to continue $L$; however, a careful analysis of our argument would give information about the residues at $s = 1$. It is possible that we would find that these residues cancel.

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REFERENCES


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