THE 3D EULER EQUATIONS WITH INFLOW, OUTFLOW AND VORTICITY BOUNDARY CONDITIONS

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Abstract. The 3D incompressible Euler equations in a bounded domain are most often supplemented with impermeable boundary conditions, which constrain the fluid to neither enter nor leave the domain. We establish well-posedness with inflow, outflow of velocity when either the full value of the velocity is specified on inflow, or only the normal component is specified along with the vorticity (and an additional constraint). We derive compatibility conditions to obtain arbitrarily high Hölder regularity of the solution, and allow multiply connected domains. Our results apply as well to impermeable boundaries, establishing higher regularity of solutions in Hölder spaces, filling a gap in the literature.

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Part I: Overview

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, possibly multiply connected, having a boundary that is at least C^2 regular. We define $\boldsymbol{n}$ to be the outward unit normal vector to the boundary, $\Gamma := \partial\Omega$, and follow the convention that for any vector field $\mathbf{v}$,

$$v^{\boldsymbol{n}} := \mathbf{v} \cdot \boldsymbol{n}, \quad \mathbf{v}^{\boldsymbol{\tau}} := \mathbf{v} - v^{\boldsymbol{n}}\boldsymbol{n} \text{ on } \Gamma. \tag{1.1}$$

Fixing $T > 0$, the Euler equations on $Q := (0, T) \times \Omega$ can be written,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

Here, $\mathbf{u}$ is the velocity field of a constant-density incompressible fluid, p its scalar pressure, $\mathbf{f}$ the divergence-free external force tangential to the boundary, and $\mathbf{u}_0$ the initial velocity.

To complete the system of equations in (1.2) we impose inflow, outflow boundary conditions in the spirit of [2]. We partition the boundary Γ into three portions, Γ_+, Γ_-, and Γ_0, corresponding to inflow, outflow, and impermeability, respectively. Each portion consists of a finite number of components (with $\Gamma_0 = \emptyset$ or $\Gamma_0 = \Gamma$ allowed—see Remark 12.1). We fix a vector field $\mathbf{U}$ on $[0, T] \times \Gamma$ and assume that

$$U^{\boldsymbol{n}} < 0 \text{ on } \Gamma_+, \qquad U^{\boldsymbol{n}} > 0 \text{ on } \Gamma_-, \qquad U^{\boldsymbol{n}} = 0 \text{ on } \Gamma_0.$$

We then define inflow, outflow boundary conditions as

$$\begin{cases} u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0, T] \times \Gamma, \\ \mathbf{u} = \mathbf{U} & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{1.3}$$

We also impose on $\mathbf{U}$ the constraint that $\int_{\Gamma_+} U^{\boldsymbol{n}} = -\int_{\Gamma_-} U^{\boldsymbol{n}}$, required to allow $\operatorname{div} \mathbf{u} = 0$.

(We choose to impose inflow, outflow boundary conditions in terms of a vector field $\mathbf{U}$ defined on all of Γ—in fact on all of Ω—because it will be productive for us to view $\mathbf{U}$ as a background flow as done in [7, 24, 28]. If we wish, we can choose $\mathbf{U}$ to be divergence-free as done in [7], though this is not necessary for our purposes.)

Defining the vorticity,

$$\boldsymbol{\omega} := \operatorname{curl} \mathbf{u},$$

applying curl to both sides of $(1.2)_1$ yields the vorticity equation,

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \mathbf{g} := \operatorname{curl} \mathbf{f}. \tag{1.4}$$

It follows from (1.4) that the vorticity is transported and stretched (pushedforward) by the flow map for $\mathbf{u}$ (when $\mathbf{g} \equiv 0$).

In particular, the vorticity is brought into the domain from the inflow boundary, making inflow, outflow substantially more difficult to treat than impermeable boundaries: the mechanism for generating vorticity on the inflow boundary must be understood and controlled. This is a key reason for using Hölder spaces, as there is no loss of regularity of the trace of the vorticity on the boundary over that in the domain.

Higher regularity solutions for inflow, outflow boundary conditions are employed, for instance, in Prandtl-type boundary layer expansions (such as [7, 28] and work in progress of the authors). The validity of such expansions for inflow, outflow boundary conditions results from a stability mechanism of injection, suction in boundary layers. These applications were the original motivation for this work: because of this, in Appendix C we give the explicit form of the compatibility conditions for those works.

More commonly, (1.2) is supplemented with *impermeable boundary conditions*, $\mathbf{u}\cdot\boldsymbol{n} = 0$, on all of Γ, meaning that fluid neither enters nor exits the domain. This places one constraint on the velocity field, as is usual for first-order equations. The condition in $(1.3)_2$, however, specifies the full velocity on the inflow boundary. This condition is natural in view of (1.4), which demonstrates that the vorticity is brought into the domain from the inflow boundary.

The system of equations we study, then, are (1.2) with (1.3):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega, \\ u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0,T]\times\Gamma, \\ \mathbf{u} = \mathbf{U} & \text{on } [0,T]\times\Gamma_+. \end{cases} \tag{1.5}$$

We can state the main result of this paper informally as follows, where **throughout, we fix $\boldsymbol{\alpha} \in (\mathbf{0},\mathbf{1})$**:

Theorem (Informal statement of main result)**.** *Assume that for some integer $N \geqslant 0$, $\mathbf{u}_0$ is a divergence-free vector field in the classical Hölder space $C^{N+1,\alpha}(\Omega)$, satisfies (1.3), and satisfies a compatibility condition to be described below. There is a $T > 0$ such that there exists a solution to (1.5) with $\mathbf{u}(t) \in C^{N+1,\alpha}(\Omega)$ for all $t \in [0,T]$.*

To state our main result rigorously, we must define the function spaces in which we will work, determine proper conditions on the forcing, and determine the required compatibility conditions. In addition, a careful study of the pressure will be needed.

Function spaces. For any $N \geqslant 0$ we define the affine hyperplanes of $C^{N+1,\alpha}(\Omega)$ and $C^{N+1,\alpha}(Q)$,

$$\begin{aligned} C^{N+1,\alpha}_\sigma(\Omega) &:= \{\mathbf{u} \in C^{N+1,\alpha}(\Omega)\colon \operatorname{div}\mathbf{u} = 0, \mathbf{u}\cdot\boldsymbol{n} = U^{\boldsymbol{n}}(0) \text{ on } \Gamma\}, \\ C^{N+1,\alpha}_\sigma(Q) &:= \{\mathbf{u} \in C^{N+1,\alpha}(Q)\colon \operatorname{div}\mathbf{u} = 0, \mathbf{u}\cdot\boldsymbol{n} = U^{\boldsymbol{n}} \text{ on } [0,T]\times\Gamma\}. \end{aligned} \tag{1.6}$$

Since only the normal component of $\mathbf{u}$ is specified on the entire boundary, only the boundary condition in $(1.5)_4$ is included in the definition of these spaces.

We also employ the classical space,

$$H := \{\mathbf{u} \in L^2(\Omega)^3\colon \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\} = H_0 \oplus H_c, \tag{1.7}$$

where

$$H_c := \{\mathbf{v} \in H : \operatorname{curl}\mathbf{v} = 0\}, \quad H_0 := H_c^\perp. \tag{1.8}$$

For $\mathbf{u} \in H$, $P_{H_c}\mathbf{u}$ is termed the *harmonic* component of $\mathbf{u}$.

We define the boundary values (via $\mathbf{U}$) and the forcing $\mathbf{f}$ for all time on $Q_\infty := [0,\infty)\times\Omega)$. We will prove existence only for short time.

Definition 1.1. *We say that the data has regularity N for an integer $N \geqslant 0$ if*

- *Γ is $C^{N+2,\alpha}$, $\mathbf{U} \in C^{N+2,\alpha}_\sigma(Q_\infty)$, $\mathbf{f} \in C^{N+1,\alpha}(Q_\infty) \cap C([0,\infty); H_0)$;*
- *$\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$, $\mathbf{u}_0^{\boldsymbol{\tau}} = \mathbf{U}_0^{\boldsymbol{\tau}}$ on Γ_+.*

We assumed that $\mathbf{U}$ has one more derivative than $\mathbf{u}$ for two somewhat related reasons, as explained in Remarks 3.2 and 9.4.

Compatibility conditions. The vorticity generated at the inflow boundary is carried by the flow into the interior; at the same time, the flow pushes the initial vorticity forward in time. The interaction between these two sources of vorticity may potentially lead to a singularity. The main thrust of this work is to show that it is possible to avoid such singularities, at least for short time, by imposing suitable conditions on the data. We refer to these conditions as *compatibility conditions*, satisfying two primary principles:

(1) They depend only upon the initial data, $\mathbf{U}$, and $\mathbf{f}$.
(2) They are compatible with being a solution to (1.5); that is, a solution to (1.5) could, in principle, satisfy them.

The conditions we develop will ensure regularity of the solution for short time. It remains an open question whether a regular solution persists for all time even in 2D.

Given $\mathbf{u}$ with data regularity N for some $N \geqslant 0$, we define the N^{th} compatibility condition,

$$\begin{aligned} &\mathrm{cond}_{-1} : \mathbf{u}_0^{\boldsymbol{\tau}} = \mathbf{U}_0^{\boldsymbol{\tau}} \text{ on } \Gamma_+, \\ &\mathrm{cond}_N : \mathrm{cond}_{N-1} \text{ and } \partial_t^{N+1}\mathbf{U}^{\boldsymbol{\tau}}|_{t=0} = \widetilde{\partial}_t^{N+1}\mathbf{u}_0^{\boldsymbol{\tau}} \text{ on } \Gamma_+. \end{aligned} \tag{1.9}$$

For integers $n \geqslant 0$, we define $\widetilde{\partial}_t^n \mathbf{u}_0$ inductively by setting $\widetilde{\partial}_t^0 \mathbf{u}_0 = \mathbf{u}_0$, while for $n \geqslant 1$, we take the time derivative of $\widetilde{\partial}_t^{n-1}\mathbf{u}$ at time zero and replace each instance of $\partial_t \mathbf{u}$ in the resulting expression by $-\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla q + \mathbf{f}(0)$. Here, q is an approximate pressure, whose detailed description, along with a more complete description of compatibility conditions in general, we present in Section 3.

For $N = 0$, (1.9) is the compatibility condition in (1.10), (1.11) of Chapter 4 of [2]:

$$\mathrm{cond}_0 : \partial_t \mathbf{U}^{\boldsymbol{\tau}}|_{t=0} = [-\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla q + \mathbf{f}(0)]^{\boldsymbol{\tau}} \text{ on } \Gamma_+.$$

Main result. We can now rigorously state the main result of this paper as follows:

Theorem 1.2. *Assume the data has regularity N for some integer $N \geqslant 0$ as in Definition 1.1 and satisfies cond_N of (1.9). There is a $T > 0$ such that there exists a solution $(\mathbf{u}, p)$ to (1.5) with $(\mathbf{u}, \nabla p) \in C_\sigma^{N+1,\alpha}(Q) \times C^{N,\alpha}(Q)$, which is unique up to an additive constant for the pressure.*

Remark 1.3. *It follows from the proof of Theorem 1.2 that T is bounded below by a continuous, increasing function of the norms of Γ, $\mathbf{U}$, $\mathbf{f}$, and $\mathbf{u}_0$ in the spaces appearing in Definition 1.1. The explicit form of the estimate is, however, involved and not optimal.*

Vorticity boundary condition. We also consider solutions $(\mathbf{u}, p, \mathbf{z})$ to the Euler equations with vorticity boundary conditions, where the value of the vorticity on the inflow boundary is given by a function $\mathbf{H}$ on $[0, T] \times \Gamma_+$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{z} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega, \\ u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0, T] \times \Gamma, \\ \operatorname{curl} \mathbf{u} = \mathbf{H} & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{1.10}$$

Here, $\mathbf{z} \in H_c$ is an harmonic vector field. We can either treat it as part of the data or as part of the solution. That is, we can either: (1) choose $\mathbf{z}$ or (2) choose the harmonic component of $\mathbf{u}(t)$, from which the value of $\mathbf{z}$ can be obtained. We choose the latter option in Theorem 1.4, as it allows for the uniqueness of solutions.

Theorem 1.4. *Fix* $\mathbf{u}_c \in C^{N+1,\alpha}(Q) \cap C([0,T]; H_c)$. *Assume that the data has regularity* N *for some integer* $N \geqslant 0$ *as in Definition 1.1, that* cond_N *holds, and that* $\mathbf{u}_c(0) = P_{H_c} P_H \mathbf{u}_0$. *Also assume that* $\mathbf{H} \in C^{\max\{N,1\},\alpha}([0,T] \times \Gamma_+)$ *and*

$$H^{\boldsymbol{n}} = 0, \quad \mathrm{div}_\Gamma[U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] + \mathrm{curl}\, \mathbf{f} \cdot \boldsymbol{n} = 0 \text{ on } [0,T] \times \Gamma_+. \tag{1.11}$$

There is a $T > 0$ *such that there exists a solution* $(\mathbf{u}, p, \mathbf{z})$ *in* $C^{N+1,\alpha}_\sigma(Q) \times C^{N,\alpha}(Q) \times (C^{N+1,\alpha}(Q) \cap C([0,T]; H_c)$ *to (1.5) for which* $P_{H_c} P_H \mathbf{u} = \mathbf{u}_c$ *on* $\overline{Q}$. *If* $N \geqslant 1$ *the solution is unique up to an additive constant for the pressure. In addition,* $\mathbf{z}(0) = 0$.

Remark 2.3 explains why the condition in (1.11) is imposed; div_Γ is the divergence operator on the boundary (see Appendix B).

What is novel in our approach. There are many proofs of well-posedness of the Euler equations taking different approaches. To the authors' knowledge, all such proofs in Hölder spaces in a 3D domain with boundary, including this paper, and many in the whole space or a periodic domain, follow in the tradition of McGrath [20, 21] and Kato [11], in which the solution is obtained as a fixed point of an operator A derived from a linearization of the Euler equations, employing Schauder's fixed point theorem.

For inflow, outflow boundary conditions, this approach was taken in Chapter 4 of [2], which establishes Theorem 1.2 for $N = 0$ and simply connected domains. The operator A is derived from a linearization of the vorticity equation (1.4) with prescribed values on the inflow boundary. This leads to linear compatibility conditions based on vorticity, whereas the nonlinear boundary conditions are based on the velocity. In fact, one challenge is to ensure that the nonlinear compatibility conditions at the level of the velocity imply the linear ones at the level of the vorticity.

To handle inflow, outflow boundary conditions, the authors of [2] make many adaptations to the Kato-McGrath approach, but we would identify their two key innovations as the following:

- They obtain estimates on the operator A under the simple linear compatibility condition that on the inflow boundary, the vorticity matches the prescribed inflow vorticity at time zero (akin to the Rankine-Hugoniot condition).
- They show how to achieve the needed regularity of the inflow vorticity from the pressure.

For $N \geqslant 1$, several complications arise. We can still use the same operator A as in [2], but now the linear compatibility conditions becomes more involved (see (2.3) and (2.4)). This linear issue was resolved in [9], but deriving and relating the nonlinear compatibility conditions to the linear ones remained a significant challenge, which we address here.

Moreover, unlike the $N = 0$ case, data satisfying the $N \geqslant 1$ compatibility condition is by itself insufficient to insure that the corresponding linear compatibility condition is satisfied. To address this, we must restrict the domain of the operator A by imposing an additional condition on the time derivative of the initial velocity (as in (4.1)) and show that, in fact, the resulting domain is nonempty.

The estimates on the operator A that result become much more complex for the higher regularity we treat here. This is in contrast to proving existence in the full space or a periodic domain, where one can bootsrap as in Section 4.4 of [16], which takes advantage of the simple form of Biot-Savart kernel for the full space. And in 2D, where the vorticity equation has no stretching term, one can bootstrap as Marchioro and Pulvirenti do in [19] (which originates in their earlier text [18]).

Instead, we must obtain existence directly in the higher-regularity spaces, and the resulting estimates are much more involved than the $N = 0$ case.

Indeed, even in the impermeable boundary case, which is also covered by our results, there is, to the authors' knowledge, no result in the literature for higher regularity in Hölder spaces (for higher regularity in Sobolev spaces, see, for example, the seminal works [12, 29]). Hence, we fill a gap in the literature even for the impermeable case.

Other Prior work. In addition to [2], we also drew ideas from [15], which proves well-posedness of the 3D Euler equations for impermeable boundary conditions in Hölder spaces (the equivalent of our $N = 0$ regularity). We mention also the work of Petcu [24], who presents a version of the argument in Chapter 4 of [2], specializing it to a 3D channel with a constant $\mathbf{U}$, which simplifies and makes clearer some of the arguments in [2].

Section 1.4 of [17] contains an extensive survey of results, both 2D and 3D, related to the problem we are studying here. We also mention the 2D work of Boyer and Fabrie [3, 4] and the recent works [5, 23].

Vorticity boundary conditions were studied in 2D by Yudovich in [10]. We refer in addition to the historical comments in Section 1.4 of [17] concerning partial results in 3D.

Structure of this paper. This paper consists of three parts, along with three appendices.

In Part I, following this introduction, we begin in Section 2 by summarizing results from [9] on the linearization of (1.5), a key tool at the heart of all of our arguments. In Section 3, we explore in-depth the nonlinear compatibility conditions cond_N as they apply to (1.5) and their counterparts for the linearized equations. We then give the proof of our main result, Theorem 1.2, in Section 4. This proof relies upon three propositions, Propositions 4.5 to 4.7: the rest of the paper is devoted to proving these propositions.

In Part II, we summarize additional background material from [9] and present identities and estimates on the flow map, on the vorticity generated on the boundary, and on the pressure.

In Part III, we use results primarily from the second part to prove Proposition 4.5, then leverage it to obtain Proposition 4.6. We also give the proof of Proposition 4.7. In the final section of this part, we describe how Theorem 1.4 follows from a simplification of the estimates obtained in Part II.

Appendix A contains a number of estimates in Hölder spaces, some very standard, some specific to this paper. In Appendix B we construct a convenient coordinate system in an ε-neighborhood of Γ_+. We use this system to develop properties of the operators ∇_Γ, div_Γ, and curl_Γ we use in the body of the paper. This allows us to treat the various calculations on the boundary in a coordinate-free manner, which makes the calculations more transparent. Finally, in Appendix C, we discuss the compatibility conditions in the special case in which $\mathbf{U}^{\boldsymbol{\tau}} \equiv 0$ and $U^{\boldsymbol{n}}$ is constant along Γ_+ (as occurs in [7, 28]).

We have structured this paper so as to allow the reader to grasp the overall structure of the proof of Theorem 1.2 without it being obscured by the many technical details. It is possible to read only Part I and get the gist of the proof. A more in-depth reading would involve at least examining the key pressure estimates in Section 9 and a reading of [9], to understand how the linear compatibility conditions arise.

On notation. Our notation, while fairly standard, has a few subtleties. If M is a matrix, M^i_n refers to the entry in row i of M, column n; v^i refers to the i^{th} entry in the vector $\mathbf{v}$, which we always treat as a column vector for purposes of multiplication. If M and N are matrices of the same dimensions then $M \cdot N := M^i_n N^i_n$, where here, and elsewhere, we use implicit sum notation. If $\mathbf{u}$ and $\mathbf{v}$ are vectors then the matrix $\mathbf{u} \otimes \mathbf{v}$ has components $[\mathbf{u} \otimes \mathbf{v}]^i_n = u^i v^n$.

We define the divergence of a matrix row-by-row, so $\operatorname{div} M$ is the column vector with components $[\operatorname{div} M]^i = \partial_n M^i_n$. Hence, $[\operatorname{div}[\mathbf{u} \otimes \mathbf{v}]]^i = \operatorname{div}[\mathbf{u} \otimes \mathbf{v}]^i = \partial_n (u^i v^n)$, where ∂_n is the derivative with respect to the n^{th} spatial variable. The notation ∇ means the gradient with respect to the spatial variables only; by D we mean the gradient with respect to all variables, time and space. When applied to the flow map $\eta(t_1, t_2, \mathbf{x})$, we write $\partial_{t_1} \eta$, $\partial_{t_2} \eta$ to mean the derivative with respect to the first, second time variable. Finally, for vector fields $\mathbf{u}$ and $\mathbf{v}$, we will interchangeably write $\mathbf{u} \cdot \nabla \mathbf{v}$ and $\nabla \mathbf{v}\, \mathbf{u}$, as they both are vectors with i^{th} component $u^m \partial_m v^i$.

For any tangent vector field $\mathbf{v}$ on Γ, $\mathbf{v}^\perp = \boldsymbol{n} \times \mathbf{v}$ is the tangent vector field $\mathbf{v}$ on Γ rotated 90 degrees counterclockwise around the normal vector $\boldsymbol{n}$ when viewed from outside Ω.

2. The linearized problem

The linearized Euler equations corresponding to the vorticity form of $(1.5)_1$ are

$$\begin{cases} \partial_t \overline{\boldsymbol{\omega}} + \mathbf{u} \cdot \nabla \overline{\boldsymbol{\omega}} - \overline{\boldsymbol{\omega}} \cdot \nabla \mathbf{u} = \mathbf{g} & \text{in } Q, \\ \overline{\boldsymbol{\omega}} = \mathbf{H} & \text{on } [0,T] \times \Gamma_+, \\ \overline{\boldsymbol{\omega}}(0) = \overline{\boldsymbol{\omega}}_0 & \text{on } \Omega. \end{cases} \tag{2.1}$$

Here, $\mathbf{H}$ is given on $[0,T] \times \Gamma_+$, $\overline{\boldsymbol{\omega}}_0$ is given on Ω, $\mathbf{u}$ and $\mathbf{g}$ are **given** on Q, and (2.1) is to be solved for $\overline{\boldsymbol{\omega}}$. In application, we will set $\overline{\boldsymbol{\omega}}_0 = \boldsymbol{\omega}_0 := \operatorname{curl} \mathbf{u}(0)$, though then $\overline{\boldsymbol{\omega}}(t) \neq \operatorname{curl} \mathbf{u}(t)$ in general for $t > 0$.

We employ the following three types of solution to (2.1):

(1) *Classical Eulerian* or simply *classical* solutions to (2.1), by which we mean that $(2.1)_1$ holds pointwise, and each term is continuous.

(2) *Weak Eulerian solutions*, defined as follows:

Definition 2.1. *We say that $\overline{\boldsymbol{\omega}} \in C(\overline{Q})$ is a weak (Eulerian) solution to (2.1) if $\overline{\boldsymbol{\omega}} = \mathbf{H}$ on $[0,T] \times \Gamma_+$ pointwise, $\overline{\boldsymbol{\omega}}(0) = \overline{\boldsymbol{\omega}}_0$ in $C^{N,\alpha}$, and $\partial_t \overline{\boldsymbol{\omega}} + \operatorname{div}(\overline{\boldsymbol{\omega}} \otimes \mathbf{u}) - \overline{\boldsymbol{\omega}} \cdot \nabla \mathbf{u} = \mathbf{g}$ in $\mathcal{D}'(Q)$.*

Note that $\overline{\boldsymbol{\omega}}$ has sufficient time and boundary regularity that we do not need to enforce the initial and boundary conditions weakly. Also, $\overline{\boldsymbol{\omega}} \otimes \mathbf{u}$ is a regular distribution, so $\operatorname{div}(\overline{\boldsymbol{\omega}} \otimes \mathbf{u})$ is a distribution even for $N = 0$.

(3) *Lagrangian solutions*, adapted to accommodate the inflow of vorticity from Γ_+. Because we must first introduce some concepts related to this inflow, we defer to Definition 7.4.

As shown in [9], the constraint,

$$\partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0, \tag{2.2}$$

is required to obtain a solution to (2.1) for which $\overline{\boldsymbol{\omega}}(t)$ lies in the range of the curl. We hence define the linear compatibility conditions,

$$\begin{aligned} &\text{lincond}_0 : \quad \mathbf{H}(0) = \boldsymbol{\omega}_0 \text{ on } \Gamma_+, \\ &\text{lincond}_1 : \quad \text{lincond}_0 \text{ and } \partial_t \mathbf{H}|_{t=0} = \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 + \mathbf{g}(0) \text{ on } \Gamma_+, \end{aligned} \tag{2.3}$$

where $\mathbf{u}_0 := \mathbf{u}(0)$. In lincond_1, we formally replaced $\partial_t \overline{\boldsymbol{\omega}}(0)$ with the value it would have were $\overline{\boldsymbol{\omega}}$ an actual classical solution to (2.1). Continuing this process inductively on higher derivatives, we define a formal operator $\widetilde{\partial}_t$ (see Definition 3.3 for the details), and define, for all $N \geqslant 2$,

$$\text{lincond}_N: \quad \text{lincond}_{N-1} \ \text{ and } \partial_t^N \mathbf{H}|_{t=0} = \widetilde{\partial}_t^N \boldsymbol{\omega}_0 \text{ on } \Gamma_+. \tag{2.4}$$

We define the space

$$\begin{aligned} \mathring{C}_\sigma^{N+1,\alpha}(Q) := \{\mathbf{u}\colon Q \to \mathbb{R}^3\colon \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}, \partial_t^j \partial_{\mathbf{x}}^\gamma \mathbf{u} \in C^\alpha(Q), \\ j + |\gamma| \leqslant N+1, j \leqslant N\}, \end{aligned} \tag{2.5}$$

endowed with the natural norm induced by the regularity of its elements. That is, $\mathring{C}_\sigma^{N+1,\alpha}(Q)$ is defined as $C_\sigma^{N+1,\alpha}(Q)$, but we require one fewer derivative in time.

Theorem 2.2 ([9]). *Assume that the data has regularity N for some $N \geqslant 0$ and that*

- $\mathbf{g} := \operatorname{curl} \mathbf{f}$,
- $\mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$,
- $\mathbf{H} \in C^{\max\{N,1\},\alpha}([0,T] \times \Gamma_+)$,
- lincond_N *holds,*
- $\boldsymbol{\omega}_0$ *is in the range of the curl,*
- *(2.2) is satisfied on* $(0,T) \times \Gamma_+$.

There exists a solution $\overline{\boldsymbol{\omega}}$ to (2.1) in $C^{N,\alpha}(Q)$, such that $\overline{\boldsymbol{\omega}}(t)$ is in the range of the curl for all $t \in [0,T]$. When $N \geqslant 1$, the solution is classical Eulerian and unique. When $N = 0$, the solution is Lagrangian and is also the unique weak Eulerian solution as in Definition 2.1 for which $\overline{\boldsymbol{\omega}}(t)$ is in the range of the curl for all $t \in [0,T]$.

Moreover, there exists a unique $\mathbf{v} \in C_\sigma^{N+1,\alpha}(Q)$ with $\operatorname{curl} \mathbf{v} = \overline{\boldsymbol{\omega}}$ and $\mathbf{v}(0) = \mathbf{u}_0$, and a mean-zero pressure field π with $\nabla \pi \in C^{N,\alpha}(Q)$ satisfying

$$\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{u} \cdot (\nabla \mathbf{v})^T + \nabla \pi = \mathbf{f}. \tag{2.6}$$

Recalling (1.8), the harmonic component $\mathbf{v}_c$ of $\mathbf{v}$ is given explicitly as

$$\mathbf{v}_c(t) := P_{H_c} \mathbf{u}(0) + \int_0^t P_{H_c} \mathbf{f}(s)\, ds - \int_0^t P_{H_c} P_H \left(\boldsymbol{\Omega}(s) \mathbf{u}(s)\right) ds, \tag{2.7}$$

where the antisymmetric matrix $\boldsymbol{\Omega} := \nabla K[\overline{\boldsymbol{\omega}}] - (\nabla K[\overline{\boldsymbol{\omega}}])^T$. Here, K is the Biot-Savart operator, as in Section 6.

Remark 2.3. *As applied to the solution of the linearized problem given by Theorem 2.2, the condition in (2.2) is a condition on the data, not on the solution, since $\mathbf{u}$ is given. Applied to the fully nonlinear problem, however, the appearance of $\mathbf{u}^{\boldsymbol{\tau}}$ in (2.2) makes (2.2) a condition on the solution. Eliminating the term involving $\mathbf{u}^{\boldsymbol{\tau}}$ by requiring that the normal component of the vorticity on inflow vanish gives (1.11), which is a condition on the data, $\mathbf{u}_0$, $\mathbf{f}$, $\mathbf{U}$, alone at time zero.*

In what follows, we will use $\overline{\boldsymbol{\omega}}$ as a Lagrangian solution, but we will need to estimate $\mathbf{v}$, which is obtained from the Eulerian solution. Hence, it is crucial that Eulerian and Lagrangian solutions agree.

3. Compatibility conditions: linear and nonlinear

For the linear problem (2.1), $\mathbf{H}$ is a *given* value of the vorticity on the inflow boundary. For the nonlinear problem (1.5) that we wish to solve, however, $\mathbf{H}$ at the inflow boundary must be obtained from the flow itself. We start with a formula for $\mathbf{H}$ that holds if $(\mathbf{u}, p)$ is a classical solution to (1.5).

Proposition 3.1. *Assume that* $(\mathbf{u}, p)$ *satisfies* $(1.5)_1$ *in a classical sense and let* $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u}$*. Then on* $[0,T] \times \Gamma$*,*

$$u^{\boldsymbol{n}} \boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \mathbf{f} \right]^\perp + \left(\operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \right) \mathbf{u}^{\boldsymbol{\tau}}, \qquad \omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}.$$

Here, ∇_Γ *is the tangential derivative, and* $\operatorname{curl}_\Gamma$ *is the curl operator on the boundary. (See Appendix B.)*

Proof. As on p. 155 of [2], we start with the Gromeka-Lamb form of the Euler equations,

$$\partial_t \mathbf{u} + \nabla \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) - \mathbf{u} \times \boldsymbol{\omega} - \mathbf{f} = 0. \tag{3.1}$$

The equivalence of (3.1) and $(1.5)_1$ follows from the identity,

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\mathbf{u} \times \boldsymbol{\omega} + \frac{1}{2} \nabla |\mathbf{u}|^2, \tag{3.2}$$

which holds as long as $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$.

From Lemma B.2

$$[\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}} [\boldsymbol{\omega}^{\boldsymbol{\tau}}]^\perp - \omega^{\boldsymbol{n}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp,$$

so restricting (3.1) to $[0,T] \times \Gamma_+$, we have

$$\partial_t \mathbf{u}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) - u^{\boldsymbol{n}} [\boldsymbol{\omega}^{\boldsymbol{\tau}}]^\perp + \omega^{\boldsymbol{n}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp - \mathbf{f}^{\boldsymbol{\tau}} = 0.$$

Hence, since $(\mathbf{v}^\perp)^\perp = -\mathbf{v}$ for any tangent vector $\mathbf{v}$,

$$u^{\boldsymbol{n}} \boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp + \omega^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}}.$$

The proof is completed by observing that $\omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}$ by (B.2). ☐

We see from Proposition 3.1 that for a solution to $(1.5)_{1\text{-}4}$ with $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u}$, we have

$$\boldsymbol{\omega} = \mathbf{W}[\mathbf{u}, p] \text{ on } [0,T] \times \Gamma_+, \tag{3.3}$$

where $\mathbf{W}[\mathbf{u}, p]$ is defined on $[0,T] \times \Gamma_+$ by

$$\begin{aligned} \mathbf{W}^{\boldsymbol{\tau}}[\mathbf{u}, p] &:= \frac{1}{U^{\boldsymbol{n}}} \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp + \frac{1}{U^{\boldsymbol{n}}} \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \\ W^{\boldsymbol{n}}[\mathbf{u}, p] &:= \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}. \end{aligned} \tag{3.4}$$

Now let $\mathbf{u}$ be any element of $\mathring{C}_\sigma^{N+1,\alpha}(Q)$, not necessarily a solution of (1.5). We seek to define a function $\mathbf{H}$ in $C^{N,\alpha}([0,T] \times \Gamma_+)$ as a modification of the expression for $\mathbf{W}[\mathbf{u}, p]$ in such a way that when the data has regularity N, at least the following two properties hold:

(P1) $\mathbf{H}$ at time zero can be defined in terms of the initial data and $\mathbf{U}$ only.

(P2) If $(\mathbf{u}, p)$ solves $(1.5)_{1\text{-}4}$ and $\mathbf{H} = \mathbf{W}[\mathbf{u}, p]$ on $[0,T] \times \Gamma_+$ then $(\mathbf{u}, p)$ satisfies $(1.5)_5$ as well—and so solves (1.5).

We define the function $\mathbf{H}$ for all $N \geqslant 0$ as done in [2] for $N = 0$. First obtain q from $\mathbf{u}$ via

$$\begin{cases} \Delta q = -\operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u}) & \text{in } \overline{Q}, \\ \nabla q \cdot \boldsymbol{n} = -\partial_t U^{\boldsymbol{n}} - N[\mathbf{u}] & \text{on } [0,T] \times \Gamma, \end{cases} \tag{3.5}$$

where

$$N[\mathbf{u}] := \begin{cases} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} & \text{on } [0,T]\times(\Gamma_-\cup\Gamma_0), \\ (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} + \operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}})) & \text{on } [0,T]\times\Gamma_+. \end{cases} \tag{3.6}$$

We explore the properties of $N[\mathbf{u}]$ in Section 8, but it is clear from its definition that if $(\mathbf{u}, p)$ solves (1.5) then $N[\mathbf{u}] = (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$ on $[0,T]\times\Gamma$, so that $\nabla q = \nabla p$ on $\overline{Q}$.

Finally, define $\mathbf{H}$ on $[0,T]\times\Gamma_+$ by replacing $\mathbf{u}^{\boldsymbol{\tau}}$ with $\mathbf{U}^{\boldsymbol{\tau}}$ in all terms in the expression for $\mathbf{W}[\mathbf{u}, p]$ having a derivative on $\mathbf{u}^{\boldsymbol{\tau}}$. This gives

$$\begin{aligned} \mathbf{H}^{\boldsymbol{\tau}} &:= \frac{1}{U^{\boldsymbol{n}}}\left[-\partial_t\mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(q + \frac{1}{2}|\mathbf{U}|^2\right) + \mathbf{f}^{\boldsymbol{\tau}}\right]^\perp + \frac{1}{U^{\boldsymbol{n}}}\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}}\mathbf{u}^{\boldsymbol{\tau}}, \\ H^{\boldsymbol{n}} &:= \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}}, \end{aligned} \tag{3.7}$$

and we see that property $(P1)$ of $\mathbf{H}$ holds. We show property (P2) in Proposition 4.7.

Remark 3.2. *Because we assumed that $\mathbf{U}$ has higher regularity than $\mathbf{u}$, the function $\mathbf{H}$ has one more derivative than $\mathbf{W}[\mathbf{u}, p]$ in (3.4). This higher regularity will persist in the limiting solution, where $\mathbf{H}$ equals $\mathbf{W}[\mathbf{u}, p]$. Such higher regularity is needed to solve the linearized problem in Theorem 2.2 only for $N = 0$, but we will see later that it is also needed to handle the pressure estimates for all $N \geqslant 0$: see Remark 9.4.*

Now, if $(\mathbf{u}, p)$ solves $(1.5)_{1\text{-}4}$ and $\boldsymbol{\omega} := \operatorname{curl}\mathbf{u}$, then, of course,

$$\begin{aligned} \partial_t\boldsymbol{\omega}(0) &= \boldsymbol{\omega}_0\cdot\nabla\mathbf{u}_0 - \mathbf{u}_0\cdot\nabla\boldsymbol{\omega}_0 + \mathbf{g}, \\ \partial_t\mathbf{u}(0) &= -\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \nabla q_0 + \mathbf{f}, \end{aligned} \tag{3.8}$$

where $\mathbf{g} := \operatorname{curl}\mathbf{f}$. This simple observation is behind both cond_N and lincond_N, which are based upon applying ∂_t, $N-1$ times, each time replacing $\partial_t\mathbf{u}$ or $\partial_t\boldsymbol{\omega}$ with the relation in (3.8), thereby replacing all time derivatives with spatial derivatives. The resulting relation would be an identity for any actual solution to the Euler equations, and cond_N consists of assuming that the identity holds at time zero. We now describe this process precisely.

Definition 3.3. *Let $N \geqslant 0$ and assume that the data has regularity N as in Definition 1.1, and let $\mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$ with $\mathbf{u}(0) = \mathbf{u}_0$. Because the forcing and $\mathbf{U}$ are independent of the solution, we simply define $\widetilde{\partial}_t^n\mathbf{f} := \partial_t^n\mathbf{f}$, $\widetilde{\partial}_t^n\mathbf{g} := \partial_t^n\mathbf{g}$, and $\widetilde{\partial}_t^n\mathbf{U} = \partial_t^n\mathbf{U}$, where we recall that $\mathbf{g} := \operatorname{curl}\mathbf{f}$. In accord with (3.8), we define*

$$\widetilde{\partial}_t\mathbf{u} := -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla q + \mathbf{f}, \quad \widetilde{\partial}_t\boldsymbol{\omega} := -\mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \mathbf{g},$$

where q satisfies (3.5).

We then define

$$\begin{aligned} \widetilde{\partial}_t^2\mathbf{u} &:= -\widetilde{\partial}_t(\mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\widetilde{\partial}_t q + \partial_t\mathbf{f}, \\ \widetilde{\partial}_t^2\boldsymbol{\omega} &:= -\widetilde{\partial}_t\mathbf{u}\cdot\nabla\boldsymbol{\omega} - \mathbf{u}\cdot\nabla\widetilde{\partial}_t\boldsymbol{\omega} + \widetilde{\partial}_t\boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\cdot\nabla\widetilde{\partial}_t\mathbf{u} + \partial_t\mathbf{g}, \end{aligned} \tag{3.9}$$

where

$$\widetilde{\partial}_t(\mathbf{u}\cdot\nabla\mathbf{u}) := \widetilde{\partial}_t\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\widetilde{\partial}_t\mathbf{u},$$

and define $\widetilde{\partial}_t q$ to be the unique mean-zero solution to (see Remark 3.4, below)

$$\begin{cases} \Delta\widetilde{\partial}_t q = -\operatorname{div}\widetilde{\partial}_t(\mathbf{u}\cdot\nabla\mathbf{u}) & \text{in } \overline{Q}, \\ \nabla\widetilde{\partial}_t\cdot\boldsymbol{n} = -\partial_t U^{\boldsymbol{n}} - \widetilde{\partial}_t N[\mathbf{u}] & \text{on } [0,T]\times\Gamma, \end{cases}$$

with

$$\widetilde{\partial}_t N[\mathbf{u}] := \begin{cases} \widetilde{\partial}_t(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} & \text{on } [0,T]\times(\Gamma_-\cup\Gamma_0), \\ \widetilde{\partial}_t(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} + \operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\widetilde{\partial}_t\mathbf{u}^{\boldsymbol{\tau}} - \partial_t\mathbf{U}^{\boldsymbol{\tau}})) & \text{on } [0,T]\times\Gamma_+. \end{cases}$$

We note, then, that

$$\widetilde{\partial}_t^2\mathbf{u} = -(-\mathbf{u}\cdot\nabla\mathbf{u} - \nabla q + \mathbf{f})\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla(-\mathbf{u}\cdot\nabla\mathbf{u} - \nabla q + \mathbf{f}) - \nabla\widetilde{\partial}_t q + \partial_t\mathbf{f}.$$

For $\widetilde{\partial}_t^n$, we repeat this process inductively, up to order $N+1$ for $\widetilde{\partial}_t\mathbf{u}$ and order N for $\widetilde{\partial}_t\boldsymbol{\omega}$.

Remark 3.4. *In the inductive extension of $\widetilde{\partial}_t^n q$ in Definition 3.3, we can see that $\widetilde{\partial}_t^n q$ is the unique mean-zero solution to*

$$\begin{cases} \Delta\widetilde{\partial}_t^n q = -\operatorname{div}\widetilde{\partial}_t^n(\mathbf{u}\cdot\nabla\mathbf{u}) & \text{in } \overline{Q}, \\ \nabla\widetilde{\partial}_t^n q\cdot\boldsymbol{n} = -\partial_t^n U^{\boldsymbol{n}} - \widetilde{\partial}_t^n N[\mathbf{u}] & \text{on } [0,T]\times\Gamma, \end{cases} \tag{3.10}$$

where

$$\widetilde{\partial}_t^n N[\mathbf{u}] := \begin{cases} \widetilde{\partial}_t^n(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} & \text{on } [0,T]\times(\Gamma_-\cup\Gamma_0), \\ \widetilde{\partial}_t^n(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} + \operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\widetilde{\partial}_t^n\mathbf{u}^{\boldsymbol{\tau}} - \partial_t^n\mathbf{U}^{\boldsymbol{\tau}})) & \text{on } [0,T]\times\Gamma_+. \end{cases}$$

Then

$$\int_\Gamma \left[\partial_t^n U^{\boldsymbol{n}} + \widetilde{\partial}_t^n N[\mathbf{u}]\right] = \int_\Omega \operatorname{div}\widetilde{\partial}_t^n(\mathbf{u}\cdot\nabla\mathbf{u}),$$

since $\operatorname{div}\mathbf{U} = 0$ and $\operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\widetilde{\partial}_t^n\mathbf{u}^{\boldsymbol{\tau}} - \partial_t^n\mathbf{U}^{\boldsymbol{\tau}}))$ integrates to zero over each boundary component. Hence, (3.10) is solvable.

In Definition 3.3, $\widetilde{\partial}_t^n$ does not represent a derivative. Rather, it is a shorthand notation to properly account for the combinatorial nature of lincond_N and cond_N. From the manner in which $\widetilde{\partial}_t q$ was defined, we have

$$\text{cond}_{n-1} \implies \widetilde{\partial}_t^n\mathbf{u}_0\cdot\boldsymbol{n} = \partial_t^n U^{\boldsymbol{n}}(0) \text{ on } \Gamma. \tag{3.11}$$

Moreover, if $(\mathbf{u}, p)$ is a solution to (1.5) with $(\mathbf{u}, \nabla p) \in C_\sigma^{N+1,\alpha}(Q)\times C^{N,\alpha}(Q)$ then $\widetilde{\partial}_t^n\mathbf{u} = \partial_t^n\mathbf{u}$ on $\overline{Q}$ for all $n \leqslant N+1$, $\widetilde{\partial}_t^n\boldsymbol{\omega} = \partial_t^n\boldsymbol{\omega}$ on $\overline{Q}$ for all $n \leqslant N$, and $\widetilde{\partial}_t^n N[\mathbf{u}] = \partial_t^n(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$ on $[0,T]\times\Gamma$ for all $n \leqslant N$.

Unlike actual time derivatives, we cannot write $\widetilde{\partial}_t(\widetilde{\partial}_t\mathbf{u}) = \widetilde{\partial}_t^2\mathbf{u}$, for we have not even defined how $\widetilde{\partial}_t$ would act on $\widetilde{\partial}_t\mathbf{u}$. But the following simple proposition, which will be useful for treating cond_N for $N \geqslant 1$, follows immediately from the definition of $\widetilde{\partial}_t^n$ in Definition 3.3:

Proposition 3.5. *Let $\mathbf{u}$ be as in Definition 3.3 for some $N \geqslant 1$ and let $t \in [0,T]$. If $\partial_t^n\mathbf{u} = \widetilde{\partial}_t^n\mathbf{u}$ on $\{t\}\times\overline{\Omega}$ for all $0 \leqslant n \leqslant N$, then*

$$\partial_t\widetilde{\partial}_t^n\mathbf{u} = \widetilde{\partial}_t^{n+1}\mathbf{u} \text{ on } \{t\}\times\overline{\Omega} \text{ for } 1 \leqslant n \leqslant N.$$

Proposition 3.6 shows that, formally, $\widetilde{\partial}_t^n \operatorname{curl}\mathbf{u} = \operatorname{curl}\widetilde{\partial}_t^n\mathbf{u}$.

Proposition 3.6. *Let $\mathbf{u}$ be as in Definition 3.3. Then $\operatorname{div}\widetilde{\partial}_t^n\mathbf{u} = 0$ for all $0 \leqslant n \leqslant N$, and*

$$\widetilde{\partial}_t^n\boldsymbol{\omega} = \operatorname{curl}\widetilde{\partial}_t^n\mathbf{u} \text{ for all } 0 \leqslant n \leqslant N-1. \tag{3.12}$$

For all $0 \leqslant n \leqslant N-1$, $\widetilde{\partial}_t^n\boldsymbol{\omega}$ is in the range of the curl. Finally, if $\mathbf{u} \in C_\sigma^{N+1,\alpha}(Q)$ then $\operatorname{div}\widetilde{\partial}_t^{N+1}\mathbf{u} = 0$, (3.12) also holds for $n = N$, and $\widetilde{\partial}_t^N\boldsymbol{\omega}$ is in the range of the curl.

Proof. We constructed the pressure q in Definition 3.3 so that $\operatorname{div} \widetilde{\partial}_t^n \mathbf{u} = 0$. Then, for $n = 1$, (3.12) follows from the identity, $\operatorname{curl}(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla q) = \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u}$.

For $n = 2$, we will use the identity,

$$\operatorname{curl}(\mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot (\nabla \mathbf{u})^T) = \mathbf{u} \cdot \nabla \operatorname{curl} \mathbf{v} - \operatorname{curl} \mathbf{u} \cdot \nabla \mathbf{v},$$

valid for any $\mathbf{u}, \mathbf{v} \in C^2(\Omega)$ with $\operatorname{div} \mathbf{u} = 0$, which we prove in Lemma 3.10. We will also use,

$$\widetilde{\partial}_t \mathbf{u} \cdot \nabla (\operatorname{curl} \mathbf{u})^T + \operatorname{curl} \mathbf{u} \cdot (\nabla \widetilde{\partial}_t \mathbf{u})^T = \nabla (\widetilde{\partial}_t \mathbf{u} \cdot \operatorname{curl} \mathbf{u}).$$

Since $\operatorname{curl} \nabla = 0$, we know that the curl of the left-hand side is zero. From (3.9) and (3.12) for $n = 1$, we can write

$$\begin{aligned}
\widetilde{\partial}_t^2 \boldsymbol{\omega} &= -\widetilde{\partial}_t \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \mathbf{u} \cdot \nabla \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}) + \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}) \cdot \nabla \mathbf{u} + \boldsymbol{\omega} \cdot \nabla \widetilde{\partial}_t \mathbf{u} + \mathbf{g} \\
&= \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}) \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}) + \boldsymbol{\omega} \cdot \nabla \widetilde{\partial}_t \mathbf{u} - \widetilde{\partial}_t \mathbf{u} \cdot \nabla \boldsymbol{\omega} + \mathbf{g} \\
&= -\operatorname{curl}(\widetilde{\partial}_t \mathbf{u} \cdot \nabla \mathbf{u} + \widetilde{\partial}_t \mathbf{u} \cdot (\nabla \mathbf{u})^T) - \operatorname{curl}(\mathbf{u} \cdot \nabla \widetilde{\partial}_t \mathbf{u} + \mathbf{u} \cdot (\nabla \widetilde{\partial}_t \mathbf{u})^T) + \mathbf{g} \\
&= \operatorname{curl}(-\widetilde{\partial}_t \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \widetilde{\partial}_t \mathbf{u} - \nabla \widetilde{\partial}_t q + \partial_t \mathbf{f}) = \operatorname{curl} \widetilde{\partial}_t^2 \mathbf{u}.
\end{aligned}$$

Equality in (3.12) follows inductively for higher values of n.

It follows directly from (3.12) that $\widetilde{\partial}_t^n \boldsymbol{\omega}$ is in the range of the curl for all $0 \leqslant n \leqslant N - 1$. Finally, if $\mathbf{u} \in C_\sigma^{N+1,\alpha}(Q)$—as opposed to $\mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$, as in Definition 3.3—then $\mathbf{u}$ and $\boldsymbol{\omega}$ have one more time derivative, giving that $\operatorname{div} \widetilde{\partial}_t^{N+1} \mathbf{u} = 0$, (3.12) also holds for $n = N$, and $\widetilde{\partial}_t^N \boldsymbol{\omega}$ is in the range of the curl. □

Since $\mathbf{u}$ is given in the linearized problem, $\operatorname{lincond}_N$ is a condition on the data. For the nonlinear problem, a different condition is needed to avoid the appearance of $\partial_t^N \mathbf{u}^{\boldsymbol{\tau}}|_{t=0}$ in the expression for $\partial_t^N \mathbf{H}^{\boldsymbol{\tau}}|_{t=0}$. We begin the exploration of this issue by examining closely the $N = 0$ case.

Using Lemma B.2 along with $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = H^{\boldsymbol{n}}$, on $[0, T] \times \Gamma_+$, we have

$$-[[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}}]^\perp = U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}} - H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(q + \frac{1}{2} |\mathbf{U}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp,$$

so,

$$\begin{aligned}
[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} &= \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(q + \frac{1}{2} |\mathbf{U}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} \\
&= \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(q + \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right).
\end{aligned}$$

Then using the vector identity in (3.2)

$$\nabla_\Gamma \left(q + \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} = [\mathbf{u} \cdot \nabla \mathbf{u} + \nabla q - \mathbf{f}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} = -\widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + [\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}}. \tag{3.13}$$

Hence,

$$[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} = \partial_t \mathbf{U}^{\boldsymbol{\tau}} - \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) + [\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}}. \tag{3.14}$$

Note that (3.14) holds on all of $[0, T] \times \Gamma_+$ for any $\mathbf{u} \in \mathring{C}_\sigma^{1,\alpha}(Q)$ when the data has regularity 0, without assuming any compatibility conditions.

Proposition 3.7. *Assume the data has regularity* 0, $\mathbf{u} \in \mathring{C}_\sigma^{1,\alpha}(Q)$, *and* cond_0 *in (1.9) holds. Then* $\operatorname{lincond}_0$ *in (2.4) holds.*

Proof. **All the calculations in this proof apply at time zero on Γ_+.**

We have $\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} = 0$ by cond$_0$. Since also $\mathbf{u}(0) = \mathbf{U}(0)$ on Γ_+, we know that $\nabla_\Gamma |\mathbf{U}|^2 = \nabla_\Gamma |\mathbf{u}|^2$, and (3.14) reduces to $[\mathbf{U} \times \mathbf{H}]^{\boldsymbol{\tau}} = [\mathbf{U} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}}$, or,

$$[\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})]^{\boldsymbol{\tau}} = 0.$$

Also from (3.7)$_2$, $H^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} = \omega^{\boldsymbol{n}}$. Then, since $H^{\boldsymbol{n}} = \omega^{\boldsymbol{n}}$ and only $(\mathbf{H} - \boldsymbol{\omega})^{\boldsymbol{\tau}}$ contributes to $\boldsymbol{n} \times (\mathbf{H} - \boldsymbol{\omega})$, we can apply the vector identity, $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ to give

$$\begin{aligned} 0 &= \boldsymbol{n} \times [\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})]^{\boldsymbol{\tau}} = \boldsymbol{n} \times [\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})] \\ &= [\boldsymbol{n} \cdot (\mathbf{H} - \boldsymbol{\omega})]\mathbf{U} - [\boldsymbol{n} \cdot \mathbf{U}](\mathbf{H} - \boldsymbol{\omega}) = -U^{\boldsymbol{n}}(\mathbf{H} - \boldsymbol{\omega}). \end{aligned}$$

Since $U^{\boldsymbol{n}}$ never vanishes on Γ_+, we conclude that $\mathbf{H} = \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, meaning that lincond$_0$ is satisfied. □

The next proposition shows that our choice of $\mathbf{H}$ does, in fact, satisfy the constraint in (2.2), necessary to ensure that $\operatorname{curl} \mathbf{u} = \boldsymbol{\omega}$.

Proposition 3.8. *Assume that the data has regularity* 0 *as in Definition 1.1. For* $\mathbf{u} \in \mathring{C}^{1,\alpha}_\sigma(Q)$, *the condition in (2.2) is satisfied on* $(0, T] \times \Gamma_+$.

Proof. From (3.7) and using that $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = H^{\boldsymbol{n}}$ we have

$$U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}} - H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(q + \frac{1}{2} |\mathbf{U}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp.$$

By (B.2), $\operatorname{div}_\Gamma \mathbf{v} = -\operatorname{div}_\Gamma (\mathbf{v}^\perp)^\perp = \operatorname{curl}_\Gamma \mathbf{v}^\perp$ for any tangent vector $\mathbf{v}$. Hence,

$$\begin{aligned} \partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} &= \partial_t H^{\boldsymbol{n}} + \operatorname{curl}_\Gamma [(H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}})^\perp] - \mathbf{g} \cdot \boldsymbol{n} \\ &= \partial_t \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} + \mathbf{g} \cdot \boldsymbol{n} - \mathbf{g} \cdot \boldsymbol{n} = 0, \end{aligned}$$

where $\operatorname{curl}_\Gamma \mathbf{f}^{\boldsymbol{\tau}} = (\operatorname{curl} \mathbf{f}) \cdot \boldsymbol{n} = \mathbf{g} \cdot \boldsymbol{n}$ by (B.2). This gives (2.2). □

Next, let us consider what happens if we try to extend Proposition 3.7 to cond$_N$ for $N = 1$. Returning to (3.14), suppose that $\mathbf{u} \in \mathring{C}^{1+1,\alpha}_\sigma(Q)$. Differentiating both sides in time gives

$$\begin{aligned} [\partial_t \mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \partial_t \mathbf{H}]^{\boldsymbol{\tau}} &= \partial_{tt} \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \partial_t \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) \\ &\quad + [\partial_t \mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \partial_t \boldsymbol{\omega}]^{\boldsymbol{\tau}} \end{aligned} \tag{3.15}$$

on $[0, T] \times \Gamma_+$. We know from the $N = 0$ result that if cond$_0$ holds then $\mathbf{H} = \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, so two terms above cancel, leaving

$$[\mathbf{u} \times \partial_t \mathbf{H}]^{\boldsymbol{\tau}} = \left[\partial_{tt} \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \partial_t \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) \right] + [\mathbf{u} \times \partial_t \boldsymbol{\omega}]^{\boldsymbol{\tau}} \text{ on } \{0\} \times \Gamma_+. \tag{3.16}$$

If we could satisfy the hypotheses of Proposition 3.5 then we would also have that $\partial_t \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} = \widetilde{\partial}_t^2 \mathbf{u}^{\boldsymbol{\tau}}$. Assuming additionally cond$_1$, the term in brackets would vanish. If, finally, we could replace $\partial_t \boldsymbol{\omega}$ in this expression with $\widetilde{\partial}_t \boldsymbol{\omega}$ then, arguing just as in the proof of Proposition 3.7, it would follow that cond$_1$ implies lincond$_1$. But with our definition of $\mathbf{H}$, we cannot make this replacement. In order to extend Proposition 3.7 to cond$_N$ for $N = 1$, we need to make one further assumption, leading to the following proposition for all $N \geqslant 1$ (and see Remark 4.1):

Proposition 3.9. *Assume that the data has regularity N as in Definition 1.1 for some $N \geqslant 1$ with $\mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$. Suppose that* cond_N *in (1.9) holds and that also $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$ on $\overline{\Omega}$ for all $1 \leqslant n \leqslant N$. Then* $\operatorname{lincond}_N$ *in (2.4) holds.*

Proof. Let $N = 1$. From Proposition 3.7, we know that $\operatorname{lincond}_0$ holds. From Proposition 3.6, $\widetilde{\partial}_t \boldsymbol{\omega}_0 = \operatorname{curl} \widetilde{\partial}_t \mathbf{u}_0 = \operatorname{curl} \partial_t \mathbf{u}(0) = \partial_t \operatorname{curl} \mathbf{u}(0) = \partial_t \boldsymbol{\omega}(0)$, and from Proposition 3.5 we know that $\partial_t \widetilde{\partial}_t \mathbf{u} = \widetilde{\partial}_t^2 \mathbf{u}$ at time zero. Thus, the term in the brackets in (3.16) vanishes because of cond_1, and we are left with

$$[\mathbf{u} \times \partial_t \mathbf{H}]^{\boldsymbol{\tau}} = [\mathbf{u} \times \partial_t \boldsymbol{\omega}]^{\boldsymbol{\tau}} \text{ on } \{0\} \times \Gamma_+.$$

As in the proof of Proposition 3.7, this gives that $\partial_t \mathbf{H} = \partial_t \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, and hence that $\partial_t \mathbf{H} = \widetilde{\partial}_t \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, which is $\operatorname{lincond}_1$.

The result for $N \geqslant 2$ follows inductively. □

We used the following lemma in the proof of Proposition 3.6:

Lemma 3.10. *For any $\mathbf{u}, \mathbf{v} \in C^2(\Omega)^3$ with $\operatorname{div} \mathbf{u} = 0$,*

$$\operatorname{curl}(\mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot (\nabla \mathbf{u})^T) = \mathbf{u} \cdot \nabla \operatorname{curl} \mathbf{v} - \operatorname{curl} \mathbf{v} \cdot \nabla \mathbf{u}.$$

Proof. Follows from a direct calculation. □

Generating Compatible Initial Data. We can construct examples of initial data satisfying the compatibility conditions as follows: choose any $\mathbf{u}_0$ and $\mathbf{f}$ having sufficient regularity, obtain q_0 from $\mathbf{u}_0$ via (3.5), then choose $\mathbf{U}^{\boldsymbol{\tau}}(0)$ so that on Γ_+ we have $\mathbf{U}(0) = \mathbf{u}_0$ and the values of $\partial_t \mathbf{U}(0), \dots, \partial_t^{N+1} \mathbf{U}(0)$ are chosen in accordance with the compatibility condition. See also Appendix C.

4. Proof of well-posedness with inflow, outflow

In this section, we give the proof of Theorem 1.2. We prepare for the proof by defining an operator A whose fixed point will be a solution to (1.5), and then define a subspace of $\mathring{C}_\sigma^{N+1,\alpha}(Q)$ in which the fixed point will lie. We then present the three key propositions on which the proof of Theorem 1.2 relies, before finally giving the proof itself.

The operator A. Fixing $\mathbf{u}_0 \in C_\sigma^{N+1,\alpha}(\Omega)$ satisfying cond_N, we define

$$\operatorname{Dom}_N(A) := \{\mathbf{u} \in C_\sigma^{N+1,\alpha}(Q) \colon \mathbf{u}(0) = \mathbf{u}_0,\ \partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0 \text{ on } \overline{\Omega},\ 0 \leqslant n \leqslant N\}, \tag{4.1}$$

where $\widetilde{\partial}_t^n$ is as in Definition 3.3. We will show in Lemma 6.4 that $\operatorname{Dom}_N(A)$, which will serve as the domain of the operator A, is a nonempty, convex subset of $\mathring{C}_\sigma^{N+1,\alpha}(Q)$.

Remark 4.1. *The condition in* $\operatorname{Dom}_N(A)$ *that $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$ on $\overline{\Omega}$ for all $1 \leqslant n \leqslant N$ arises from Proposition 3.9. For $N = 0$, $\mathbf{H}(0)$ depends only upon the data and only $\mathbf{H}(0)$ appears in* $\operatorname{lincond}_0$*, so there is no need to restrict the domain of A beyond $C_\sigma^{1,\alpha}(Q)$. For $N \geqslant 1$, we must impose $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$ on $\overline{\Omega}$ for all $1 \leqslant n \leqslant N$ as an additional condition and show that the resulting domain is, in fact, nonempty, as we do in Lemma 6.4.*

To define A, let $\mathbf{u} \in \operatorname{Dom}_N(A)$ and define $\mathbf{H}$ as in (3.7). We know from Proposition 3.9 that $\operatorname{lincond}_N$ is satisfied for any $\mathbf{u} \in \operatorname{Dom}_N(A)$, so we can obtain from Theorem 2.2 the unique solution $\overline{\boldsymbol{\omega}} \in C^{N,\alpha}(Q)$ to (2.1) with $\overline{\boldsymbol{\omega}}_0 = \boldsymbol{\omega}_0 = \operatorname{curl} \mathbf{u}_0$. Proposition 3.8 shows that (2.2) is satisfied, so by Theorem 2.2, $\overline{\boldsymbol{\omega}}$ is in the range of the curl and there exists a unique velocity field $\mathbf{v} \in C_\sigma^{N+1,\alpha}(Q)$ and pressure π with $\operatorname{curl} \mathbf{v} = \overline{\boldsymbol{\omega}}$ satisfying $\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{u} \cdot (\nabla \mathbf{v})^T + \nabla \pi = \mathbf{f}$. Finally, we set

$$A\mathbf{u} := \mathbf{v}, \tag{4.2}$$

and define also

$$\Lambda \mathbf{u} := \operatorname{curl} A\mathbf{u} = \overline{\boldsymbol{\omega}}. \tag{4.3}$$

Proposition 4.2. *A maps $\operatorname{Dom}_N(A)$ to itself.*

Proof. Let $\mathbf{u} \in \operatorname{Dom}_N(A)$ and let $\mathbf{v} = A\mathbf{u}$. Theorem 2.2 shows that $\mathbf{v} \in C_\sigma^{N+1,\alpha}(Q)$ and $\mathbf{v}(0) = \mathbf{u}_0$, so it remains only to show that $\partial_t^n \mathbf{v}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$ for $1 \leqslant n \leqslant N$.

Suppose $N = 1$. Then since $\mathbf{v}(0) = \mathbf{u}(0)$, (2.6) gives

$$\partial_t \mathbf{v}(0) = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T - \nabla \pi(0) + \mathbf{f}(0).$$

But $\mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T = (1/2) \nabla |\mathbf{u}_0|^2$, so we have

$$\partial_t \mathbf{v}(0) = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla r + \mathbf{f}(0)$$

for some "pressure" r. But r is recovered in the same manner as p, which is the same as q at time zero. We see, then, that $\partial_t \mathbf{v}(0) = \widetilde{\partial}_t \mathbf{u}_0$.

The result for $N > 1$ follows inductively. ∎

We will apply Schauder's fixed point theorem to obtain the existence of a fixed point of A, but this requires that A be continuous. Estimates on A in [9] would give that A is bounded as a map from $\operatorname{Dom}_N(A)$ to $\operatorname{Dom}_N(A)$ in the $\mathring{C}_\sigma^{N+1,\alpha}(Q)$ norm, as long as we can obtain sufficient control of the pressure so as to control $\mathbf{H}$. But A, which is nonlinear, need not be continuous in the $\mathring{C}_\sigma^{N+1,\alpha}(Q)$ norm. To ensure continuity, we need to work in some new spaces, which we introduce next.

Definition 4.3. *For a fixed $\beta_1, \beta_2 \in (0,1)$ and any integer $N \geqslant 0$, let*

$$\begin{aligned} X^N_{\beta_1,\beta_2} &:= \{\mathbf{u} \in C^{N,\beta_1}(Q) \colon \operatorname{curl} \mathbf{u} \in C^{N,\beta_2}(Q)\}, \\ \|\mathbf{u}\|_{X^N_{\beta_1,\beta_2}} &:= \|\mathbf{u}\|_{C^{N,\beta_1}(Q)} + \|\operatorname{curl} \mathbf{u}\|_{C^{N,\beta_2}(Q)}. \end{aligned}$$

Remark 4.4. *It will follow from Lemma 6.3 that $X^N_{\alpha,\alpha} = \mathring{C}_\sigma^{N+1,\alpha}(Q)$.*

Fixing $\alpha' \in (\alpha, 1)$, we will show that A is continuous as a map from $X^N_{\beta,\beta} \cap \operatorname{Dom}_N(A)$ to $X^N_{\beta,\beta} \cap \operatorname{Dom}_N(A)$ for any $\beta < \alpha$, and there exists a convex set K lying in $X^N_{\alpha',\alpha} \cap \operatorname{Dom}_N(A)$ that is a compact subset of $X^N_{\alpha',\alpha}$ that is fixed by A. Applying Schauder's fixed point theorem gives the existence of a fixed point. We will show a posteriori that the full inflow, outflow boundary conditions in $(1.5)_{4,5}$ are satisfied.

In constructing solutions, $X^N_{\alpha,\alpha} = \mathring{C}_\sigma^{N+1,\alpha}(Q)$ would seem the most natural. Then, once a solution is obtained, the Euler equations themselves easily yield one more derivative in time, giving a solution in $C_\sigma^{N+1,\alpha}(Q)$. Indeed, this is how it works for the linearized problem, (2.1).

But there are two difficulties for the full problem: We need the extra time regularity of $X^N_{\alpha',\alpha}$ to establish (non-classical) estimates on the pressure, and A is not continuous in $X^N_{\alpha',\alpha}$.

Three key Propositions. We will show that Theorem 1.2 follows from Propositions 4.5 to 4.7. To streamline the presentation, we defer the proofs of these technical lemmas to later sections.

Proposition 4.5. *Assume that the data has regularity $N \geqslant 0$ and $\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$. For any $M > \|\mathbf{u}_0\|_{C^{N+1,\alpha}_\sigma(\Omega)}$ there exists $T > 0$ for which the set*

$$K := \{\mathbf{u} \in X^N_{\alpha',\alpha} \cap \mathrm{Dom}_N(A) \colon \|\mathbf{u}\|_{X^N_{\alpha',\alpha}} \leqslant M\} \tag{4.4}$$

is invariant under A. That is, $\mathbf{u} \in \mathrm{Dom}_N(A)$ with $\|\mathbf{u}\|_{X^N_{\alpha',\alpha}} \leqslant M$ implies that $A\mathbf{u} \in \mathrm{Dom}_N(A)$ with $\|A\mathbf{u}\|_{X^N_{\alpha',\alpha}} \leqslant M$.

Proof. Given in Section 10. □

Proposition 4.6. *For any $\beta \in (0, \alpha)$, $A \colon K \to K$ is continuous in the $X^N_{\beta,\beta}$ norm.*

Proof. Given in Section 11, and follows from Proposition 4.5. □

Proposition 4.7. *Assume that $(\mathbf{u}, p) \in C^{1,\alpha}_\sigma(Q) \times C^\alpha(Q)$ solves $(1.5)_{1\text{-}4}$ and $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u} = \mathbf{H}$ on $[0, T] \times \Gamma_+$, with $\mathbf{H}$ given in (3.7). Then $(1.5)_5$ also holds.*

Proof. Given in Section 12. □

Proof of well-posedness. Theorem 1.2 we now see is a consequence of Propositions 4.6 and 4.7:

Proof of Theorem 1.2. Choose any $\beta \in (0, \alpha)$. Because $C^{N,\alpha}$ is compactly embedded in $C^{N,\beta}$, and also using Lemma 6.4, below, we see that K is a convex compact subset of $X^N_{\beta,\beta}$. By Proposition 4.6, A is continuous as a map from K to K in the $X^N_{\beta,\beta}$ norm, and so has a fixed point $\mathbf{u}$ by Schauder's Fixed Point Theorem. It follows that $A\mathbf{u} = \mathbf{u}$ with $\mathbf{u} \in X_{\alpha',\alpha}$ and hence, in particular, $\mathbf{u} \in C^{N+1,\alpha}_\sigma(Q)$.

Since $\mathbf{v} := A\mathbf{u} = \mathbf{u}$, Theorem 2.2 implies that $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}$ for some pressure p. Hence, $(\mathbf{u}, p)$ is a solution to $(1.5)_{1\text{-}4}$. But since $\mathbf{u} = A\mathbf{u}$, we have $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u} = \mathbf{H}$. Proposition 4.7 thus gives that $(1.5)_5$ holds, so $(\mathbf{u}, p)$ is a solution to (1.5).

To prove uniqueness, let $(\mathbf{u}_1, p_1)$, $(\mathbf{u}_2, p_2)$ be two solutions to (1.5) with the same initial velocity in $C^{1,\alpha}$ (so we prove uniqueness for $N = 0$ and it then follows for all $N \geqslant 0$). Letting $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, subtracting $(1.5)_1$ for $(\mathbf{u}_2, p_2)$ from $(1.5)_1$ for $(\mathbf{u}_1, p_1)$,

$$\partial_t \mathbf{w} + \mathbf{u}_1 \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_2 + \nabla(p_1 - p_2) = 0.$$

Multiplying by $\mathbf{w}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 = -\int_\Omega (\mathbf{w} \cdot \nabla \mathbf{u}_2) \cdot \mathbf{w} - \frac{1}{2}\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2 \leqslant \|\nabla \mathbf{u}_2\|_{L^\infty(Q)} \|\mathbf{w}\|^2 - \frac{1}{2}\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2.$$

But,

$$-\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2 = -\int_\Gamma (\mathbf{u}_1 \cdot \boldsymbol{n}) |\mathbf{w}|^2 = -\int_{\Gamma_-} (\mathbf{u}_1 \cdot \boldsymbol{n}) |\mathbf{w}|^2 \leqslant 0,$$

since $\mathbf{w} = 0$ on Γ_+, $\mathbf{u}_1 \cdot \boldsymbol{n} = 0$ on Γ_0, and $\mathbf{u}_1 \cdot \boldsymbol{n} > 0$ on Γ_-. Hence,

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leqslant 2\|\nabla \mathbf{u}_2\|_{L^\infty(Q)} \|\mathbf{w}\|^2,$$

and we conclude that $\mathbf{w} = 0$ by Grönwall's Lemma, giving the uniqueness in Theorem 1.2. □

When $\Gamma_0 = \Gamma$—that is, when classical impermeable boundary conditions are imposed on the entire boundary—Theorem 1.2 gives well-posedness of the 3D Euler equations in $C^{N,\alpha}(Q)$ for any $N \geqslant 0$. The proof simplifies, as we discuss briefly in Remark 12.1.

Part II: Preliminary Estimates

Organization of Part II. We introduce in Section 5 some conventions that we will use throughout the remainder of this paper to streamline the presentation. We summarize in Sections 6 and 7 some of the results from [9], describe the generation of vorticity on the boundary in Section 8, and obtain critical estimates on the pressure in Section 9.

5. Some conventions

Pressures. We employ three distinct pressure functions:

- p: The "true" pressure recovered by (9.1), appearing in a solution to $(1.5)_{1\text{-}4}$.
- q: The "approximate" pressure recovered by (3.5), used to obtain $\mathbf{H}$ on $[0, T] \times \Gamma_+$.
- π: The "linearized" pressure of (2.6), obtained by recovering the velocity from the vorticity for the linearized Euler equations.

The true and approximate pressures, p and q, are key, with the majority of our estimates involving q.

Constants. To simplify notation, we write M as a universal but unspecified bound on $\|\mathbf{u}\|_{X^N_{\alpha',\alpha}}$. Thus, we assume that

$$\|\mathbf{u}\|_{X^N_{\alpha',\alpha}} \leqslant M \text{ for some } M \geqslant 1 \tag{5.1}$$

in what follows. (Having $M \geqslant 1$ simplifies the form of some estimates.)

Definition 5.1. *We define the following two types of positive "constant":*

$$\begin{aligned} c_0 &= c_0(\|\mathbf{u}_0\|_{C^{N+1,\alpha}_\sigma(\Omega)}, U^{-1}_{min}, \|\mathbf{U}\|_{C^{N+2,\alpha}_\sigma(Q)}, \|\operatorname{curl} \mathbf{f}\|_{C^{N,\alpha}(\Omega)}), \\ c_X &= c_X(c_0, M), \end{aligned}$$

where

$$U_{min} := \min\{|U^{\boldsymbol{n}}(t, \mathbf{x})| \colon (t, \mathbf{x}) \in [0, T] \times \Gamma_+\}. \tag{5.2}$$

Both c_0 and c_X are continuous, increasing functions of each of their arguments. Each appearance of c_0 and c_X may have different values, even within the same expression.

In the process of obtaining constants c_0 and c_X, it will be clear that they increase with their arguments. The value of c_0 will increase with T because all of its arguments increase with T; in particular, c_0 determines inversely the size of the initial data.

Remark 5.2. *Many of our estimates contain factors of the form $C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3}$, $0 < e_1 < e_2 < e_3$, where C_1, C_2, and C_3 may depend upon the norms of the data or the solution, but have no explicit dependence on time. To simplify matters, we will assume that $T \leqslant T_0$ for some fixed but arbitrarily large $T_0 > 0$. Then*

$$\begin{aligned} &C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3} \leqslant C_1 T^{e_1} + C_2 T^{e_1} T_0^{e_2 - e_1} + C_3 T^{e_1} T_0^{e_3 - e_1} \leqslant C' T^{e_1}, \\ &C' := (1 + T_0^{e_2 - e_1} + T_0^{e_3 - e_1}) \max\{C_1, C_2, C_3\}. \end{aligned}$$

Hence, in the final forms of estimates, we will only keep the lowest exponents of T and, similarly, of $|t_1 - t_2|$ for $t_1, t_2 \in [0, T]$.

6. Recovering velocity from vorticity

We need a few facts from [9] related to the Biot-Savart law, which we present now. We use the spaces H, H_c, and H_0 of (1.7) and (1.8),

Lemma 6.1. *Assume that Γ is $C^{n,\alpha}$-regular and let X be any function space contained in $C^{n,\alpha}(\Omega)^3$. For any $\mathbf{v} \in H$,*

$$\|P_{H_c}\mathbf{v}\|_X \leqslant C(X)\|\mathbf{v}\|_H$$

and if also $\mathbf{v} \in X$ then

$$\|\mathbf{v}\|_X \leqslant \|P_{H_0}\mathbf{v}\|_X + C(X)\|\mathbf{v}\|_H, \quad \|P_{H_0}\mathbf{v}\|_X \leqslant \|\mathbf{v}\|_X + C(X)\|\mathbf{v}\|_H.$$

Letting $h \in C^{n,\alpha}(\Gamma)$ for some $n \geqslant 1$, we define the subspace,

$$C^{n,\alpha}_{\sigma,h} := \{\mathbf{u} \in C^{n,\alpha}(\Omega)\colon \operatorname{div}\mathbf{u} = 0, \mathbf{u}\cdot\boldsymbol{n} = h \text{ on } \Gamma\}.$$

Corollary 6.2. *If $\mathbf{u}_1, \mathbf{u}_2 \in C^{n,\alpha}_{\sigma,h}$ for some $n \geqslant 1$ have the same vorticity and harmonic component then $\mathbf{u}_1 = \mathbf{u}_2$.*

For any $\boldsymbol{\omega}$ in the range of the curl, $\operatorname{curl}(H^1_0(\Omega)^3)$, there exists a unique $\mathbf{u} = K[\boldsymbol{\omega}] \in H_0 \cap H^1(\Omega)^3$ for which $\operatorname{curl}\mathbf{u} = \boldsymbol{\omega}$. The operator K, which recovers the unique divergence-free vector field in H_0 having a given curl, encodes the Biot-Savart law.

There exists a vector field $\boldsymbol{\mathcal{V}}$ as regular as $\mathbf{U}$ with $\operatorname{div}\boldsymbol{\mathcal{V}} = 0$, $\operatorname{curl}\boldsymbol{\mathcal{V}} = 0$, and $\boldsymbol{\mathcal{V}}\cdot\boldsymbol{n} = U^{\boldsymbol{n}}$ on $[0,T]\times\Gamma$. We define

$$K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] := K[\boldsymbol{\omega}] + \boldsymbol{\mathcal{V}}. \tag{6.1}$$

Define the solution space for vorticity,

$$V^{N,\alpha}_\sigma(Q) := \{\boldsymbol{\omega}\colon C^{N,\alpha}(Q)\colon \boldsymbol{\omega}(t) \in \operatorname{curl}(H^1(\Omega)^3) \text{ for all } t \in [0,T]\}.$$

Lemma 6.3. *Assume that $\mathbf{U} \in C^{N+2,\alpha}_\sigma(Q)$ and Γ is C^{N+2}. For all $t \in [0,T]$, $K_{U^{\boldsymbol{n}}(t)}$ maps $C^{N,\alpha}(\Omega)\cap\operatorname{curl}(H^1(\Omega)^3)$ continuously onto $C^{N+1,\alpha}_{\sigma,U^{\boldsymbol{n}}(t)} \cap (H_0 + \boldsymbol{\mathcal{V}}(t))$ and maps $W^{N,p}(\Omega)\cap\operatorname{curl}(H^1(\Omega)^3)$ continuously into $W^{N+1,p}(\Omega)$ for any $p \in (1,\infty)$. Also, $K_{U^{\boldsymbol{n}}}$ maps $V^{N,\alpha}_\sigma(Q)$ continuously onto*

$$\mathring{C}^{N+1,\alpha}_{\sigma,0}(Q) := \{\mathbf{u} \in \mathring{C}^{N+1,\alpha}_\sigma(Q)\colon \mathbf{u}(t) \in H_0 + \boldsymbol{\mathcal{V}}(t) \text{ for all } t \in [0,T]\}.$$

We now have the tools needed to to prove Lemma 6.4:

Lemma 6.4. *Assuming cond_N holds, $\operatorname{Dom}_N(A)$ is a nonempty, convex subset of $\mathring{C}^{N+1,\alpha}_\sigma(Q)$.*

Proof. We first show that $\operatorname{Dom}_N(A)$ is convex. Let $a, b \in [0,1]$ with $a + b = 1$, let $\mathbf{v}$, $\mathbf{w}$ be in $\operatorname{Dom}_N(A)$, and let $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$. Then $\mathbf{u}(0) = a\mathbf{u}_0 + b\mathbf{u}_0 = \mathbf{u}_0$, and so also cond_N is satisfied. Similarly, $\partial^n_t\mathbf{u}|_{t=0} = a\partial^n_t\mathbf{v}|_{t=0} + b\partial^n_t\mathbf{w}|_{t=0} = a\widetilde{\partial}^n_t\mathbf{u}_0 + b\widetilde{\partial}^n_t\mathbf{u}_0 = \widetilde{\partial}^n_t\mathbf{u}_0$. It follows that $\operatorname{Dom}_N(A)$ is convex.

To show that $\operatorname{Dom}_N(A)$ is nonempty, let $\boldsymbol{\omega}_0 := \operatorname{curl}\mathbf{u}_0$ and define

$$\boldsymbol{\omega}(t) := \boldsymbol{\omega}_0 + \sum_{n=1}^N \frac{t^n}{n!}\widetilde{\partial}^n_t\boldsymbol{\omega}_0,$$

so that for all $0 \leqslant n \leqslant N$, $\partial^n_t\boldsymbol{\omega}(0) = \widetilde{\partial}^n_t\boldsymbol{\omega}_0$. Because $\boldsymbol{\omega}(t)$ is in the range of the curl for all $t \in [0,T]$ by Proposition 3.6, we can define

$$\mathbf{u}(t) := K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] + \sum_{n=0}^N \frac{t^n}{n!}P_{H_c}\widetilde{\partial}^n_t\mathbf{u}_0,$$

which we note lies in $C_\sigma^{N+1,\alpha}(Q)$. Then $\mathbf{u}(0) = \mathbf{u}_0$ by Corollary 6.2, since they have the same vorticity and harmonic component and both lie in $C_\sigma^{N+1,\alpha}(\Omega)$. Moreover, for $1 \leqslant n \leqslant N$,

$$\operatorname{curl} \partial_t^n \mathbf{u}(0) = \partial_t^n \boldsymbol{\omega}(0) = \widetilde{\partial}_t^n \boldsymbol{\omega}_0 = \operatorname{curl} \widetilde{\partial}_t^n \mathbf{u}_0$$

by Proposition 3.6. Also, $P_{H_c} \partial_t^n \mathbf{u}(0) = P_{H_c} \widetilde{\partial}_t^n \mathbf{u}_0$. That is, $\partial_t^n \mathbf{u}(0)$ and $\widetilde{\partial}_t^n \mathbf{u}_0$ have the same curl and same harmonic component, and $\partial_t^n \mathbf{u}(0)$ and, by (3.11), $\widetilde{\partial}_t^n \mathbf{u}_0$ lie in $C^{n,\alpha}_{\sigma,\partial_t U^{\boldsymbol{n}}}$. Hence, it follows from Corollary 6.2 that $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$, and we see that $\mathbf{u} \in \operatorname{Dom}_N(A)$, demonstrating that $\operatorname{Dom}_N(A)$ is nonempty. □

The estimates we will need are given in Lemma 6.5.

Lemma 6.5. *Assume $\mathbf{U} \in C_\sigma^{N+1,\alpha}(Q)$. Let $\boldsymbol{\omega} \in C^\alpha(\Omega)$ be a divergence-free vector field on Ω having vanishing external fluxes. For any $\mathbf{u} \in H$ there exists $\mathbf{u}_c \in H_c$ such that $\mathbf{u} := K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] + \mathbf{u}_c$, and for all $t \in [0, T]$,*

$$\begin{aligned}
\|\mathbf{u}(t)\|_{W^{N+1,p}(\Omega)} &\leqslant C\|\boldsymbol{\omega}(t)\|_{W^{N,p}(\Omega)} + \|\mathbf{U}(t)\|_{W^{N+1,p}(\Omega)} + \|\mathbf{u}_c(t)\|_{W^{N+1,p}(\Omega)}, \\
\|\mathbf{u}(t)\|_{C_\sigma^{N+1,\alpha}(\Omega)} &\leqslant C\|\boldsymbol{\omega}(t)\|_{C^{N,\alpha}(\Omega)} + \|\mathbf{U}(t)\|_{C^{N+1,\alpha}(\Omega)} + \|\mathbf{u}_c(t)\|_{C^{N+1,\alpha}(\Omega)}, \\
\|\nabla \mathbf{u}(t)\|_{L^p(\Omega)} &\leqslant C_p\|\boldsymbol{\omega}(t)\|_{L^p(\Omega)} + \|\nabla \mathbf{U}(t)\|_{L^p(\Omega)} + \|\nabla \mathbf{u}_c(t)\|_{L^p(\Omega)}, \\
\|\mathbf{u}(t)\|_{L^p(\Omega)} &\leqslant C_p\|\boldsymbol{\omega}(t)\|_{L^p(\Omega)} + \|\mathbf{U}(t)\|_{L^p(\Omega)} + \|\mathbf{u}_c(t)\|_{L^p(\Omega)}
\end{aligned}$$

for all $p \in (1, \infty)$. In each case, the final term can be replaced by $C\|\mathbf{u}\|_H$.

Proof. The first three inequalities follow from Lemma 6.3. The fourth inequality follows from the third and Poincaré's inequality, since elements of H have mean zero. Lemma 6.1 allows us to replace each of the final terms by $C\|\mathbf{u}\|_H$. □

In Section 10, we will require a bound on $\|\mathbf{u}\|_{C^{N+1}(Q)}$ that is better than just M of (5.1). To obtain such a bound, first observe that, setting $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$,

$$\|\boldsymbol{\omega}\|_{L^2(Q)} \leqslant \left(\int_0^T M^2 \right)^{\frac{1}{2}} \leqslant M T^{\frac{1}{2}}.$$

In analogy with $\mathring{C}_\sigma^{N+1,\alpha}(Q)$, we define $\mathring{C}_\sigma^{N+1}(Q)$ to be the space $C_\sigma^{N+1}(Q)$, but with one fewer time derivatives, and similarly for $\mathring{C}^{N+1}(Q)$. Then, using Lemmas 6.5 and A.5, for any $0 < \beta < \alpha$,

$$\begin{aligned}
\|\mathbf{u}\|_{\mathring{C}_\sigma^{N+1}(Q)} &\leqslant \|\boldsymbol{\mathcal{V}}\|_{\mathring{C}^{N+1}(Q)} + \|\mathbf{u} - \boldsymbol{\mathcal{V}}\|_{\mathring{C}^{N+1}(Q)} \leqslant c_0 + \|\mathbf{u} - \boldsymbol{\mathcal{V}}\|_{\mathring{C}^{N+1}(Q)} \\
&\leqslant c_0 + C_\beta \|\boldsymbol{\omega}\|_{C^{N,\beta}} \leqslant c_0 + C_\beta \|\boldsymbol{\omega}\|_{L^\infty(Q)} + C_\beta \|\boldsymbol{\omega}\|^a_{C^{N,\alpha}(Q)} \|\boldsymbol{\omega}\|^{1-a}_{L^2(Q)} \qquad (6.2) \\
&\leqslant c_0 + C_\beta \|\boldsymbol{\omega}\|_{L^\infty(Q)} + C_\beta M T^b,
\end{aligned}$$

where $0 < b < 1$ (its exact value being unimportant). Here, we used our assumptions that $M \geqslant 1$ and $T \leqslant T_0$ to simplify the form of the estimates coming from Lemma A.5 (see Remark 5.2).

7. Flow map estimates

The pushforward of the initial vorticity by the flow map meets, along a hypersurface S in Q, the pushforward of the vorticity generated on the inflow boundary. This requires some analysis at the level of the flow map. For the most part, the analysis in [9], which we summarize here, suffices. The coarse bounds developed on the flow map in [9], however, would only be sufficient for us to obtain small data existence of solutions: for the short time result for general data that we desire, we will require more explicit and refined bounds, which we develop in Lemma 7.2.

We assume throughout this section that $\mathbf{U} \in C_\sigma^{N+2,\alpha}(Q), \mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$ for some $N \geqslant 0$. As in [9], we extend $\mathbf{u}$ to be defined on all of $\mathbb{R} \times \mathbb{R}^3$ using an extension operator like that in Theorem 5′, chapter VI of [25]. This extension need not be divergence-free, and is used only as a matter of convenience in stating results; it is only the value of $\mathbf{u}$ on $\overline{Q}$ that ultimately concerns us.

We define $\eta\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ to be the unique flow map for $\mathbf{u}$, so that $\partial_{t_2}\eta(t_1, t_2; \mathbf{x}) = \mathbf{u}(t_2, \eta(t_1, t_2; \mathbf{x}))$. Then $\eta(t_1, t_2; \mathbf{x})$ is the position that a particle starting at time t_1 at position $\mathbf{x} \in \mathbb{R}^3$ will be at time t_2 as it moves under the action of the velocity field $\mathbf{u}$.

For any $(t, \mathbf{x}) \in \overline{Q}$ let

- $\boldsymbol{\gamma}(t, \mathbf{x})$ be the point on Γ_+ at which the flow line through x at time t intersects Γ_+;
- let $\tau(t, \mathbf{x})$ be the time at which that intersection occurs.

For our purposes, we can leave τ and $\boldsymbol{\gamma}$ undefined if the flow line never intersects with Γ_+.

Remark 7.1. *We will often drop the $(t, \mathbf{x})$ arguments on τ and $\boldsymbol{\gamma}$ for brevity.*

We define the hypersurface,

$$S := \{(t, \mathbf{x}) \in \overline{Q}\colon \tau(t, \mathbf{x}) = 0\},$$

which is nonempty since it contains at least $\Gamma_+ \times \{0\}$, and the open sets $U_\pm \subseteq Q$,

$$\begin{aligned} U_- &:= \{(t, \mathbf{x}) \in Q\colon (t, \mathbf{x}) \notin \text{ domain of } \tau, \boldsymbol{\gamma}\}, \\ U_+ &:= \{(t, \mathbf{x}) \in Q\colon \tau(t, \mathbf{x}) > 0\}. \end{aligned}$$

Then S is of class $C^{N+1,\alpha}$ as a hypersurface in Q and $S(t) := \{\mathbf{x} \in \Omega\colon (t, \mathbf{x}) \in S\}$ is of class $C^{N+1,\alpha}$ as a surface in Ω.

The estimates on the flow map in Lemma 7.2 are more explicit than in [9], where we required only coarse estimates. We note that η has one more derivative in both time variables than has $\mathbf{u}$, which we can see in the explicit estimates. In Lemma 7.2, $\dot{C}^\alpha(Q)$ is the homogeneous Hölder norm and the subscripts $\mathbf{x}$ and t refer to norms only in those variables (see (A.3) for detailed definitions).

Lemma 7.2. *The flow map $\eta \in C^{N+1,\alpha}([0, T]^2 \times \mathbb{R}^3)$. Define $\mu\colon U_+ \to [0, T] \times \Gamma_+$ by*

$$\mu(t, \mathbf{x}) = (\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})).$$

The functions τ, $\boldsymbol{\gamma}$, μ lie in $C^{N+1,\alpha}(\overline{U}_+ \setminus \{0\} \times \Gamma_+)$. Moreover,

$$\begin{aligned} \|\partial_{t_1}\eta(t_1, t_2; \mathbf{x})\|_{L^\infty_{\mathbf{x}}} &\leqslant \|\mathbf{u}\|_{L^\infty(Q)} h(t_1, t_2), \\ \|\nabla\eta(t_1, t_2; \mathbf{x})\|_{L^\infty_{\mathbf{x}}} &\leqslant h(t_1, t_2), \\ \|\nabla\eta(0, t_2; \mathbf{x})\|_{\dot{C}^\alpha_{t_2}(Q)} &\leqslant \|\nabla\mathbf{u}\|_{L^\infty(Q)} h(0, T) T^{1-\alpha}, \\ \|\nabla\eta(0, t_2; \mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}(Q)} &\leqslant h(0, T)^{1+2\alpha} \int_0^T \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds, \\ \|\nabla\eta(0, T; \mathbf{x})\|_{\dot{C}^\alpha(Q)} &\leqslant e^{(1+2\alpha)MT} M T^{1-\alpha}, \end{aligned} \tag{7.1}$$

where

$$h(t_1, t_2) := \exp\left|\int_{t_1}^{t_2} \|\nabla\mathbf{u}(s)\|_{L^\infty}\, ds\right| \leqslant e^{MT}.$$

Also,

$$\|D\mu\|_{L^\infty(Q)} \leqslant CU_{min}^{-1}[1 + \|\mathbf{u}\|^2_{L^\infty(Q)}]h(0,T), \tag{7.2}$$

where U_{min} is as in (5.2).

More generally, for any $N \geqslant 0$, defining $\exp^n$ to be $\exp$ composed with itself n times,

$$\begin{aligned}
\|\partial_{t_1}^{N+1}\eta(t_1,t_2;\mathbf{x})\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant C\|\mathbf{u}\|_{C^N(Q)}\exp^{N+1}(MT),\\
\|\nabla^{N+1}\eta(t_1,t_2;\mathbf{x})\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant \exp^{N+1}(MT),\\
\|\nabla^{N+1}\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{t_2}(Q)} &\leqslant \|\nabla^{N+1}\mathbf{u}\|_{L^\infty(Q)}\exp^{N+1}(MT)T^{1-\alpha},\\
\|\nabla^{N+1}\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}(Q)} &\leqslant \exp^{N+1}(CMT)\int_0^T \|\nabla^{N+1}\mathbf{u}(s)\|_{\dot{C}^\alpha}\,ds,\\
\|\nabla^{N+1}\eta(0,T;\mathbf{x})\|_{\dot{C}^\alpha(Q)} &\leqslant \exp^{N+1}(CMT)MT^{1-\alpha},\\
\|D^{N+1}\mu\|_{L^\infty(Q)} &\leqslant c_0[1+\|\mathbf{u}\|^{2(N+1)}_{C^N(Q)}]\exp^{N+1}(MT).
\end{aligned} \tag{7.3}$$

Proof. We will apply Lemma A.2 multiple times without explicit reference.

Taking the gradient of the integral expression in (3.2) of [9],

$$\nabla\eta(t_1,t_2;\mathbf{x}) = I + \int_{t_1}^{t_2} \nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\,ds. \tag{7.4}$$

Thus,

$$\|\nabla\eta(t_1,t_2;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} \leqslant 1 + \left|\int_{t_1}^{t_2}\|\nabla\mathbf{u}(s)\|_{L^\infty}\|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}}\,ds\right|.$$

Grönwall's Lemma, applied with fixed t_1, gives $(7.1)_2$. Lemma 3.1 of [9] gives $\partial_{t_1}\eta(t_1,t_2;\mathbf{x}) = -\mathbf{u}(t_1,\mathbf{x})\cdot\nabla\eta(t_1,t_2;\mathbf{x})$, from which $(7.1)_1$ follows.

It also follows from (7.4) that

$$\begin{aligned}
\|\nabla\eta(0,t_2;\mathbf{x})\|_{C(\dot{Q})^\alpha_{t_2}} &\leqslant \sup_{t_2\neq t_2'}\frac{\|\nabla\mathbf{u}\|_{L^\infty(Q)}\|\nabla\eta\|_{L^\infty(Q)}}{|t_2-t_2'|^\alpha}|t_2-t_2'|\\
&\leqslant \|\nabla\mathbf{u}\|_{L^\infty(Q)}h(0,T)T^{1-\alpha},
\end{aligned}$$

giving $(7.1)_3$.

Returning once more to (7.4),

$$\|\nabla\eta(t_1,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \leqslant \int_0^{t_2}\|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\,ds.$$

But, using Lemma A.1,

$$\begin{aligned}
&\|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\\
&\quad\leqslant \|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\|_{\dot{C}^\alpha_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} + \|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\|_{L^\infty_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\\
&\quad\leqslant \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\|\eta(t_1,s;\mathbf{x})\|^\alpha_{Lip_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} + \|\nabla\mathbf{u}(s)\|_{L^\infty}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\\
&\quad\leqslant \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}h(t_1,s)^{2\alpha} + \|\nabla\mathbf{u}(s)\|_{L^\infty}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}},
\end{aligned}$$

so

$$\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}$$

$$\begin{aligned}
&\leqslant \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{\dot{C}^\alpha} h(0,s)^{2\alpha}\, ds + \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{L^\infty(\Omega)} \|\nabla \eta(0,s;\mathbf{x})\|_{\dot{C}^\alpha_\mathbf{x}}\, ds \\
&\leqslant h(0,t_2)^{2\alpha} \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds + \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{L^\infty(\Omega)} \|\nabla \eta(0,s;\mathbf{x})\|_{\dot{C}^\alpha_\mathbf{x}}\, ds.
\end{aligned}$$

Applying Grönwall's Lemma gives

$$\begin{aligned}
\|\nabla \eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_\mathbf{x}} &\leqslant \left[h(0,t_2)^{2\alpha} \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds \right] \exp \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{L^\infty(\Omega)}\, ds \\
&= h(0,t_2)^{1+2\alpha} \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds,
\end{aligned}$$

which is $(7.1)_4$.

From Lemma 3.5 of [9],

$$\begin{aligned}
&\partial_t \tau = -U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})^{-1} \partial_{t_1} \eta(t,\tau;\mathbf{x}) \cdot \boldsymbol{n}(\boldsymbol{\gamma}), && \nabla \tau = -U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})^{-1} (\nabla \eta(t,\tau;\mathbf{x}))^T \boldsymbol{n}(\boldsymbol{\gamma}), \\
&\partial_t \boldsymbol{\gamma} = \partial_{t_1} \eta(t,\tau;\mathbf{x}) + \partial_t \tau \mathbf{u}(\tau,\boldsymbol{\gamma}), && \nabla \boldsymbol{\gamma} = \mathbf{u}(\tau,\boldsymbol{\gamma}) \otimes \nabla \tau + \nabla \eta(t,\tau;\mathbf{x}).
\end{aligned}$$

We use these expressions to calculate,

$$\begin{aligned}
\|\partial_t \tau\|_{L^\infty(Q)} &\leqslant C U_{min}^{-1} \|\partial_{t_1} \eta\|_{L^\infty(Q)} \leqslant C U_{min}^{-1} \|\mathbf{u}\|_{L^\infty(Q)} h(0,T), \\
\|\nabla \tau\|_{L^\infty(Q)} &\leqslant C U_{min}^{-1} \|\nabla \eta\|_{L^\infty(Q)} \leqslant C U_{min}^{-1} h(0,T), \\
\|\partial_t \boldsymbol{\gamma}\|_{L^\infty(Q)} &\leqslant C U_{min}^{-1} \|\partial_{t_1} \eta\|_{L^\infty(Q)} + \|\mathbf{u}\|_{L^\infty(Q)} \|\partial_t \tau\|_{L^\infty(Q)} \\
&\leqslant C U_{min}^{-1} [\|\mathbf{u}\|_{L^\infty(Q)} + \|\mathbf{u}\|^2_{L^\infty(Q)}] h(0,T), \\
\|\nabla \boldsymbol{\gamma}\|_{L^\infty(Q)} &\leqslant \|\mathbf{u}\|_{L^\infty(Q)} \|\nabla \tau\|_{L^\infty(Q)} + \|\nabla \eta\|_{L^\infty(Q)} \leqslant [1 + C U_{min}^{-1} \|\mathbf{u}\|_{L^\infty(Q)}] h(0,T).
\end{aligned}$$

Summing these estimates gives the bound on $D\mu = (\partial_t \mu, \nabla \mu)$.

The bounds for higher N follow from inductive extension of these arguments. □

Remark 7.3. *The exact bounds in Lemma 7.2 are not so important, but it is important that M only appear in the exponentials, while other factors contain norms of $\mathbf{u}$ lower than $X^N_{\alpha',\alpha}$, as these can be bounded a little better (by (6.2), primarily).*

We are now in a position to give the definition of a Lagrangian solution to (2.1), as it appears in [9]. For this purpose, define

$$\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0(t,\mathbf{x}) := \eta(t,0;\mathbf{x}). \tag{7.5}$$

As with τ and $\boldsymbol{\gamma}$ (see Remark 7.1) we will often drop the $(t,\mathbf{x})$ arguments on $\boldsymbol{\gamma}_0$.

Definition 7.4 (Lagrangian solution to (2.1))**.** *Define $\overline{\boldsymbol{\omega}}_\pm$ and $\mathbf{G}_\pm$ on $U_\pm$ by*

$$\begin{aligned}
&\overline{\boldsymbol{\omega}}_-(t,\mathbf{x}) = \nabla \eta(0,t;\boldsymbol{\gamma}_0) \overline{\boldsymbol{\omega}}_0(\boldsymbol{\gamma}_0) + \mathbf{G}_+(t,\mathbf{x}), \\
&\overline{\boldsymbol{\omega}}_+(t,\mathbf{x}) = \nabla \eta(\tau,t;\boldsymbol{\gamma}) \mathbf{H}(\tau,\boldsymbol{\gamma}) + \mathbf{G}_-(t,\mathbf{x}), \\
&\mathbf{G}_-(t,\mathbf{x}) := \int_0^t \nabla \eta(s,t;\eta(t,s;\mathbf{x})) \mathbf{g}(s,\eta(t,s;\mathbf{x}))\, ds, \\
&\mathbf{G}_+(t,\mathbf{x}) := \int_{\tau(t,\mathbf{x})}^t \nabla \eta(s,t;\eta(t,s;\mathbf{x})) \mathbf{g}(s,\eta(t,s;\mathbf{x}))\, ds.
\end{aligned} \tag{7.6}$$

Then $\overline{\boldsymbol{\omega}}$ defined by $\overline{\boldsymbol{\omega}}|_{U_\pm} = \overline{\boldsymbol{\omega}}_\pm$ is called a Lagrangian solution to (2.1).

In (7.6), we left the value of $\overline{\boldsymbol{\omega}}$ along S unspecified. Under the assumptions of Theorem 2.2, $\overline{\boldsymbol{\omega}}_\pm$ can be extended along S so that $\overline{\boldsymbol{\omega}}$ lies in $C^{N,\alpha}(Q)$, and the bounds on $U_\pm$ combine to give estimates on $\overline{\boldsymbol{\omega}}$ in $C^{N,\alpha}(Q)$.

8. The nonlinear term on the boundary

Proposition 8.2 gives coordinate-free expressions for $(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$ on Γ. The proof of Proposition 8.2 is most readily obtained using the boundary coordinates introduced in Appendix B, so we defer it to that appendix.

Definition 8.1. *For any tangent vector field* $\mathbf{v}$ *on* Γ*, define* $\mathbf{v}^\perp$ *to be* $\mathbf{v}$ *rotated 90 degrees counterclockwise around the normal vector when viewed from outside* Ω *(so* $\mathbf{v}^\perp = \boldsymbol{n}\times\mathbf{v}$*).*

We write the gradient and divergence on the boundary as ∇_Γ and $\operatorname{div}_\Gamma$, as in Appendix B.

Proposition 8.2. *Assume that* Γ *is* C^2*. Let* $\mathbf{u}$ *be a divergence-free differentiable vector field, let* $u^{\boldsymbol{n}} = \mathbf{u}\cdot\boldsymbol{n}$*, and, as in (1.1), let* $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{u} - u^{\boldsymbol{n}}\boldsymbol{n}$*. Let* κ_1, κ_2 *be the principal curvatures on* Γ*. On* $[0,T]\times\Gamma$*, we have*

$$(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} = -u^{\boldsymbol{n}}\operatorname{div}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}\cdot\nabla_\Gamma u^{\boldsymbol{n}} - (\kappa_1+\kappa_2)(u^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}}\cdot\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}. \tag{8.1}$$

Here, $\mathcal{A}$ *is the shape operator on the boundary: for any tangential vector field,* $\mathcal{A}\mathbf{v}$ *is the directional derivative of* $\boldsymbol{n}$ *in the direction of* $\mathbf{v}$*, which is also a tangential vector field.*

The nonlinear term on the boundary is key to recovering the pressure, as we will see in the next section. It was for these purposes that we used $N[\mathbf{u}]$ given in (3.6) to define the approximate pressure in (3.5). Using that $\mathbf{u}^{\boldsymbol{n}} = \mathbf{U}^{\boldsymbol{n}}$, substituting the expression in (8.1) for $(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$, and using (B.1), we see that on Γ_+,

$$N[\mathbf{u}] = -U^{\boldsymbol{n}}\operatorname{div}_\Gamma\mathbf{U}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}\cdot\nabla_\Gamma U^{\boldsymbol{n}} - (\kappa_1+\kappa_2)(U^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}}\cdot\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}, \tag{8.2}$$

so $N[\mathbf{u}]$ has no derivatives on $\mathbf{u}^{\boldsymbol{\tau}}$. Nonetheless, integrating $(3.6)_2$ by parts along each boundary component using Lemma B.1, we see that

$$\int_\Gamma N[\mathbf{u}] = \int_\Gamma (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}, \tag{8.3}$$

which will allow us to use $N[\mathbf{u}]$ in place of $(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$ in the Neumann boundary condition on the pressure in Section 9.

9. Pressure Estimates

We can determine the pressure from the velocity by taking the divergence of $(1.5)_1$ and using that $\operatorname{div}\mathbf{u} = 0$, which yields

$$\begin{cases} \Delta p = -\nabla\mathbf{u}\cdot(\nabla\mathbf{u})^T & \text{in } \Omega,\\ \nabla p\cdot\boldsymbol{n} = \partial_t\mathbf{u}\cdot\boldsymbol{n} - (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{9.1}$$

On Γ_0, as we can see from (8.1), $\nabla p\cdot\boldsymbol{n} = -\mathbf{u}^{\boldsymbol{\tau}}\cdot\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}$ ($= -\kappa|\mathbf{u}|^2$ in 2D). Hence, when $\Gamma = \Gamma_0$, standard Schauder estimates imply that ∇p and $\mathbf{u}$ have the same spatial regularity. This is the impermeable boundary case. But for inflow, outflow boundary conditions, the expression for $\nabla p\cdot\boldsymbol{n}$ contains spatial derivatives of $\mathbf{u}$, as we can see from (8.1), and elliptic theory gives only a pressure gradient having one fewer spatial derivative than the velocity. (Because $\mathbf{u}\cdot\boldsymbol{n} = U^{\boldsymbol{n}}$ on all of Γ, the time derivative in $(9.1)_2$ does not impact the regularity of p.)

We see, then, that impermeable boundary conditions are very special, and with inflow, outflow we should not expect to obtain a gradient pressure field with the same regularity as that of $\mathbf{u}$. This is not in itself a problem, for as we can see from (3.7), we only need the pressure gradient to have the same regularity as the vorticity to generate vorticity on the boundary. We will need higher regularity, however, to obtain a fixed point for the operator $A\colon X_{\alpha',\alpha} \to X_{\alpha',\alpha}$.

We circumvent this difficulty using the simple but clever technique in [2]: we replace the boundary condition in $(9.1)_2$ using $N[\mathbf{u}]$ of (3.6), solving instead, (3.5) for the pressure q. We see from (8.3) that the required compatibility condition coming from $\int_\Gamma \nabla q \cdot \boldsymbol{n} = \int_\Omega \Delta q = \int_\Omega \operatorname{div}(-\partial_t \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u})$ remains satisfied when using $-\partial_t U^{\boldsymbol{n}} - N[\mathbf{u}]$ in place of $-\partial_t u^{\boldsymbol{n}} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$ on Γ.

Lemma 9.1. *Suppose that Ω' is a compact subset of $\Omega \cup \Gamma_+$, $\Omega' \neq \emptyset$. For any $n \geqslant 0$,*

$$\|f\|_{W^{n+2,r}(\Omega')} \leqslant C \left[\|\Delta f\|_{W^{n,r}(\Omega)} + \|\nabla f \cdot \boldsymbol{n}\|_{W^{n+1-\frac{1}{r},r}(\Gamma_+)} + \|f\|_{L^r(\Omega)} \right] \tag{9.2}$$

for $f \in W^{n,r}(\Omega)$, where $r \in (1, \infty)$.

Proof. These bounds for $n = 0$ are stated near the bottom of page 174 of [2], but let us say a few words about them. First, they are derived from combining an interior estimate away from all boundaries with an estimate that includes only Γ_+. Second, [2] treats the $N = 0$ case, and we use (15.1.5) of [1] for the $N \geqslant 1$ case. □

We start in Propositions 9.2 and 9.3 by controlling only the spatial derivatives of q.

Proposition 9.2. *Let $r \in [2, \infty)$, $t_1, t_2 \in [0, T]$, and q solve (3.5) for some $\mathbf{u} \in X_{\alpha',\alpha}$ with q normalized so that*

$$\int_\Omega q|q|^{r-2} = 0. \tag{9.3}$$

Then

$$\begin{aligned} \|q(t)\|_{L^r(\Omega)} &\leqslant C_1, \\ \|q(t_1) - q(t_2)\|_{L^r(\Omega)} &\leqslant C_2 \|\mathbf{u}\|_{X_{\alpha',\alpha}} |t_1 - t_2|^{\alpha'}, \end{aligned} \tag{9.4}$$

where

$$C_1 := C \left[\|\mathbf{U}\|^2_{X_{\alpha',\alpha}} + \|\mathbf{u}\|^2_{L^\infty(Q)} \right], \qquad C_2 := C \left[\|\mathbf{U}\|_{L^\infty(Q)} + \|\mathbf{u}\|_{L^\infty(Q)} \right],$$

the constant C depending only upon Ω and r.

Proof. We adapt the argument on pages 175-176 of [2]. For now we suppress the time variable.

Let β be the unique mean-zero solution to

$$\begin{cases} \Delta\beta = q|q|^{r-2} & \text{in } \Omega, \\ \nabla\beta \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases}$$

where the normalization of q in (9.3) gives solvability. Letting $r' = r/(r-1)$, which we note is Hölder conjugate to r, Lemma 9.1 gives

$$\|\beta\|_{W^{2,r'}(\Omega)} \leqslant C \| |q|^{r-1} \|_{L^{r'}(\Omega)} = C \|q\|^{r-1}_{L^r(\Omega)}.$$

Then,

$$\|q\|^r_{L^r(\Omega)} = (\Delta\beta, q) = -(\nabla\beta, \nabla q) + \int_\Gamma (\nabla\beta \cdot \boldsymbol{n}) q = (\Delta q, \beta) - \int_\Gamma (\nabla q \cdot \boldsymbol{n})\beta.$$

Now,

$$(\Delta q, \beta) = -(\operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u}), \beta) = (\mathbf{u} \cdot \nabla \mathbf{u}, \nabla \beta) - \int_\Gamma ((\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n})\beta.$$

But,

$$\begin{aligned}(\mathbf{u}\cdot\nabla \mathbf{u}, \nabla \beta) &= \int_\Omega u^i \partial_i u^j \partial_j \beta = \int_\Omega u^i \partial_i (u^j \partial_j \beta) - \int_\Omega u^i u^j \partial_i \partial_j \beta \\ &= \int_\Omega \mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla \beta) - (\mathbf{u} \otimes \mathbf{u}, \nabla\nabla\beta) = -\int_\Gamma U^{\boldsymbol{n}}(\mathbf{u} \cdot \nabla \beta) - (\mathbf{u} \otimes \mathbf{u}, \nabla\nabla\beta)\end{aligned}$$

and, using (3.6)

$$\begin{aligned}-\int_\Gamma ((\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n})\beta &= \int_\Gamma (\partial_t U^{\boldsymbol{n}} + \nabla p \cdot \boldsymbol{n})\beta + \int_{\Gamma_+} \operatorname{div}_\Gamma (U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}})) \\ &= \int_\Gamma (\partial_t U^{\boldsymbol{n}} + \nabla p \cdot \boldsymbol{n})\beta,\end{aligned}$$

the integral over Γ_+ vanishing by Lemma B.1. Hence,

$$\|q\|^r_{L^r(\Omega)} = -(\mathbf{u} \otimes \mathbf{u}, \nabla\nabla\beta) + \int_\Gamma (\partial_t U^{\boldsymbol{n}} \beta - U^{\boldsymbol{n}}(\mathbf{u} \cdot \nabla \beta)). \tag{9.5}$$

We thus have the bound,

$$\|q\|^r_{L^r(\Omega)} \leqslant \|\mathbf{u}\|_{L^\infty} \|\mathbf{u}\|_{L^r} \|\beta\|_{W^{2,r'}} + \|\partial_t U^{\boldsymbol{n}}\|_{L^r(\Gamma)} \|\beta\|_{L^{r'}(\Gamma)} - \int_\Gamma U^{\boldsymbol{n}}(\mathbf{u} \cdot \nabla \beta).$$

But,

$$\begin{aligned}-\int_\Gamma U^{\boldsymbol{n}}(\mathbf{u} \cdot \nabla \beta) &\leqslant \|\mathbf{U}\|_{L^{r'}([0,T]\times\Gamma)} \|\mathbf{u}\|_{L^\infty(Q)} \|\nabla\beta\|_{L^r(\Gamma)} \\ &\leqslant C\|\mathbf{U}\|_{L^\infty(Q)} \|\mathbf{u}\|_{L^\infty(Q)} \|\nabla \beta\|_{W^{1,r}(\Omega)} \leqslant C\|\mathbf{U}\|_{L^\infty(Q)} \|\mathbf{u}\|_{L^\infty(Q)} \|\beta\|_{W^{2,r}(\Omega)}.\end{aligned} \tag{9.6}$$

We see, then, that

$$\|q\|^r_{L^r(\Omega)} \leqslant C_1 \|\beta\|_{W^{2,r'}} \leqslant C_1 \|q\|^{r-1}_{L^r(\Omega)},$$

from which $(9.4)_1$ follows.

To obtain $(9.4)_2$ we argue the same way, bounding now $\overline{q} := q(t_1) - q(t_2)$ and using $\partial_t U^{\boldsymbol{n}}(t_1) - N[\mathbf{u}(t_1)] - (\partial_t U^{\boldsymbol{n}}(t_2) - N[\mathbf{u}(t_2)])$ in place of $\partial_t U^{\boldsymbol{n}} - N[\mathbf{u}]$ evaluated at a single time. And now β solves

$$\begin{cases}\Delta \beta = \overline{q}|\overline{q}|^{r-2} & \text{in } \Omega, \\ \nabla \beta \cdot \boldsymbol{n} = 0 & \text{on } \Gamma.\end{cases}$$

In place of (9.5), we find

$$\|\overline{q}\|^r_{L^r(\Omega)} = -(\mathbf{u}(t_1) \otimes \mathbf{u}(t_1) - \mathbf{u}(t_2) \otimes \mathbf{u}(t_2), \nabla\nabla\beta) - \int_\Gamma U^{\boldsymbol{n}}((\mathbf{u}(t_1) - \mathbf{u}(t_2)) \cdot \nabla \beta), \tag{9.7}$$

where we note that the boundary integral involving $\partial_t U^{\boldsymbol{n}} \beta$ appearing in (9.5) cancels.

For the first term on the right-hand side of (9.7), we use that

$$\|\mathbf{u}(t_1) \otimes \mathbf{u}(t_1) - \mathbf{u}(t_2) \otimes \mathbf{u}(t_2)\|_{L^r(\Omega)} \leqslant [\|\mathbf{u}(t_1)\|_{L^\infty} + \|\mathbf{u}(t_2)\|_{L^\infty}] \, \|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{L^{r'}(\Omega)}.$$

But, applying Lemma A.8 with $N = 0$,

$$\begin{aligned} \|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{L^r(\Omega)} &\leqslant C\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{L^\infty(\Omega)} \leqslant C\|\mathbf{u}\|_{C^\alpha(Q)}|t_1 - t_2|^{\alpha'} \\ &\leqslant C\|\mathbf{u}\|_{X_{\alpha',\alpha}}|t_1 - t_2|^{\alpha'}, \end{aligned} \tag{9.8}$$

so

$$-(\mathbf{u}(t_1) \otimes \mathbf{u}(t_1) - \mathbf{u}(t_2) \otimes \mathbf{u}(t_2), \nabla\nabla\beta) \leqslant C\|\mathbf{u}\|_{L^\infty(Q)}\|\mathbf{u}\|_{X_{\alpha',\alpha}}|t_1 - t_2|^{\alpha'}\|\beta\|_{W^{2,r}(\Omega)}.$$

For the boundary integral in (9.7), we obtain as in (9.6),

$$-\int_\Gamma U^{\boldsymbol{n}}((\mathbf{u}(t_1) - \mathbf{u}(t_2)) \cdot \nabla\beta) \leqslant C\|\mathbf{U}\|_{L^\infty(Q)}\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{L^\infty(Q)}\|\beta\|_{W^{2,r}(\Omega)}.$$

Combining these bounds, we see that

$$\|\overline{q}\|_{L^r(\Omega)} \leqslant C\left[\|\mathbf{u}\|_{L^\infty(Q)} + \|\mathbf{U}\|_{L^\infty(Q)}\right]\|\mathbf{u}\|_{X_{\alpha',\alpha}}|t_1 - t_2|^{\alpha'},$$

which is $(9.4)_2$. □

Proposition 9.3. *Assume that the data has regularity N and let Ω' be as in Lemma 9.1. Let $\mathbf{u} \in X^N_{\alpha',\alpha} \cap \mathrm{Dom}_N(A)$ and let q solving (3.5) be normalized as in (9.3) with $r = 3/(1-\alpha)$. Then*

$$\begin{aligned} \|q(t_1) - q(t_2)\|_{W^{N+2,r}(\Omega')} &\leqslant c_X|t_1 - t_2|^\alpha, \\ \|\nabla q(t_1) - \nabla q(t_2)\|_{C^{N,\alpha}(\Omega')} &\leqslant c_X|t_1 - t_2|^\alpha \end{aligned} \tag{9.9}$$

for all $t_1, t_2 \in [0, T]$.

Proof. We first prove $(9.9)_1$. Defining $\overline{q} := q(t_1) - q(t_2)$ and applying Lemma 9.1, we have

$$\|\overline{q}\|_{W^{N+2,r}(\Omega')} \leqslant C\left[\|\Delta\overline{q}\|_{W^{N,r}(\Omega)} + \|\nabla\overline{q} \cdot \boldsymbol{n}\|_{W^{N+1-\frac{1}{r},r}(\Gamma_+)} + \|\overline{q}\|_{L^r(\Omega)}\right].$$

Now,

$$\begin{aligned} \Delta\overline{q} &= \nabla\mathbf{u}(t_2) \cdot (\nabla\mathbf{u}(t_2))^T - \nabla\mathbf{u}(t_1) \cdot (\nabla\mathbf{u}(t_1))^T \\ &= \nabla(\mathbf{u}(t_2) - \mathbf{u}(t_1)) \cdot (\nabla\mathbf{u}(t_2))^T + \nabla\mathbf{u}(t_1) \cdot (\nabla(\mathbf{u}(t_2) - \mathbf{u}(t_1)))^T. \end{aligned}$$

Thus, for $N = 0$,

$$\|\Delta\overline{q}\|_{L^r(\Omega)} \leqslant C\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{L^r(\Omega)}\left[\|\nabla\mathbf{u}(t_1)\|_{L^\infty(\Omega)} + \|\nabla\mathbf{u}(t_2)\|_{L^\infty(\Omega)}\right].$$

For $N \geqslant 1$, since $Nr > 3N \geqslant 3$, $W^{N,r}$ is an algebra, so

$$\|\Delta\overline{q}\|_{W^{N,r}(\Omega)} \leqslant C\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{W^{N,r}(\Omega)}\left[\|\nabla\mathbf{u}(t_1)\|_{W^{N,r}(\Omega)} + \|\nabla\mathbf{u}(t_2)\|_{W^{N,r}(\Omega)}\right].$$

In either case, we have

$$\|\Delta\overline{q}\|_{W^{N,r}(\Omega)} \leqslant C\|\nabla\mathbf{u}\|_{L^\infty(0,T;W^{N,r}(\Omega))}\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{W^{N,r}(\Omega)}.$$

But, setting $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$,

$$\mathbf{u}(t_1) - \mathbf{u}(t_2) = K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}(t_1)] - K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}(t_2)] = K[\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)] + \mathbf{w}, \tag{9.10}$$

where

$$\mathbf{w} = \boldsymbol{\mathcal{V}}(t_1) - \boldsymbol{\mathcal{V}}(t_2) + \mathbf{u}_c(t_1) - \mathbf{u}_c(t_2).$$

Hence, applying Lemma 6.5,

$$\|\nabla\mathbf{u}(t_1) - \nabla\mathbf{u}(t_2)\|_{W^{N,r}(\Omega)} \leqslant C\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{W^{N,r}(\Omega)} + C\|\mathbf{w}\|_{W^{N,r}(\Omega)}. \tag{9.11}$$

Applying Lemma A.8,

$$\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{W^{N,r}(\Omega)} \leqslant \|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{C^N(\Omega)} \leqslant \|\boldsymbol{\omega}\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha.$$

Using Lemma A.8 again,

$$\begin{aligned}\|\mathbf{w}\|_{W^{N,r}(\Omega)} &\leqslant C\|\mathbf{w}\|_{C^N(\Omega)} \leqslant \|\mathbf{w}\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha \\ &\leqslant \|\boldsymbol{\mathcal{V}}\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha + \|\mathbf{u}\|_{L^\infty(0,T;H)}|t_1 - t_2|^\alpha \leqslant c_X|t_1 - t_2|^\alpha,\end{aligned}$$

where we also used Lemma 6.1. Hence,

$$\|\nabla\mathbf{u}(t_1) - \nabla\mathbf{u}(t_2)\|_{W^{N,r}(\Omega)} \leqslant c_X|t_1 - t_2|^\alpha. \tag{9.12}$$

On Γ_+,

$$\nabla\overline{q}\cdot\boldsymbol{n} = \partial_t U^{\boldsymbol{n}}(t_1) - \partial_t U^{\boldsymbol{n}}(t_2) + N[\mathbf{u}(t_2)] - N[\mathbf{u}(t_1)],$$

and we can see from the expression for $N[\mathbf{u}]$ in (8.2)—the key point being that on Γ_+, $N[\mathbf{u}]$ has no derivatives on $\mathbf{u}^{\boldsymbol{\tau}}$—that applying Lemma A.8 again,

$$\begin{aligned}&\|\nabla\overline{q}\cdot\boldsymbol{n}\|_{W^{N+1-\frac{1}{r},r}(\Gamma_+)} \\ &\quad\leqslant C\|\nabla\overline{q}\cdot\boldsymbol{n}\|_{C^{N+1}(\Gamma_+)} \leqslant \left[\|\mathbf{U}\|^2_{C^{N+2,\alpha}(Q)} + \|\mathbf{u}\|_{C^{N+1,\alpha}(Q)}\|\mathbf{U}\|_{C^{N+1,\alpha}(Q)}\right]|t_1 - t_2|^\alpha \\ &\quad\leqslant c_X|t_1 - t_2|^\alpha,\end{aligned}$$

where in the last inequality we used a bound like that in (9.12).

Along with Proposition 9.2, these bounds give $(9.9)_1$.

Since we set $r = 3/(1-\alpha)$, Sobolev embedding gives $W^{1,r}(\Omega') \subseteq C^\alpha(\Omega')$. Applying $(9.9)_1$ gives $(9.9)_2$. □

Remark 9.4. *It is only in the bound on* $\|\nabla\overline{q}\cdot\boldsymbol{n}\|_{W^{N+1-\frac{1}{r},r}(\Gamma_+)}$ *in the proof of Proposition 9.3 that we use the higher regularity of* $\mathbf{U}$ *over that of* $\mathbf{u}$.

To account for time derivatives $\partial_t^k q$, $k \leqslant N+1$, we note that (3.5) becomes

$$\begin{cases}\Delta\partial_t^k q = -\partial_t^k(\nabla\mathbf{u}\cdot(\nabla\mathbf{u})^T) & \text{in } \Omega,\\ \nabla\partial_t^k q\cdot\boldsymbol{n} = -\partial_t^{k+1}U^{\boldsymbol{n}} - \partial_t^k N[\mathbf{u}] & \text{on } \Gamma,\end{cases}$$

and the same analysis in Propositions 9.2 and 9.3 applies to $\partial_t^k q$. Then, from the key bound in (9.9), letting $Q' = [0,T]\times\Omega'$, we have

$$\|q(t_1) - q(t_2)\|_{C^{N+1,\alpha}(Q'))} \leqslant c_X|t_1 - t_2|^\alpha. \tag{9.13}$$

Moreover, applying the interpolation inequality in Lemma A.4 and Proposition 9.2, we have,

$$\begin{aligned}\|q(t_2) - q(t_1)\|_{C^{N+1}([0,T]\times\Gamma_+)} &\leqslant C\|q(t_1) - q(t_2)\|^a_{C^{N+1,\alpha}(Q')}\|q(t_2) - q(t_1)\|^{1-a}_{L^2(\Omega')} \\ &\leqslant C\left[c_X|t_1 - t_2|^\alpha\right]^a\left[c_X M|t_1 - t_2|^{\alpha'}\right]^{1-a} \leqslant c_X|t_2 - t_1|^{\alpha''},\end{aligned} \tag{9.14}$$

where $\alpha < \alpha'' := \alpha a + \alpha'(1-a) < \alpha'$ (using the value of a for $N+1$ in Lemma A.4). In the second inequality, we applied Proposition 9.2 and (9.13).

Then from (9.13) and (9.14) and using that $\|\nabla_\Gamma q(t_1) - \nabla_\Gamma q(t_2)\|_{C^\alpha(\Gamma_+))} \leqslant \|\nabla q(t_1) - \nabla q(t_2)\|_{C^\alpha(\Gamma_+))}$, we can apply Lemma A.7 with

$$F_1(t) = c_X t^\alpha, \qquad F_2(t) = c_X t^{\alpha''}$$

to obtain

$$\begin{aligned}\|\nabla_\Gamma q(t)\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} &\leqslant \|\nabla q(0)\|_{\dot{C}^{N,\alpha}(\Gamma_+)} + c_X T^{\alpha''} + c_X T^{\alpha} + c_X T^{\alpha''-\alpha} \\ &\leqslant c_0 + c_X T^a.\end{aligned} \tag{9.15}$$

We used here that $\nabla q(0)$ depends only upon the initial data along with Remark 5.2.

Part III: Estimates on the Operator A

Organization of Part III. In Section 10 we give the proof of Proposition 4.5 by first obtaining sufficient estimates on the operator A using (primarily) the pressure estimates from Section 9 along with the estimates on the flow map from Section 7. In Section 11, we use these estimates on A and the invariant set of Proposition 4.5 to prove Proposition 4.6. In Section 12, we give the proof of Proposition 4.7. In the final section of Part III, we prove Theorem 1.4.

10. An invariant set

We now make a series of estimates leading in Proposition 4.5 to the existence of an invariant set in $X^N_{\alpha',\alpha}$ for the operator A.

Proposition 10.1. *Assume that for $N \geqslant 0$ the data has regularity N, cond_N holds, and that $\mathbf{u} \in X^N_{\alpha',\alpha} \cap \text{Dom}_N(A)$. Then*

$$\begin{aligned}\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} &\leqslant \|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)} + MT^\alpha \leqslant c_0 + MT^\alpha, \\ \|\mathbf{H}\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} &\leqslant c_0 + c_X T^a,\end{aligned}$$

where $a = \min\{\alpha, \alpha'' - \alpha\} > 0$ (α'' is as in (9.14)).

Proof. By cond_0, $\mathbf{H}(0) = \boldsymbol{\omega}_0$ on Γ_+. Then, letting $\boldsymbol{\omega} = \text{curl}\,\mathbf{u}$, we have,

$$\begin{aligned}\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} &\leqslant \|\mathbf{H}((t,\mathbf{x}) - \mathbf{H}(0,\mathbf{x})\|_{L^\infty([0,T]\times\Gamma_+)} + \|\mathbf{H}(0,\mathbf{x})\|_{L^\infty(\Gamma_+)} \\ &\leqslant \sup_{[0,T]\times\Gamma_+} |\mathbf{H}(t,\mathbf{x}) - \mathbf{H}(0,\mathbf{x})| + \|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)} \\ &\leqslant \|\mathbf{H}\|_{\dot{C}^\alpha_t([0,T]\times\Gamma_+)} T^\alpha + \|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)} \leqslant \|\boldsymbol{\omega}\|_{\dot{C}^\alpha_t(Q)} T^\alpha + \|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)}.\end{aligned}$$

From (3.7), we can write,

$$\mathbf{H}^{\boldsymbol{\tau}} = \delta_1 + \delta_2 - \nabla_\Gamma q, \quad H^{\boldsymbol{n}} = \text{curl}_\Gamma\,\mathbf{U}^{\boldsymbol{\tau}},$$

where

$$\delta_1 := \frac{1}{U^{\boldsymbol{n}}}\left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(\frac{1}{2}|\mathbf{U}|^2\right) + \mathbf{f}\right]^\perp, \quad \delta_2 := \frac{1}{U^{\boldsymbol{n}}}\,\text{curl}_\Gamma\,\mathbf{U}^{\boldsymbol{\tau}}\mathbf{u}^{\boldsymbol{\tau}}.$$

Since $\mathbf{U} \in C^{N+2,\alpha}_\sigma(Q)$, we see that $\|\delta_1\|_{\dot{C}^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0$ and, applying Corollary A.9,

$$\begin{aligned}\|\delta_2\|_{\dot{C}^{N,\alpha}([0,T]\times\Gamma_+)} &\leqslant C\|\mathbf{u}^{\boldsymbol{\tau}}_0\|_{\dot{C}^{N,\alpha}(\Gamma_+)} + C\|\mathbf{u}^{\boldsymbol{\tau}}(t) - \mathbf{u}^{\boldsymbol{\tau}}_0\|_{\dot{C}^{N,\alpha}([0,T]\times\Gamma_+)} T^\alpha \\ &\leqslant c_0 + C\|\mathbf{u}\|_{C^{N,\alpha}(Q)} T^\alpha \leqslant c_0 + C\|\mathbf{u}\|_{X^N_{\alpha',\alpha}} T^\alpha \leqslant c_0 + c_X T^\alpha.\end{aligned}$$

With (9.15), then, we see that

$$\|\mathbf{H}\|_{\dot{C}^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0 + c_X T^a. \qquad \square$$

Proposition 10.2. *Assume that the data has regularity* $N \geqslant 0$ *and that* $\mathbf{u} \in X^N_{\alpha',\alpha} \cap \mathrm{Dom}_N(A)$*. With* Λ *as in (4.3),*

$$\|\Lambda \mathbf{u}\|_{L^\infty(Q)} \leqslant [\|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)} + MT^\alpha]e^{MT} \leqslant [c_0 + MT^\alpha]e^{MT},$$
$$\|\Lambda \mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant (1 + c_0)F_c(M, T) + c_X T^a,$$

for some $a > 0$*, where* F_c *is continuous and increasing in its arguments with* $F_c(M, 0) = c_0$*.*

Proof. First assume no forcing. Let $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$ and recall the definition of $\boldsymbol{\gamma}_0$ in (7.5). From (7.6), we can write, $\overline{\boldsymbol{\omega}} := \Lambda\mathbf{u} = \overline{\boldsymbol{\omega}}_\pm$ on $U_\pm$, where

$$\begin{aligned}\overline{\boldsymbol{\omega}}_-(t, \mathbf{x}) &= \nabla\eta(0, t; \boldsymbol{\gamma}_0)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0) \text{ on } U_-,\\ \overline{\boldsymbol{\omega}}_+(t, \mathbf{x}) &= \nabla\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x}))\mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) \text{ on } U_+.\end{aligned} \tag{10.1}$$

It follows, using Lemma 7.2 and Proposition 10.1, that

$$\|\overline{\boldsymbol{\omega}}_-(t, \mathbf{x})\|_{L^\infty(U_-)} \leqslant \|\nabla\eta\|_{L^\infty(Q)}\|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)} \leqslant \|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)}e^{MT},$$
$$\|\overline{\boldsymbol{\omega}}_+(t, \mathbf{x})\|_{L^\infty(U_+)} \leqslant \|\nabla\eta\|_{L^\infty(Q)}\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} \leqslant [\|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)} + MT^\alpha]e^{MT},$$

which yields our bound on $\|\Lambda\mathbf{u}\|_{L^\infty(Q)}$.

Let us now first treat the case $N = 0$, to get a better understanding of the estimates involved. Using Lemma 7.2 along with Lemmas A.1 and A.2, we see that

$$\begin{aligned}\|\overline{\boldsymbol{\omega}}_-\|_{C^\alpha(U_-)} &\leqslant \|\nabla\eta(0, t; \boldsymbol{\gamma}_0)\|_{C^\alpha(U_-)}\|\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0)\|_{C^\alpha(U_-)}\\ &\leqslant \|\nabla\eta(0, t; \cdot)\|_{C^\alpha(Q)}[\|\nabla\boldsymbol{\gamma}_0\|^\alpha_{L^\infty(U_-)}]^2\|\boldsymbol{\omega}_0\|_{C^\alpha(\Omega)}\\ &\leqslant \|\boldsymbol{\omega}_0\|_{C^\alpha(\Omega)}[1 + Me^{(1+2\alpha)MT}T^{1-\alpha}]e^{2MT}.\end{aligned}$$

Similarly,

$$\|\overline{\boldsymbol{\omega}}_+(t, \mathbf{x})\|_{C^\alpha(U_+)} \leqslant \|\nabla\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x}))\|_{C^\alpha(U_+)}\|\mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x}))\|_{C^\alpha(U_+)}.$$

Using Lemmas 7.2 and A.2,

$$\begin{aligned}&\|\nabla\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x}))\|_{C^\alpha(U_+)} \leqslant \|\nabla\eta(t_1, t_2; \mathbf{x})\|_{C^\alpha([0,T]^2\times\Omega)}[1 + \|D\mu\|_{L^\infty(Q)}]^\alpha\\ &\qquad \leqslant [e^{MT} + e^{(1+2\alpha)MT}MT^{1-a}][1 + \|D\mu\|_{L^\infty(Q)}]^\alpha\end{aligned}$$

and, using Lemma 7.2 and Proposition 10.1,

$$\begin{aligned}\|\mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x}))\|_{C^\alpha(U_+)} &\leqslant \|\mathbf{H}\|_{C^\alpha([0,T]\times\Gamma_+)}[1 + \|D\mu\|_{L^\infty(Q)}]^\alpha\\ &\leqslant [1 + \|D\mu\|_{L^\infty(U_+)}]^{2\alpha}[c_0 + c_X T^a].\end{aligned}$$

Again using Lemma 7.2, we see that

$$\|\overline{\boldsymbol{\omega}}_+(t, \mathbf{x})\|_{C^\alpha(U_+)} \leqslant [e^{MT} + e^{(1+2\alpha)MT}MT^{1-\alpha}][c_0 + c_X T^a][1 + \|D\mu\|_{L^\infty(Q)}]^{2\alpha}.$$

From (6.2) and our bound above on $\|\overline{\boldsymbol{\omega}}\|_{L^\infty(Q)}$, we have, for some $b < 1$,

$$\|\mathbf{u}\|_{L^\infty(Q)} \leqslant c_0 + [c_0 + MT^\alpha]e^{MT} + CMT^b,$$

so (7.2) gives

$$1 + \|D\mu\|_{L^\infty(Q)} \leqslant c_0[1 + \|\mathbf{u}\|^2_{L^\infty(Q)}]e^{MT} \leqslant c_0[[c_0 + MT^\alpha]^2 e^{2MT}]e^{MT} + CM^2T^{2b}.$$

Hence,

$$\begin{aligned}&\|\overline{\boldsymbol{\omega}}_+(t, \mathbf{x})\|_{C^\alpha(U_+)}\\ &\qquad \leqslant c_0[e^{MT} + e^{(1+2\alpha)MT}MT^{1-\alpha}][c_0 + c_X T^a][[c_0 + MT^\alpha]^4 e^{4\alpha MT} + M^4T^{4b}].\end{aligned}$$

But we know from Theorem 2.2 that $\overline{\boldsymbol{\omega}} \in C^\alpha(Q)$, because we assumed cond_0: hence, taking the maximum of the bounds for $\overline{\boldsymbol{\omega}}_\pm$ on $U_\pm$ leads to an estimate of the form,

$$\|\Lambda \mathbf{u}\|_{C^\alpha(Q)} \leqslant (1 + c_0) F_c(M, T) + c_X T^{a'},$$

where $a' > 0$, and where $F_c(M, 0) = c_0$. Including forcing only adds a $c_X T$ term to the bound, as we can see from (7.6), so an estimate of the same form holds with forcing.

Now consider $N \geqslant 1$. The expressions for $\overline{\boldsymbol{\omega}}_\pm$ in (10.1) each consist of two factors. We first apply Leibniz's product rule to these expressions then apply the chain rule to each term. For $\overline{\boldsymbol{\omega}}_+$, if β is a time-space multi-index with $|\beta| = N$, then $D^\beta \overline{\boldsymbol{\omega}}_+$ consists of a finite sum of terms of the form,

$$D^{\beta_1} \nabla \eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x})) D^{\beta_2} \mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) \prod_{\ell=1}^{n} D^{\beta_3^\ell} \mu(t, \mathbf{x}) \text{ on } U_+,$$

where $\beta_1 + \beta_2 = \beta$ and $\sum_{\ell=1}^n |\beta_3^\ell| = |\beta|$. The factors can be controlled by Proposition 10.1, Lemma 7.2, and (6.2). Following the similar process for $D^\beta \overline{\boldsymbol{\omega}}_-$ leads to an estimate for $\|\Lambda \mathbf{u}\|_{C^{N,\alpha}(Q)}$ of the same form as for $\|\Lambda \mathbf{u}\|_{C^\alpha(Q)}$. □

Having established our many estimates, we can now give the proof of Proposition 4.5.

Proof of Proposition 4.5. For $\mathbf{u} \in K$, Proposition 10.2 gives

$$\|\Lambda \mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant (1 + c_0) F_c(M, T) + c_X T^a.$$

Recalling, from the comment following Definition 5.1, that c_0 may increase with T, let $c_0(0) > 0$ be its value for $T = 0$. Start by choosing any

$$M > M_0 := \max\{(3(1 + c_0(0)))^{\frac{1}{a}}, 3\|P_{H_c} \mathbf{u}_0\|_{C_\sigma^{N+1,\alpha}(\Omega)}, 1\}, \tag{10.2}$$

which gives $(1 + c_0(0)) F_c(M, 0) < M/3$. Next, by continuity there exists $T > 0$ such that

$$(1 + c_0) F_c(M, T) \leqslant \frac{M}{3}.$$

We can choose $T > 0$ small enough that

$$c_X T^a \leqslant \frac{M}{3}.$$

It follows that

$$\|\operatorname{curl} A\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant \frac{2M}{3}.$$

Then, because $M > 3\|P_{H_c} \mathbf{u}_0\|_{C_\sigma^{N+1,\alpha}(\Omega)}$, we see from (2.7) that $\|A\mathbf{u}\|_{X_{\alpha',\alpha}} \leqslant M$, after again decreasing T if necessary. □

11. Continuity of the operator A

To prove Proposition 4.6, we first make some definitions and establish a few lemmas.

Throughout this section, we let M, T, and K be given as in Proposition 4.5. We assume that $\mathbf{u}_1, \mathbf{u}_2$ are two vector fields in K and, for $j = 1, 2$, we let $\boldsymbol{\omega}_j = \operatorname{curl} \mathbf{u}_j$, with η_j, τ_j, $\boldsymbol{\gamma}_j$, $U_\pm^j$, and S_j defined for the velocity field $\mathbf{u}_j$. We let $V_\pm = U_\pm^1 \cap U_\pm^2$, $W = Q \setminus (V_+ \cup V_-)$.

By virtue of Lemma 7.2, we have, for $j = 1, 2$,

$$\|\eta_j(0, \cdot; \cdot)\|_{C^{N+1,\alpha}(Q} \leqslant C(T, M). \tag{11.1}$$

We generally to not state the the dependence of constants on T and M, which are fixed and hence have no impact on the proof of Proposition 4.6. We do state such dependence explicitly, however, when it makes the nature of the bound being derived clearer. We define $\mu_j \colon U_+ \to [0,T] \times \Gamma_+$ by $\mu_j(t,\mathbf{x}) = (\tau_j(t,\mathbf{x}), \boldsymbol{\gamma}_j(t,\mathbf{x}))$. We let

$$\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mu := \mu_1 - \mu_2.$$

We fix $\beta \in (0,\alpha]$ arbitrarily and let

$$\theta_\beta := \|\mathbf{w}\|_{X_{\beta,\beta}} = \|\mathbf{w}\|_{C^\beta(Q)} + \|\operatorname{curl}\mathbf{w}\|_{C^\beta(Q)}. \tag{11.2}$$

Lemma 11.1. *We have,*

$$\|\mu\|_{L^\infty(V_+)} \leqslant C(T,M)T\theta_\beta.$$

Proof. We know from Lemma 3.5 of [9] that μ_j is transported by the flow map for $\mathbf{u}_j$; that is,

$$\begin{aligned}\partial_t\mu_1 + \mathbf{u}_1 \cdot \nabla\mu_1 = 0,\\ \partial_t\mu_2 + \mathbf{u}_2 \cdot \nabla\mu_2 = 0.\end{aligned}$$

Hence,

$$\partial_t\mu + \mathbf{u}_1 \cdot \nabla\mu = -\mathbf{w}\cdot\nabla\mu_2,$$

or,

$$\frac{d}{dt}\mu(t,\eta_1(0,t;\mathbf{x})) = -(\mathbf{w}\cdot\nabla\mu_2)(t,\eta_1(0,t;\mathbf{x})).$$

Integrating in time, using that $\mu(t,\eta_1(0,t;\mathbf{x}))|_{t=0} = 0$, and employing Lemma 7.2 gives

$$\begin{aligned}\mu(t,\eta_1(0,t;\mathbf{x})) = -\int_0^t g(\mathbf{w}\cdot\nabla\mu_2)(s,\eta_1(0,s;\mathbf{x})) &\leqslant \|\mathbf{w}\|_{L^\infty(Q)}\|\nabla\mu_2\|_{L^\infty(Q)}\\ &\leqslant C(T,M)\theta_\beta.\end{aligned}$$

□

Lemma 11.2. *We have*

$$\begin{aligned}\|\eta_1 - \eta_2\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant C(T,M)T\theta_\beta,\\ \|\nabla\eta_1 - \nabla\eta_2\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant C(T,M)T[\theta_\beta + \theta_\beta^\alpha].\end{aligned}$$

Proof. We have,

$$\eta_1(t_1,t_2;\mathbf{x}) - \eta_2(t_1,t_2;\mathbf{x}) = \int_{t_1}^{t_2} [\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x})) - \mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))]\, ds.$$

Fixing t_1, using (11.1), Lemma A.2, Lemma A.3, and applying Minkowski's integral inequality gives

$$\begin{aligned}&|\eta_1(t_1,t;\mathbf{x}) - \eta_2(t_1,t;\mathbf{x})|\\ &\quad\leqslant \int_{t_1}^t \|\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x})) - \mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))\|_{L^\infty}\, ds\\ &\qquad + \int_{t_1}^t \|\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x})) - \mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))\|_{L^\infty}\, ds\\ &\quad\leqslant \int_{t_1}^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{L^\infty}\, ds + \int_{t_1}^t \|\mathbf{u}_1(s)\|_{\dot{C}^1}\|\eta_1(t_1,s;\mathbf{x}) - \eta_2(t_1,s;\mathbf{x})\|_{L^\infty}\, ds\end{aligned}$$

$$\leqslant T\theta_\beta + C(T,M)\int_{t_1}^{t} \|\eta_1(t_1,s;\mathbf{x}) - \eta_2(t_1,s;\mathbf{x})\|_{L^\infty}\, ds.$$

Taking the supremum over $\mathbf{x}$ and applying Grönwall's Lemma gives

$$\|\eta_1(t_1,t;\mathbf{x}) - \eta_2(t_1,t;\mathbf{x})\|_{C([0,T];L^\infty(\Omega))} \leqslant Te^{C(M,T)T}\theta_\beta.$$

Since this holds uniformly for all $t_1, t \in [0,T]$, we obtain the first bound.

Similarly, starting from

$$\begin{aligned}\nabla\eta_1(t_1,t;\mathbf{x}) - \nabla\eta_2(t_1,t;\mathbf{x}) &= \int_{t_1}^{t} [\nabla_{\mathbf{x}}(\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))) - \nabla_{\mathbf{x}}(\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x})))]\, ds\\ &= \int_{t_1}^{t} [\nabla\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x}) - \nabla\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))\nabla\eta_2(t_1,s;\mathbf{x})]_{L^\infty}\, ds,\end{aligned}$$

we find

$$\begin{aligned}&|\nabla\eta_1(t_1,t;\mathbf{x}) - \nabla\eta_2(t_1,t;\mathbf{x})|\\ &\quad\leqslant \int_{t_1}^{t} \|\nabla\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x}) - \nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x})\|_{L^\infty}\, ds\\ &\qquad + \int_{t_1}^{t} \|(\nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x})) - \nabla\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x})))\nabla\eta_2(t_1,s;\mathbf{x})\|_{L^\infty}\, ds\\ &\qquad + \int_{t_1}^{t} \|\nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))(\nabla\eta_1(t_1,s;\mathbf{x}) - \nabla\eta_2(t_1,s;\mathbf{x}))\|_{L^\infty}\, ds\\ &\quad\leqslant \int_{t_1}^{t} \|\mathbf{u}_1(s)\|_{\dot{C}^\alpha}\|\eta_1(t_1,s;\mathbf{x}) - \eta_2(t_1,s;\mathbf{x})\|_{L^\infty}^\alpha \|\nabla\eta_1(x)\|_{L^\infty}\, ds\\ &\qquad + \int_{t_1}^{t} \|\nabla\mathbf{u}_1(s) - \nabla\mathbf{u}_2(s)\|_{L^\infty}\|\nabla\eta_2(s)\|_{L^\infty}\, ds\\ &\qquad + \int_{t_1}^{t} \|\mathbf{u}_1(s)\|_{\dot{C}^1}\|\nabla\eta_1(t_1,s;\mathbf{x}) - \nabla\eta_2(t_1,s;\mathbf{x})\|_{L^\infty}\, ds\\ &\quad\leqslant C(T,M)[Te^{(C(T,M)T}\theta_\beta]^\alpha T + C(M,T)T\theta_\beta\\ &\qquad + C(M,T)\int_{t_1}^{t} \|\nabla\eta_1(t_1,s;\mathbf{x}) - \nabla\eta_2(t_1,s;\mathbf{x})\|_{L^\infty}\, ds.\end{aligned}$$

In the last inequality, we used Lemma 6.5 to conclude that $\|\nabla\mathbf{u}_1(s) - \nabla\mathbf{u}_2(s)\|_{L^\infty(\Omega)} \leqslant \|\nabla\mathbf{w}(s)\|_{C^{1,\beta}(\Omega)} \leqslant C\|\operatorname{curl}\mathbf{w}(s)\|_{C^\beta(\Omega)} + C\|\mathbf{w}(s)\|_H \leqslant C\theta_\beta$. Taking the supremum over $\mathbf{x}$ and applying Grönwall's Lemma as before gives the second bound. □

Lemma 11.3. *Letting $|W|$ be the Lebesgue measure of $W := Q \setminus (V_+ \cup V_-)$, we have*

$$|W| \leqslant C(T,M)T^2\theta_\beta.$$

Proof. The set $W(t) := \{\mathbf{x} \in \Omega\colon (t,\mathbf{x}) \in W\}$ consists of all points lying between $S_1(t)$ and $S_2(t)$. Any $\mathbf{x}_1 \in S_1(t)$ is of the form $\mathbf{x}_1 = \eta_1(0,t;\mathbf{y})$ for some $\mathbf{y} \in \Gamma_+$, and by Lemma 11.2, the point $\mathbf{x}_2 = \eta_2(0,t;\mathbf{y})$ is within a distance $\delta = C(T,M)T\theta_\beta$ of $\mathbf{x}_1$. That is, any point in $S_1(t)$ is within a distance δ of $S_2(t)$ and the relation is symmetric. So

$$W(t) \subseteq W_\delta(t) := \{x \in \Omega\colon \operatorname{dist}(x, S_1(t)) \leqslant \delta\}.$$

As we observed in Section 7, $S_1(t)$ is at least $C^{1,\alpha}$ regular as a surface in Ω, and so has finite Hausdorff measure; hence, we can see that $|W_\delta(t)| \leqslant C\delta$. Moreover, this constant can depend upon T and M, but is bounded over $[0,T]$, for as also observed in Section 7, S_1 is at least $C^{1,\alpha}$ regular as a hypersurface in Q. Thus, $|W| \leqslant T|W_\delta(t)| \leqslant C(T,M)T^2\theta_\beta$. □

Proof of Proposition 4.6. Let $\mathbf{u}_1, \mathbf{u}_2 \in K$. We will obtain a bound in the following three steps:

(A) Bound the difference in vorticities, $\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$, assuming zero forcing.
(B) Account for forcing in the bound on $\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$.
(C) Account for the harmonic component of $\mathbf{u}_1$ and $\mathbf{u}_2$ to bound $A\mathbf{u}_1 - A\mathbf{u}_2$.

(A) Vorticity: Let $f \in C^{N,\alpha}(Q)$. By Lemma A.5,

$$\|f\|_{C^{N,\beta}(Q)} \leqslant \|f\|_{L^\infty(Q)} + F(\|f\|_{C^{N,\alpha}(Q)})\|f\|_{L^2(Q)}^{1-a}, \tag{11.3}$$

where $F(x) = x^{a_1} + x^{a_N} + x^{a'}$, a_n is given in Lemma A.4, and a' is given in Lemma A.5. The exponent a depends upon whether $\|f\|_{L^2(Q)}$ is greater or less than 1. Applying (11.3) with $f := \Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$, we see that

$$\|\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2\|_{C^{N,\beta}(Q)} \leqslant \|\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2\|_{L^\infty(Q)} + C(M)\|\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2\|_{L^2(Q)}^{1-a}, \tag{11.4}$$

since $F(\|\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2\|_{C^{N,\alpha}(Q)}) \leqslant M^{a_1} + M^{a_N} + M^{a'} \leqslant C(M)$. We conclude that to prove the continuity of Λ in the $C^{N,\beta}(Q)$ norm it suffices to obtain a bound on $\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$ in $L^\infty(Q)$.

Letting $(t,\mathbf{x}) \in Q$, we must estimate $|\Lambda\mathbf{u}_1(t,\mathbf{x}) - \Lambda\mathbf{u}_2(t,\mathbf{x})|$. This involves three cases: **(1)** $(t,\mathbf{x}) \in V_-$, **(2)** $(t,\mathbf{x}) \in V_+$, **(3)** $(t,\mathbf{x}) \in W$, which we consider separately. We argue first without forcing.

(1) Define, for $(t,\mathbf{x}) \in V_-$, $j = 1,2$,

$$\boldsymbol{\gamma}_0^j = \boldsymbol{\gamma}_0^j(t,\mathbf{x}) := \eta_j(t,0;\mathbf{x}). \tag{11.5}$$

From (7.6), we can write,

$$\Lambda\mathbf{u}_1(t,\mathbf{x}) - \Lambda\mathbf{u}_2(t,\mathbf{x}) = \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^2) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1)\cdot(\nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2)),\\ I_2 &:= (\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1) - \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^2))\cdot\nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2). \end{aligned}$$

We also make the decomposition, $I_1 = \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1)\cdot(I_1^1 + I_1^2)$, where

$$\begin{aligned} I_1^1 &:= \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1) - \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^2),\\ I_1^2 &:= \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^2) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2). \end{aligned}$$

Then,

$$\|I_1\|_{L^\infty(V_-)} \leqslant \|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)}\left(\|I_1^1\|_{L^\infty(V_-)} + \|I_1^2\|_{L^\infty(V_-)}\right),$$

with

$$\begin{aligned} \|I_1^1\|_{L^\infty(V_-)} &\leqslant \|\nabla\eta_1(0,t;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}(\Omega)}\|\eta_1(t,0;\cdot) - \eta_2(t,0;\cdot)\|_{L^\infty(\Omega)}^\alpha\\ &\leqslant C(T,M)T[T\theta_\beta]^\alpha \leqslant C(T,M)T^{1+\alpha}\theta_\beta^\alpha,\\ \|I_1^2\|_{L^\infty(V_-)} &\leqslant \|\nabla\eta_1(0,t;\cdot) - \nabla\eta_2(0,t;\cdot)\|_{L^\infty(\Omega)} \leqslant C(T,M)T[\theta_\beta + \theta_\beta^\alpha], \end{aligned}$$

where we applied Lemma 11.2. Similarly, applying Lemmas 11.2 and A.3,

$$\begin{aligned} \|I_2\|_{L^\infty(V_-)} &\leqslant \|\boldsymbol{\omega}_0\|_{\dot{C}^\alpha(\Omega)}\|\eta_1(t,0;\cdot) - \eta_2(t,0;\cdot)\|_{L^\infty(V_-)}^\alpha\|\nabla\eta_2(0,t,\cdot)\|_{L^\infty(V_-)}\\ &\leqslant C(T,M)M[C(T,M)T\theta_\beta]^\alpha. \end{aligned}$$

Dropping the dependence upon M or the initial data, which play no role here, we conclude

$$\|\Lambda \mathbf{u}_1(t,\mathbf{x}) - \Lambda \mathbf{u}_2(t,\mathbf{x})\|_{L^\infty(V_-)} \leqslant C(T)[\theta_\beta + \theta_\beta^\alpha].$$

(2) For $(t,\mathbf{x}) \in V_+$, we have

$$\begin{aligned}\Lambda \mathbf{u}_1(t,\mathbf{x}) - \Lambda \mathbf{u}_2(t,\mathbf{x}) &= \mathbf{H}_1(\mu_1(t,\mathbf{x})) \cdot \nabla\eta_1(\tau_1(t,\mathbf{x}), t; \boldsymbol{\gamma}_1(t,\mathbf{x})) \\ &\quad - \mathbf{H}_2(\mu_2(t,\mathbf{x})) \cdot \nabla\eta_2(\tau_2(t,\mathbf{x}), t; \boldsymbol{\gamma}_2(t,\mathbf{x})) \\ &= J_1 + J_2 + J_3,\end{aligned}$$

where $\mathbf{H}_j(t,\mathbf{x})$ is defined in (3.7) for $\mathbf{u}_j$, and

$$\begin{aligned}J_1 &:= \mathbf{H}_1(\mu_1(t,\mathbf{x})) \cdot (\nabla\eta_1(\tau_1(t,\mathbf{x}), t; \boldsymbol{\gamma}_1(t,\mathbf{x})) - \nabla\eta_2(\tau_1(t,\mathbf{x}), t; \boldsymbol{\gamma}_1(t,\mathbf{x}))), \\ J_2 &:= \mathbf{H}_1(\mu_1(t,\mathbf{x})) \cdot (\nabla\eta_2(\tau_1(t,\mathbf{x}), t; \boldsymbol{\gamma}_1(t,\mathbf{x})) - \nabla\eta_2(\tau_2(t,\mathbf{x}), t; \boldsymbol{\gamma}_2(t,\mathbf{x}))), \\ J_3 &:= (\mathbf{H}_1(\mu_1(t,\mathbf{x})) - \mathbf{H}_2(\mu_2(t,\mathbf{x})) \cdot \nabla\eta_2(\tau_2(t,\mathbf{x}), t; \boldsymbol{\gamma}_2(t,\mathbf{x})).\end{aligned}$$

Now, since $\mathbf{H}_j(s,\mathbf{y}) = \boldsymbol{\omega}_j(s,\mathbf{y})$ for $(s,\mathbf{y}) \in [0,T] \times \Gamma_+$, we have, using Lemma 11.2,

$$\|J_1\|_{L^\infty(V_+)} \leqslant \|\boldsymbol{\omega}_1\|_{L^\infty(Q)} \|\nabla\eta_1(\cdot,t;\cdot) - \nabla\eta_2(\cdot,t;\cdot))\|_{L^\infty(Q)} \leqslant C(T,M)[\theta_\beta + \theta_\beta^\alpha],$$

where we also used cond_0. For J_2, we have, using Lemmas 11.1 and A.3,

$$\begin{aligned}\|J_2\|_{L^\infty(V_+)} &\leqslant \|\boldsymbol{\omega}_1\|_{L^\infty(Q)} \|\nabla\eta_2\|_{\dot{C}^\alpha(Q)} \|(\tau_1(t,\mathbf{x}), \boldsymbol{\gamma}_1(t,\mathbf{x})) - (\tau_2(t,\mathbf{x}), \boldsymbol{\gamma}_2(t,\mathbf{x}))\|_{L^\infty(Q)}^\alpha \\ &\leqslant C(T,M)\|\mu\|_{L^\infty(U_+)}^\alpha \leqslant C(T,M)\theta_\beta^\alpha.\end{aligned}$$

For J_3, we have

$$J_3 \leqslant \|\mathbf{H}_1(\mu_1(t,\mathbf{x})) - \mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)} \|\nabla\eta_2\|_{L^\infty(Q)}.$$

But, $\|\nabla\eta_2\|_{L^\infty(Q)} \leqslant C(T,M)$ by Lemma 7.2, and, using Lemma A.3,

$$\begin{aligned}&\|\mathbf{H}_1(\mu_1(t,\mathbf{x})) - \mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)} \\ &\quad \leqslant \|\mathbf{H}_1(\mu_1(t,\mathbf{x})) - \mathbf{H}_2(\mu_1(t,\mathbf{x})\|_{L^\infty(U_+)} + \|\mathbf{H}_2(\mu_1(t,\mathbf{x})) - \mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)} \\ &\quad \leqslant \|\mathbf{H}_1 - \mathbf{H}_2\|_{L^\infty([0,T]\times\Gamma_+)} + \|\mathbf{H}_1 - \mathbf{H}_2\|_{\dot{C}^\alpha([0,T]\times\Gamma_+)} \|\mu\|_{L^\infty}^\alpha \\ &\quad \leqslant \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{L^\infty([0,T]\times\Gamma_+)} + C(T,M)\theta_\beta^\alpha \leqslant C(T,M)[\theta_\beta + \theta_\beta^\alpha],\end{aligned}$$

where in the second-to-last inequality we used the bounds on $\mathbf{H}_1$ and $\mathbf{H}_2$ from Proposition 10.1 and appealed to cond_0.

Combined, we see that

$$\|\Lambda \mathbf{u}_1(t,\mathbf{x}) - \Lambda \mathbf{u}_2(t,\mathbf{x})\|_{L^\infty(V_+)} \leqslant C(T,M)[\theta_\beta + \theta_\beta^\alpha].$$

(3) Now assume $(t,\mathbf{x}) \in W$. Applying Lemma A.10 with Lipschitz modulus of continuity, $r \mapsto \|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{\dot{C}^\alpha} r \leqslant Mr$,

$$\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)} \leqslant F_M\left(\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^2(W)}\right)$$

for a continuous function F_M with $F_M(0) = 0$. From Lemma 11.3,

$$\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^2(W)} \leqslant \|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)} |W|^{\frac{1}{2}} \leqslant CM|W|^{\frac{1}{2}} \leqslant C(T,M)\theta_\beta,$$

which then gives a bound on $\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)}$.

(B) Accounting for forcing: To treat forcing, let $\mathbf{G}_\pm^j$ be given by (7.6) for η_j. Then

$$\begin{aligned}&\|\mathbf{G}_\pm^1 - \mathbf{G}_\pm^2\|_{L^\infty(V_\pm)} \\ &\quad \leqslant \int_0^T \|\nabla\eta_1(s,t;\eta_1(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x})) - \nabla\eta_2(s,t;\eta_2(t,s;\mathbf{x}))\mathbf{g}(s,\eta_2(t,s;\mathbf{x}))\|_{L^\infty(\Omega)}\, ds.\end{aligned}$$

But,

$$\begin{aligned}
&\|\nabla\eta_1(s,t;\eta_1(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x})) - \nabla\eta_2(s,t;\eta_2(t,s;\mathbf{x}))\mathbf{g}(s,\eta_2(t,s;\mathbf{x}))\|_{L^\infty(\Omega)} \\
&\quad \leqslant \|\nabla\eta_1(s,t;\eta_1(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x})) - \nabla\eta_2(s,t;\eta_1(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x}))\|_{L^\infty(\Omega)} \\
&\qquad + \|\nabla\eta_2(s,t;\eta_1(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x})) - \nabla\eta_2(s,t;\eta_2(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x}))\|_{L^\infty(\Omega)} \\
&\qquad + \|\nabla\eta_2(s,t;\eta_2(t,s;\mathbf{x}))\mathbf{g}(s,\eta_1(t,s;\mathbf{x})) - \nabla\eta_2(s,t;\eta_2(t,s;\mathbf{x}))\mathbf{g}(s,\eta_2(t,s;\mathbf{x}))\|_{L^\infty(\Omega)} \\
&\quad \leqslant \|\nabla\eta_1 - \nabla\eta_2\|_{L^\infty([0,T]^2\times\Omega)}\|\mathbf{g}\|_{L^\infty(Q)} + \|\nabla\eta_2\|_{\dot{C}^\alpha([0,T]^2\times\Omega)}\|\nabla\eta_1 - \nabla\eta_2\|^\alpha_{L^\infty([0,T]^2\times\Omega)}\|\mathbf{g}\|_{L^\infty(Q)} \\
&\qquad + \|\nabla\eta_2\|_{L^\infty([0,T]^2\times\Omega))}\|\mathbf{g}\|_{\dot{C}^\alpha}\|\eta_1 - \eta_2\|^\alpha_{L^\infty(Q)},
\end{aligned}$$

where we used Lemmas A.2 and A.3.

Since $\mathbf{g} \in L^\infty(Q)$, while $\nabla\eta_1$ and $\nabla\eta_2$ are bounded in $\dot{C}^\alpha([0,T]^2 \times \Omega)$, by Lemma 11.2 we see that

$$\|\mathbf{G}^1_\pm - \mathbf{G}^2_\pm\|_{L^\infty(V_\pm)} \leqslant CT[\theta_\beta + \theta_\beta^\alpha].$$

Hence, the inclusion of forcing does not change our bounds on $\|\Lambda\mathbf{u}_1(t,\mathbf{x}) - \Lambda\mathbf{u}_2(t,\mathbf{x})\|_{L^\infty(V_\pm)}$ in **(1)**, **(2)**. And $\mathbf{G}^1_\pm$, $\mathbf{G}^2_\pm$ are bounded on Q, so the estimate on $\|\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2\|_{L^2(W)}$ in **(3)** is also unchanged.

(C) Velocity: It remains to deal with the harmonic component of $\mathbf{v}_1 - \mathbf{v}_2$. Let $\mathbf{\Omega}_j = \nabla K[\Lambda\mathbf{u}_j] - (\nabla K[\Lambda\mathbf{u}_j])^T$, as in (2.7). We have that

$$P_{H_c}\mathbf{v}_j(t) := P_{H_c}\mathbf{u}_j(0) + \int_0^t P_{H_c}\mathbf{f}(s)\,ds - \int_0^t P_{H_c}P_H\left(\mathbf{\Omega}_j(s)\mathbf{u}_j(s)\right)\,ds.$$

By Lemma 6.1, $\|P_{H_c}\mathbf{u}\|_{L^\infty(Q)} \leqslant C\|\mathbf{u}\|_H$ for any $\mathbf{u} \in H$, so, noting that $\mathbf{v}_1(t) - \mathbf{v}_2(t) \in H$,

$$\begin{aligned}
\|P_{H_c}(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^\infty(Q)} &\leqslant \|\mathbf{u}_1(0) - \mathbf{u}_2(0)\|_H + \int_0^t \|P_H\left(\mathbf{\Omega}_1\mathbf{u}_1 - \mathbf{\Omega}_2\mathbf{u}_2\right)(s)\|_H\,ds \\
&\leqslant C\theta_\beta + \int_0^t \|\left(\mathbf{\Omega}_1\mathbf{u}_1 - \mathbf{\Omega}_2\mathbf{u}_2\right)(s)\|_{L^2(\Omega)}\,ds.
\end{aligned}$$

But,

$$\begin{aligned}
&\int_0^t \|\left(\mathbf{\Omega}_1\mathbf{u}_1 - \mathbf{\Omega}_2\mathbf{u}_2\right)(s)\|_{L^2(\Omega)}\,ds \\
&\quad \leqslant \int_0^t \|\mathbf{\Omega}_1(s)(\mathbf{u}_1 - \mathbf{u}_2)(s)\|_{L^2(\Omega)}\,ds + \int_0^t \|(\mathbf{\Omega}_1 - \mathbf{\Omega}_2)(s)\mathbf{u}_2(s)\|_{L^2(\Omega)}\,ds \\
&\quad \leqslant \int_0^t \|\mathbf{\Omega}_1(s)\|_{L^\infty(\Omega)}\|(\mathbf{u}_1 - \mathbf{u}_2)(s)\|_{L^2(\Omega)}\,ds + \int_0^t \|(\mathbf{\Omega}_1 - \mathbf{\Omega}_2)(s)\|_{L^\infty(\Omega)}\|\mathbf{u}_2(s)\|_{L^\infty(\Omega)}\,ds \\
&\quad \leqslant MT\theta_\beta,
\end{aligned}$$

where we used that the nonzero components of $\mathbf{\Omega}_j$ come from $\Lambda\mathbf{u}_j$.

Applying (11.4), we conclude that

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{L^\infty(Q)} \leqslant C(M,T)[\|\mathbf{u}_1 - \mathbf{u}_2\|_{C^\beta(Q)} + \|\mathbf{u}_1 - \mathbf{u}_2\|^\alpha_{C^\beta(Q)}],$$

which shows that $A\colon K \to K$ is continuous in the $X_{\beta,\beta}$ norm. □

12. Full inflow boundary condition satisfied

We are ready to prove Proposition 4.7, which shows that a solution satisfying $(1.5)_{1\text{-}4}$ also satisfies $(1.5)_5$. This can be done by defining $\mathbf{H}$ by (3.7) and recovering the pressure using $N[\mathbf{u}]$ of (3.6), as already observed in [2].

Proof of Proposition 4.7. Our proof is inspired by the proof of Lemma 4.2.1 pages 156-159 of [2]. Let

$$\mathbf{w} = \mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}}, \quad P := p - q.$$

By Proposition 3.1, $\boldsymbol{\omega} = \mathbf{W}[\mathbf{u}, p]$ on $[0, T] \times \Gamma_+$, where we recall that $\mathbf{W}[\mathbf{u}, p]$ is defined in (3.4). From (9.1), (3.5), and (3.6), we see that on Γ_+, $\nabla P \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w})$. Hence, P satisfies

$$\begin{cases} \Delta P = 0 & \text{in } \Omega, \\ \nabla P \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_- \cup \Gamma_0, \\ \nabla P \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w}) & \text{on } \Gamma_+. \end{cases}$$

Multiplying by P and integrating over Ω gives

$$\|\nabla P\|^2_{L^2(\Omega)} = -(\Delta P, P) + \int_{\Gamma_+} (\nabla P \cdot \boldsymbol{n})P = \int_{\Gamma_+} \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w})P = -\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \nabla_\Gamma P. \quad (12.1)$$

By (3.3) and the assumption that $\mathbf{H} = \boldsymbol{\omega}$ on Γ_+, we know that $U^{\boldsymbol{n}}[\mathbf{H}^{\boldsymbol{\tau}}]^\perp = U^{\boldsymbol{n}}[\mathbf{W}^{\boldsymbol{\tau}}[\mathbf{u}, p]]^\perp$. Using also that $(\mathbf{v}^\perp)^\perp = -\mathbf{v}$, we have, from (3.4) and (3.7), that on Γ_+,

$$\begin{aligned} \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(q + \frac{1}{2}|\mathbf{U}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} + \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp &= \mathbf{H} \\ = \boldsymbol{\omega} = \partial_t \mathbf{u}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) &- \mathbf{f}^{\boldsymbol{\tau}} + \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp. \end{aligned}$$

Subtracting the left hand side from the right hand side, we have

$$0 = \nabla_\Gamma P + \frac{1}{2}\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) + \partial_t \mathbf{w} + \operatorname{curl}_\Gamma \mathbf{w} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp.$$

But, $\omega^{\boldsymbol{n}} = H^{\boldsymbol{n}}$ on Γ_+, which gives $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}$. Hence, $\operatorname{curl}_\Gamma \mathbf{w} = 0$, so

$$\nabla_\Gamma P = -\partial_t \mathbf{w} - \frac{1}{2}\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2).$$

Returning to (12.1), we thus have

$$\|\nabla P\|^2_{L^2(\Omega)} = \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \partial_t \mathbf{w} + \frac{1}{2}\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2).$$

Now,

$$\begin{aligned} \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \partial_t \mathbf{w} &= \frac{1}{2}\int_{\Gamma_+} U^{\boldsymbol{n}} \partial_t |\mathbf{w}|^2 = \frac{1}{2}\int_{\Gamma_+} \partial_t [U^{\boldsymbol{n}}|\mathbf{w}|^2] - \frac{1}{2}\int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}\, |\mathbf{w}|^2 \\ &= \frac{1}{2}\frac{d}{dt}\int_{\Gamma_+} U^{\boldsymbol{n}}|\mathbf{w}|^2 - \frac{1}{2}\int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}\, |\mathbf{w}|^2, \end{aligned}$$

so

$$\frac{d}{dt}\int_{\Gamma_+} U^{\boldsymbol{n}}|\mathbf{w}|^2 = \int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}|\mathbf{w}|^2 - \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) + 2\|\nabla P\|^2_{L^2(\Omega)}. \quad (12.2)$$

Writing $|\mathbf{U}|^2 - |\mathbf{u}|^2 = |\mathbf{u}^{\boldsymbol{\tau}}|^2 - |\mathbf{U}^{\boldsymbol{\tau}}|^2 = \mathbf{w}\cdot\mathbf{v}$ on Γ_+, since $U^{\boldsymbol{n}} = u^{\boldsymbol{n}}$, where $\mathbf{v} := \mathbf{U}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}$, we have

$$\begin{aligned}\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) &= \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(\mathbf{w}\cdot\mathbf{v})\\ &= \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{v})\cdot\mathbf{w} + \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{w})\cdot\mathbf{v}\\ &= \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{v})\cdot\mathbf{w} - \frac{1}{2}\int_{\Gamma_+}|\mathbf{w}|^2\operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{v}).\end{aligned}$$

For the last term above, we used that $U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{w})\cdot\mathbf{v} = (1/2)U^{\boldsymbol{n}}\mathbf{v}\cdot\nabla_\Gamma|\mathbf{w}|^2$ and integrated by parts via Lemma B.1. Then because $\mathbf{v}$ and $U^{\boldsymbol{n}}$ are sufficiently regular, we have

$$\left|\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2)\right| \leqslant C\int_{\Gamma_+}|\mathbf{w}|^2.$$

Changing sign in (12.2) and integrating in time, we see that

$$\begin{aligned}\int_{\Gamma_+}|U^{\boldsymbol{n}}(t)||\mathbf{w}(t)|^2 &= -\int_{\Gamma_+} U^{\boldsymbol{n}}(t)|\mathbf{w}(t)|^2\\ &\leqslant -\int_0^t\int_{\Gamma_+}\partial_t U^{\boldsymbol{n}}|\mathbf{w}|^2 + \int_0^t\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) - 2\int_0^t\|\nabla P\|^2_{L^2(\Omega)}\\ &\leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds - 2\int_0^t\|\nabla P\|^2_{L^2(\Omega)} \leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds.\end{aligned}$$

In the first equality we used that $U^{\boldsymbol{n}} < 0$ on Γ_+, in the second equality we used that $\mathbf{w}(0) = 0$, and in the third equality we used that $\partial_t U^{\boldsymbol{n}}$ is bounded.

Now since $|U^{\boldsymbol{n}}|$ is bounded away from zero, we have

$$\int_{\Gamma_+}|\mathbf{w}(t)|^2 \leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds,$$

and we conclude from Grönwall's Lemma that $\mathbf{w}\equiv 0$. This means that $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{U}^{\boldsymbol{\tau}}$, so $(1.5)_5$ holds. ☐

Remark 12.1. *If $\Gamma_0 = \Gamma$, the classical setting of impermeable boundary conditions on the whole boundary, our proof of existence and uniqueness still applies, though a number of things trivialize. First, no vorticity is transported off of the boundary, so there is no need for the pressure estimates in Section 9, and U_- is all of Q, so many of the flow map constructs, such as S, τ, and $\boldsymbol{\gamma}$ are unnecessary. And, of course, none of the estimates involving U_+ are needed. The bound on the time of existence is still finite, however.*

13. Vorticity boundary conditions

Proof of Theorem 1.4. The proof of existence is the same as that for Theorem 1.2, though with substantial simplifications. Because $\mathbf{H}$ is given with sufficient regularity, it satisfies

$$\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} \leqslant c_0, \quad \|\mathbf{H}\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0.$$

Hence, there are no pressure estimates involved, so the condition in (1.11) immediately gives (2.2), and there is no need to appeal to Proposition 3.8. Since we only require $\mathbf{u}\cdot\boldsymbol{n} = U^{\boldsymbol{n}}$ on

Γ_+, we simplify the definition of $\mathrm{Dom}_N(A)$ in (4.1) to

$$\mathrm{Dom}_N(A) := \{\mathbf{u} \in C^{N+1,\alpha}_\sigma(Q) \colon \mathbf{u}(0) = \mathbf{u}_0\},$$

and there is no need to invoke Proposition 4.7 or Lemma 6.4. Otherwise, the remainder of the proof of existence proceeds unchanged.

For uniqueness when $N \geqslant 1$, let $\boldsymbol{\omega}_j = \operatorname{curl} \mathbf{u}_j$, $j = 1, 2$, and let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then $\mathbf{w} \in H_0$, since $\mathbf{u}_1$, $\mathbf{u}_2$ have the same prescribed harmonic component, $\mathbf{u}_c$. Let

$$\boldsymbol{\mu} := \operatorname{curl} \mathbf{w} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2.$$

Since $N \geqslant 1$, we have enough regularity to write $\partial_t \boldsymbol{\omega}_j + \mathbf{u}_j \cdot \nabla \boldsymbol{\omega}_j = \boldsymbol{\omega}_j \cdot \nabla \mathbf{u}_j + \operatorname{curl} \mathbf{f}$, and subtracting this relation for $j = 2$ from that for $j = 1$ gives

$$\partial_t \boldsymbol{\mu} + \mathbf{u}_1 \cdot \nabla \boldsymbol{\mu} + \mathbf{w} \cdot \nabla \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 \cdot \nabla \mathbf{w} + \boldsymbol{\mu} \cdot \nabla \mathbf{u}_2. \tag{13.1}$$

Multiplying by $\boldsymbol{\mu}$, integrating over Ω, and using that $(\mathbf{u}_1 \cdot \nabla \boldsymbol{\mu}, \boldsymbol{\mu}) = (1/2)(\mathbf{u}_1, \nabla |\boldsymbol{\mu}|^2)$, gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\mu}\|^2 + \frac{1}{2} \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 &= -(\mathbf{w} \cdot \nabla \boldsymbol{\omega}_2, \boldsymbol{\mu}) + (\boldsymbol{\omega}_1 \cdot \nabla \mathbf{w}, \boldsymbol{\mu}) + (\boldsymbol{\mu} \cdot \nabla \mathbf{u}_2, \boldsymbol{\mu}) \\ &\leqslant \frac{1}{2} \|\nabla \boldsymbol{\omega}_2\|_{L^\infty} \|\mathbf{w}\|^2 + \frac{1}{2} \|\boldsymbol{\mu}\|^2 + \frac{1}{2} \|\boldsymbol{\omega}_1\|_{L^\infty} \|\nabla \mathbf{w}\|^2 + \frac{1}{2} \|\boldsymbol{\mu}\|^2 + \|\nabla \mathbf{u}_2\|_{L^\infty} \|\boldsymbol{\mu}\|^2, \end{aligned} \tag{13.2}$$

where $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ here. As in the proof of Lemma 6.5, elements of H have mean zero, so by Poincaré's inequality, $\|\mathbf{w}\| \leqslant C \|\nabla \mathbf{w}\|$. Moreover, since $\mathbf{w} \in H_0$, we have $\|\nabla \mathbf{w}\| \leqslant C \|\boldsymbol{\mu}\|$ and so obtain

$$\frac{d}{dt} \|\boldsymbol{\mu}\|^2 \leqslant - \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 + C \|\boldsymbol{\mu}\|^2.$$

We note that $\nabla \boldsymbol{\omega}_2 \in L^\infty([0, T] \times \Omega)$ by the $N = 1$ existence result. But,

$$- \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 = \int_\Omega \operatorname{div} \mathbf{u}_1 \, |\boldsymbol{\mu}|^2 - \int_\Gamma U^{\boldsymbol{n}} |\boldsymbol{\mu}|^2 = - \int_{\Gamma_-} U^{\boldsymbol{n}} |\boldsymbol{\mu}|^2 \leqslant 0,$$

so we conclude from Gronwall's lemma, since $\boldsymbol{\mu}(0) = 0$, that $\boldsymbol{\mu} \equiv 0$. That is, $\mathbf{u}_1 = \mathbf{u}_2$.

Finally, from $(1.10)_1$, we have

$$\partial_t \mathbf{u}^{\boldsymbol{\tau}} + (\mathbf{u} \cdot \nabla \mathbf{u})^{\boldsymbol{\tau}} = (\mathbf{f} - \nabla p)^{\boldsymbol{\tau}} + \mathbf{z}^{\boldsymbol{\tau}}.$$

From cond_0, then, we see that $\mathbf{z}^{\boldsymbol{\tau}}(0) = 0$. Since also $z^{\boldsymbol{n}}(0) = 0$, we know that $\mathbf{z}(0) = 0$. □

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Appendix A. Hölder space lemmas

We collect here a number of estimates in Hölder spaces, which we use throughout much of this paper. We include proofs only of the less standard ones.

Lemma A.1. *Let $f, g \in C^\alpha(U)$. Then*

$$\begin{aligned}
\|fg\|_{C^\alpha} &\leqslant \|f\|_{C^\alpha}\|g\|_{C^\alpha}, \\
\|fg\|_{\dot{C}^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{\dot{C}^\alpha}, \\
\|fg\|_{C^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{\dot{C}^\alpha}, \\
&\leqslant \|f\|_{L^\infty}\|g\|_{C^\alpha} + \|g\|_{L^\infty}\|f\|_{C^\alpha}, \\
\|fg\|_{C^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{C^\alpha}.
\end{aligned}$$

Also, for any $\beta \in (0, \alpha)$, allowing $\alpha = 1$, we have the interpolation inequality,

$$\|f\|_{\dot{C}^\beta} \leqslant 2\|f\|_{\dot{C}^\alpha}^{\frac{\beta}{\alpha}}\|f\|_{L^\infty}^{1-\frac{\beta}{\alpha}}.$$

Lemma A.2. *Let U, V be open subsets of Euclidean spaces, $\alpha \in (0, 1]$, and $k \geqslant 1$ an integer. If $f \in C^{k,\alpha}(U)$ and $g \in C^{k+1,\alpha}(V)$ with $g(V) \subseteq U$ then*

$$\begin{aligned}
&\|f \circ g\|_{\dot{C}^\alpha(V)} \leqslant \|f\|_{\dot{C}^\alpha(U)}\|g\|_{Lip(V)}^\alpha, \\
&\|f \circ g\|_{C^\alpha(V)} \leqslant \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)}\|g\|_{Lip(V)}^\alpha \leqslant \|f\|_{C^\alpha(U)}\left[1 + \|g\|_{Lip(V)}^\alpha\right], \\
&\|f \circ g\|_{C^{k,\alpha}(V)} \leqslant C(k)\|f\|_{C^{k,\alpha}(U)}\left[1 + \|g\|_{C^{k+1}(V)}\right]^{k+1},
\end{aligned} \tag{A.1}$$

where Lip is the homogeneous Lipschitz semi-norm and $\dot{C}^\alpha$ is the homogeneous Hölder norm.

Lemma A.3. *Let U, V be open subsets of $\mathbb{R}^d$, $d \geqslant 1$, and let $\alpha \in (0, 1]$. Assume that the domain of f is U and the domains of g and h are V, with $g(V), h(V) \subseteq U$. Then*

$$\|f \circ g - f \circ h\|_{L^\infty(V)} \leqslant \|f\|_{\dot{C}^\alpha(U)}\|g - h\|_{L^\infty(V)}^\alpha.$$

We also have the following interpolation-like inequality:

Lemma A.4. *Let U be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, let $n \geqslant 1$, and $\nabla^n f \in C^\alpha(U)$. Then*

$$\|\nabla^n f\|_{L^\infty(U)} \leqslant C\|f\|_{C^{n,\alpha}(U)}^a\|f\|_{L^2(U)}^{1-a},$$

where

$$a = a_n = \frac{2n + d}{2n + d + 2\alpha} < 1.$$

Proof. First extend f continuously to all of $\mathbb{R}^d$ in all Sobolev and Hölder spaces, as can be done using the extension operator in Theorem $5'$, chapter VI of [25]. Applying a cutoff function, we can insure that the extension, which we continue to call f, has support with a diameter no more than twice $\operatorname{diam}(U)$.

Then

$$\|\nabla^n f\|_{L^\infty(U)} = \sup_{\mathbf{x} \in \operatorname{supp} f}|\nabla^n f(\mathbf{x})| = \sup_{\mathbf{x} \in \operatorname{supp} f}|\nabla^n f(\mathbf{x}) - \nabla^n f(\mathbf{x}_0)| \leqslant R,$$

where $\mathbf{x}_0$ is a fixed point in $(\operatorname{supp} f)^C$ and

$$R = \sup_{\mathbf{x} \in \operatorname{supp} f}|\mathbf{x} - \mathbf{x}_0|^\alpha \sup_{\mathbf{x} \in \operatorname{supp} f}\frac{|\nabla^n f(\mathbf{x}) - \nabla^n f(\mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|^\alpha} = \sup_{\mathbf{x} \in \operatorname{supp} f}|\mathbf{x} - \mathbf{x}_0|^\alpha\|\nabla^n(f(s\cdot))\|_{\dot{C}^\alpha(\mathbb{R}^d)}.$$

In particular,

$$\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leqslant R + \|f\|_{L^2(\mathbb{R}^d)} \tag{A.2}$$

for all $f \in C_0^\infty(\mathbb{R}^d)$.

Following the scaling argument in the proof of Proposition 13.3.4 of [27], we write (A.2) schematically in the form $Q \leqslant R + P$. Replacing $f(\cdot)$ with $f(s\cdot)$, we have $\nabla^n(f(s\mathbf{x})) = s^n \nabla f(s\mathbf{x})$. This gives $\|\nabla^n(f(s\cdot))\|_{L^\infty(\mathbb{R}^d)} = s^n\|\nabla f\|_{L^\infty(\mathbb{R}^d)}$ and $\|f(s\cdot)\|_{L^2(\mathbb{R}^d)} = s^{-\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^d)}$. Also, R becomes

$$\sup_{\mathbf{x}\in \operatorname{supp} f} |s\mathbf{x} - s\mathbf{x}_0|^\alpha \sup_{\mathbf{x}\in \operatorname{supp} f} s^n \frac{|\nabla^n f(s\mathbf{x}) - \nabla^n f(s\mathbf{x}_0)|}{|s\mathbf{x} - s\mathbf{x}_0|^\alpha} = s^{n+\alpha} R.$$

Thus, $Q \leqslant R + P$ becomes

$$s^n Q \leqslant s^{n+\alpha} R + s^{-\frac{d}{2}} P \implies Q \leqslant s^\alpha R + s^{-(n+\frac{d}{2})} P.$$

As in [27], we conclude that

$$\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\nabla^n f\|^a_{\dot{C}^\alpha(\mathbb{R}^d)} \|f\|^{1-a}_{L^2(\mathbb{R}^d)} \leqslant C\|\nabla^n f\|^a_{\dot{C}^\alpha(U)} \|f\|^{1-a}_{L^2(U)}$$

as long as $\alpha a = (n + \frac{d}{2})(1-a)$, which gives the stated value of a and the stated estimate, using the continuity of the extension operator. □

The inequality in Lemma A.4 is similar to that in the lemma on page 126 of [22], used by the authors of [2] (for $N = 0$).

Lemma A.5. *Let U be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, let $n \geqslant 1$, and suppose that $f \in C^{n,\alpha}(U)$. Let a_n be as in Lemma A.4. For any $\beta \in (0, \alpha)$,*

$$\begin{aligned} \|f\|_{C^{n,\beta}(U)} \leqslant \|f\|_{L^\infty(U)} &+ C\left[\|f\|^{a_1}_{C^{n,\alpha}(U)} + \|f\|^{a_n}_{C^{n,\alpha}(U)}\right]\left[\|f\|^{1-a_1}_{L^2(U)} + \|f\|^{1-a_n}_{L^2(U)}\right] \\ &+ C\|f\|^{a'}_{C^{n,\alpha}(U)}\|f\|^{1-a'}_{L^2(U)}, \end{aligned}$$

where

$$a' = (\beta/\alpha) + a_n(1 - \beta/\alpha) < 1.$$

On $[0,T] \times \Gamma_+$, by the regularity of Γ_+, we have the following equivalent formulations of Hölder norms (a simulation formulation holds for any time-space domain, such as Q and $U_\pm$):

$$\begin{aligned} \|f\|_{\dot{C}^\alpha_t([0,T]\times\Gamma_+)} &:= \sup_{\substack{(t_1,\mathbf{x})\neq(t_2,\mathbf{x}) \\ \text{in } [0,T]\times\Gamma_+}} \frac{|f(t_1,\mathbf{x}) - f(t_2,\mathbf{x})|}{|t_1 - t_2|^\alpha} = \sup_{\mathbf{x}\in\Gamma_+} \|f(\cdot,\mathbf{x})\|_{\dot{C}^\alpha([0,T])}, \\ \|f\|_{\dot{C}^\alpha_{\mathbf{x}}([0,T]\times\Gamma_+)} &:= \sup_{\substack{(t,\mathbf{x}_1)\neq(t,\mathbf{x}_2) \\ \text{in } [0,T]\times\Gamma_+}} \frac{|f(t,\mathbf{x}_1) - f(t,\mathbf{x}_2)|}{|\mathbf{x}_1 - \mathbf{x}_2|^\alpha} = \sup_{t\in[0,T]} \|f(t)\|_{\dot{C}^\alpha(\Gamma_+)}, \\ \|f\|_{\dot{C}([0,T]\times\Gamma_+)} &:= \|f\|_{\dot{C}^\alpha_t([0,T]\times\Gamma_+)} + \|f\|_{\dot{C}^\alpha_{\mathbf{x}}([0,T]\times\Gamma_+)}. \end{aligned} \tag{A.3}$$

Lemma A.6. *Let $\alpha \in (0,1]$ and assume that $f\colon [0,T] \times \Gamma_+ \to \mathbb{R}$ is a continuous function with the properties that for all $t_1, t_2 \in [0,T]$*

- $\|f(t_1) - f(t_2)\|_{\dot{C}^\alpha(\Gamma_+)} \leqslant F_1(|t_1 - t_2|)$;
- $\|f(t_1) - f(t_2)\|_{L^\infty(\Gamma_+)} \leqslant F_2(|t_1 - t_2|)$,

where F_1, F_2 are increasing continuous functions with $F_2(t) = O(t^\alpha)$. Then

$$\|f\|_{\dot{C}^\alpha([0,T]\times\Gamma_+)} \leqslant \|f(0)\|_{\dot{C}^\alpha(\Gamma_+)} + F_1(T) + \sup_{t\in[0,T]} \frac{F_2(t)}{t^\alpha}.$$

Lemma A.7. *Assume that $f \in C^{N,\alpha}([0,T]\times\Gamma_+)$ for some $N \geqslant 0$, with the properties that for all $t_1, t_2 \in [0,T]$,*

- $\|D^N f(t_1) - D^N f(t_2)\|_{\dot{C}^\alpha(\Gamma_+)} \leqslant F_1(|t_1 - t_2|)$;
- $\|D^N f(t_1) - D^N f(t_2)\|_{L^\infty(\Gamma_+)} \leqslant F_2(|t_1 - t_2|)$,

where F_1, F_2 are increasing continuous functions with $F_2(t) = O(t^\alpha)$. Then

$$\|f\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant \|f(0)\|_{C^{N,\alpha}(\Gamma_+)} + \|f(t) - f(0)\|_{C^N([0,T]\times\Gamma_+)} + CF_1(T) + C \sup_{t\in[0,T]} \frac{F_2(t)}{t^\alpha}.$$

Lemma A.8. *If $f \in C^{N,\alpha}(Q)$ for some $N \geqslant 0$ and $\alpha \in (0,1]$ then for any $t_1, t_2 \in [0,T]$,*

$$\|f(t_1) - f(t_2)\|_{C^N(\Omega)} \leqslant C\|f\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha.$$

Corollary A.9. *If $f \in C^{N,\alpha}(Q)$ for some $N \geqslant 0$ and $\alpha \in (0,1]$ then*

$$\|f(t) - f(0)\|_{C^N(Q)} \leqslant C\|f\|_{C^{N,\alpha}(Q)}T^\alpha.$$

Lemma A.10 is adapted from Lemma 8.3 of [13].

Lemma A.10. *Suppose that $f_j \colon \mathbb{R}^d \to \mathbb{R}$, $j = 1, 2$, each have the modulus of continuity μ, with $\mu\colon [0,\infty) \to [0,\infty)$ continuous and increasing with $\mu(0) = 0$. There exists a continuous increasing function $F\colon [0,\infty) \to \infty$, depending on μ, with $F(0) = 0$ for which*

$$\|f_1 - f_2\|_{L^\infty(\mathbb{R}^d)} \leqslant F(\|f_1 - f_2\|_{L^2(\mathbb{R}^d)}).$$

Proof. Fix $x \in \mathbb{R}^d$ arbitrarily and suppose that $\delta = |f_1(x) - f_2(x)| > 0$. Let y be in the ball B of radius $a = \mu^{-1}(\delta/4)$ about x, so that $|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)| \leqslant \delta/4$. Then

$$|f_1(y) - f_2(y)| \geqslant \delta - |f_1(x) - f_1(y)| - |f_2(x) - f_2(y)| = \frac{\delta}{2}.$$

Hence,

$$\|f_1 - f_2\|_{L^2(\mathbb{R}^d)} \geqslant \|f_1 - f_2\|_{L^2(B)} \geqslant \left(\int_B \left(\frac{\delta}{2}\right)^2\right)^{\frac{1}{2}} = \frac{\delta}{2}\sqrt{\pi}a,$$

or,

$$h(\delta) := \frac{\sqrt{\pi}}{2}\delta\mu^{-1}(\delta/4) \leqslant \|f_1 - f_2\|_{L^2(\mathbb{R}^d)}.$$

Since μ^{-1} must be increasing, so must h, so setting $F = h^{-1}$ (noting that $F(0) = 0$) we have

$$|f_1(x) - f_2(x)| = \delta \leqslant F(\|f_1 - f_2\|_{L^2(\mathbb{R}^d)}).$$

This inequality applies for all x even when $\delta = |f_1(x) - f_2(x)| = 0$, giving the result. □

Appendix B. Boundary differential operators

We can define differential operators up to order two on $\partial\Omega$ by treating it as a manifold having at least C^2 regularity. In this appendix, we describe the properties that we need of the first-order differential operators, ∇_Γ, $\operatorname{div}_\Gamma$, and $\operatorname{curl}_\Gamma$. We refer the reader to standard references for such operators (for instance, Section 2.2 of [26]).

We will also have the need to calculate ∇, div, and curl in 3-space, but restricted to the boundary. This can be done by introducing a convenient coordinate system in a tubular neighborhood of the boundary in such a way that on the boundary itself, the coordinates reduce to a convenient coordinate system on the boundary. This is as done, for instance, in [8], drawing upon [14], and we refer the reader to those references for details.

We can define ∇_Γ—and then from it, $\operatorname{div}_\Gamma$ and $\operatorname{curl}_\Gamma$—in a coordinate-free manner by requiring that for any $f \in C^\infty(\Gamma)$ and any smooth curve $\mathbf{x}(s)$ on Γ parameterized by arc length,

$$\nabla_\Gamma f \cdot \mathbf{x}'(0) = \lim_{s\to 0} \frac{f(\mathbf{x}(s)) - f(\mathbf{x}(0))}{s}.$$

We then define $\operatorname{div}_\Gamma$ as the adjoint of ∇_Γ, in the sense of Lemma B.1:

Lemma B.1. *Let $f \in C^1(\Gamma)$, $\mathbf{v} \in (C^1(\Gamma))^d$. Then*

$$\int_\Gamma \mathbf{v} \cdot \nabla_\Gamma f = -\int_\Gamma \operatorname{div}_\Gamma \mathbf{v}\, f.$$

Moreover,

$$\operatorname{div}_\Gamma(f\mathbf{v}) = f \operatorname{div}_\Gamma \mathbf{v} + \mathbf{v} \cdot \nabla_\Gamma f. \tag{B.1}$$

Proof. This is classical for smooth functions (see, for instance, Proposition 2.2.2 of [26]), and follows in the same way for C^1 functions, integrating by parts on the boundary in charts. □

Finally, we define (with the $\perp$ operator as in Definition 8.1)

$$\operatorname{curl}_\Gamma \mathbf{v} := -\operatorname{div}_\Gamma \mathbf{v}^\perp.$$

We collect now a few useful facts.

For $\mathbf{u}$, $\mathbf{v}$ tangent vectors,

$$(\mathbf{u} \cdot \nabla_\Gamma \mathbf{v}) \cdot \mathbf{v} = \frac{1}{a_j} u^j \partial_j v^i\, v^i = \frac{1}{2a_j} u^i \partial_j |\mathbf{v}|^2 = \frac{1}{2} \mathbf{u} \cdot \nabla |\mathbf{v}|^2,$$

so for any component Γ_n of the boundary,

$$\int_{\Gamma_n} (\mathbf{u} \cdot \nabla_\Gamma \mathbf{v}) \cdot \mathbf{v} = \frac{1}{2} \int_{\Gamma_n} \mathbf{u} \cdot \nabla_\Gamma |\mathbf{v}|^2.$$

For a vector field $\mathbf{v}$ on $\overline{\Omega}$,

$$\operatorname{curl}_\Gamma \mathbf{v}^{\boldsymbol{\tau}} = (\operatorname{curl} \mathbf{v}) \cdot \boldsymbol{n} \tag{B.2}$$

and

$$\operatorname{div} \mathbf{v} = \operatorname{div}_\Gamma \mathbf{v}^{\boldsymbol{\tau}} + \partial_{\boldsymbol{n}} v^{\boldsymbol{n}} + (\kappa_1 + \kappa_2) v^{\boldsymbol{n}} \text{ on } \Gamma. \tag{B.3}$$

Lemma B.2. *Let $\mathbf{u}, \mathbf{v}$ be vector fields on $\overline{\Omega}$. Then*

$$[\mathbf{u} \times \mathbf{v}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}} [\mathbf{v}^{\boldsymbol{\tau}}]^\perp - v^{\boldsymbol{n}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp, \quad u^{\boldsymbol{n}} \mathbf{v}^{\boldsymbol{\tau}} - v^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = [(\mathbf{v} \times \mathbf{u})^{\boldsymbol{\tau}}]^\perp.$$

Proof. We have,

$$\mathbf{u}\times\mathbf{v} = (\mathbf{u}^{\boldsymbol{n}} + \mathbf{u}^{\boldsymbol{\tau}})\times(\mathbf{v}^{\boldsymbol{n}} + \mathbf{v}^{\boldsymbol{\tau}}) = \mathbf{u}^{\boldsymbol{n}}\times\mathbf{v}^{\boldsymbol{\tau}} - \mathbf{v}^{\boldsymbol{n}}\times\mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}\times\mathbf{v}^{\boldsymbol{\tau}},$$

since $\mathbf{u}^{\boldsymbol{n}}\times\mathbf{v}^{\boldsymbol{n}} = 0$. Now, $\mathbf{u}^{\boldsymbol{\tau}}\times\mathbf{v}^{\boldsymbol{\tau}}$ is parallel to $\boldsymbol{n}$, so we see that

$$[\mathbf{u}\times\mathbf{v}]^{\boldsymbol{\tau}} = \mathbf{u}^{\boldsymbol{n}}\times\mathbf{v}^{\boldsymbol{\tau}} - \mathbf{v}^{\boldsymbol{n}}\times\mathbf{u}^{\boldsymbol{\tau}}.$$

But, $\mathbf{u}^{\boldsymbol{n}}$ is perpendicular to $\mathbf{v}^{\boldsymbol{\tau}}$, so we see that $\mathbf{u}^{\boldsymbol{n}}\times\mathbf{v}^{\boldsymbol{\tau}} = u^{\boldsymbol{n}}[\mathbf{v}^{\boldsymbol{\tau}}]^\perp$, and similarly, $\mathbf{v}^{\boldsymbol{n}}\times\mathbf{u}^{\boldsymbol{\tau}} = v^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp$. Hence, $[\mathbf{u}\times\mathbf{v}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}}[\mathbf{v}^{\boldsymbol{\tau}}]^\perp - v^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp$, giving also $u^{\boldsymbol{n}}\mathbf{v}^{\boldsymbol{\tau}} - v^{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{\tau}} = [(\mathbf{v}\times\mathbf{u})^{\boldsymbol{\tau}}]^\perp$. □

Proof of Proposition 8.2. All the following calculations are on Γ. We start with a short calculation *in rectangular coordinates*, using that $\operatorname{div}\mathbf{u} = \partial_i u^i = 0$:

$$\begin{aligned}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} &= u^i\partial_i u^j n^j = \partial_i(u^i u^j n^j) - u^j u^i \partial_i n^j = \operatorname{div}(u^{\boldsymbol{n}}\mathbf{u}) - \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\boldsymbol{n})\\ &= \operatorname{div}(u^{\boldsymbol{n}}\mathbf{u}) - \mathbf{u}^{\boldsymbol{\tau}}\cdot\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}.\end{aligned}$$

In the last equality, we used that because $\boldsymbol{n}$ does not change in the direction of $\boldsymbol{n}$,

$$\mathbf{u}\cdot\nabla\boldsymbol{n} = (\mathbf{u}^{\boldsymbol{n}}\cdot\nabla)\boldsymbol{n} + \mathbf{u}^{\boldsymbol{\tau}}\cdot\nabla\boldsymbol{n} = \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}},$$

which is a tangent vector.

From (B.3) followed by (B.1), then,

$$\begin{aligned}\operatorname{div}(u^{\boldsymbol{n}}\mathbf{u}) &= \operatorname{div}_\Gamma(u^{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{\tau}}) + \partial_{\boldsymbol{n}}(u^{\boldsymbol{n}})^2 + (\kappa_1+\kappa_2)(u^{\boldsymbol{n}})^2\\ &= u^{\boldsymbol{n}}\operatorname{div}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}\cdot\nabla_\Gamma u^{\boldsymbol{n}} + \partial_{\boldsymbol{n}}(u^{\boldsymbol{n}})^2 + (\kappa_1+\kappa_2)(u^{\boldsymbol{n}})^2.\end{aligned}$$

Using (B.3) again,

$$0 = (\operatorname{div}\mathbf{u})u^{\boldsymbol{n}} = (\operatorname{div}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} + \partial_{\boldsymbol{n}}u^{\boldsymbol{n}} + (\kappa_1+\kappa_2)u^{\boldsymbol{n}})u^{\boldsymbol{n}},$$

so

$$\partial_{\boldsymbol{n}}(u^{\boldsymbol{n}})^2 = 2u^{\boldsymbol{n}}\partial_{\boldsymbol{n}}u^{\boldsymbol{n}} = -2u^{\boldsymbol{n}}\operatorname{div}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} - 2(\kappa_1+\kappa_2)(u^{\boldsymbol{n}})^2.$$

Hence,

$$(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} = -u^{\boldsymbol{n}}\operatorname{div}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}\cdot\nabla_\Gamma u^{\boldsymbol{n}} - (\kappa_1+\kappa_2)(u^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}}\cdot\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}.$$ □

Appendix C. Compatibility conditions: special case

In [28], Temam and Wang consider a periodic domain with $\mathbf{U} = (0,0,-1)$, so $\mathbf{U}^{\boldsymbol{\tau}} = 0$ for all time. More generally, the authors of [6] consider $\mathbf{U} = -U^I\boldsymbol{n}$, where $U^I > 0$ is constant, so $\mathbf{U}^{\boldsymbol{\tau}} = 0$ on Γ_+ for all time. The compatibility conditions simplify in these settings.

Proposition C.1. *Assume that $\mathbf{U}^{\boldsymbol{\tau}} \equiv 0$ and $U^{\boldsymbol{n}}$ is spatially constant along Γ_+ ($U^{\boldsymbol{n}}$ need not be constant in time). Then the compatibility condition cond_N for $N \geqslant 0$ is*

$$\partial_t^j\mathbf{f}^{\boldsymbol{\tau}}|_{t=0} = \partial_t^j\nabla_\Gamma p|_{t=0} - U_0^{\boldsymbol{n}}(\partial_t^j\boldsymbol{\omega}^{\boldsymbol{\tau}})^\perp|_{t=0} \text{ for all } 0\leqslant j\leqslant N, \tag{C.1}$$

where $\partial_t^j\nabla_\Gamma p|_{t=0}$ and $\partial_t^j\boldsymbol{\omega}|_{t=0}$ must be treated as explained following (1.9).

Proof. Since $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{U}^{\boldsymbol{\tau}} = 0$, (B.2) gives that on Γ_+,

$$\omega^{\boldsymbol{n}} = \boldsymbol{\omega}\cdot\boldsymbol{n} = \operatorname{curl}_\Gamma\mathbf{u}^{\boldsymbol{\tau}} = 0.$$

In particular, this holds at time zero. Both $\partial_t\mathbf{U}^{\boldsymbol{\tau}} = 0$ and $\operatorname{curl}_\Gamma\mathbf{U}^{\boldsymbol{\tau}} = 0$, while $|\mathbf{U}|^2 = (U^{\boldsymbol{n}})^2$ is constant on Γ_+, so also $\nabla_\Gamma|\mathbf{U}|^2 = 0$. We see, then, that $\mathbf{H}^{\boldsymbol{\tau}}$ simplifies to $\mathbf{H}^{\boldsymbol{\tau}} = (U^{\boldsymbol{n}})^{-1}\left[\mathbf{f}^{\boldsymbol{\tau}} - \nabla_\Gamma p\right]^\perp$, so lincond_0 (which follows from cond_0 by Proposition 3.7) becomes

$$\left[\mathbf{f}^{\boldsymbol{\tau}} - \nabla_\Gamma p\right]_{t=0}^\perp = U_0^{\boldsymbol{n}}\boldsymbol{\omega}_0^{\boldsymbol{\tau}},$$

which is (C.1) for $N = 0$. The inductive extension of this to higher N follows readily, leading to (C.1) for $N \geqslant 0$. $\square$

The condition in (C.1) for $N = 0$ also follows from cond_0 with slightly more work, though the inductive extension to higher N is not so transparent as it is starting from cond_0'.

Because $\operatorname{div} \mathbf{f} = 0$ with $\mathbf{f} \cdot \boldsymbol{n} = 0$ on Γ, $\mathbf{f}$ plays no role in the calculation of $\nabla_\Gamma p$ for $N = 0$. By writing the condition in (C.1) as we do, we are stressing that, given initial data one can always choose a forcing at time zero so that cond_0 is satisfied.

For all $N \geqslant 1$, though, forcing enters into the calculation of $\partial_t \nabla_\Gamma p$, when $\partial_t \mathbf{u}_0$ is replaced by $\mathbf{f} - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p_0$: even though $\mathbf{f} \cdot \boldsymbol{n} = 0$, the forcing still does not, in general, vanish from even the $N = 1$ condition. Because of this fact, the forcing is intimately entwined in cond_N for $N \geqslant 1$, appearing on both sides of the condition, even for the simplest nontrivial case considered in [28]. These same comments hold in the general setting, but are more transparent in this simplified setting.

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