OBSERVATIONS ON THE VANISHING VISCOSITY LIMIT

JAMES P. KELLIHER

Abstract. Whether, in the presence of a boundary, solutions of the Navier-Stokes equations converge to a solution to the Euler equations in the vanishing viscosity limit is unknown. In a seminal 1983 paper, Tosio Kato showed that the vanishing viscosity limit is equivalent to having sufficient control of the gradient of the Navier-Stokes velocity in a boundary layer of width proportional to the viscosity. In a 2008 paper, the present author showed that the vanishing viscosity limit is equivalent to the formation of a vortex sheet on the boundary. We present here several observations that follow on from these two papers.

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The Navier-Stokes equations for a viscous incompressible fluid in a domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, with no-slip boundary conditions can be written,

$$(NS)\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega. \end{cases}$$

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The Euler equations modeling inviscid incompressible flow on such a domain with no-penetration boundary conditions can be written,

$$(EE)\begin{cases} \partial_t \overline{u} + \overline{u}\cdot\nabla\overline{u} + \nabla p = \overline{f} & \text{in } \Omega, \\ \operatorname{div}\overline{u} = 0 & \text{in } \Omega, \\ \overline{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

Here, $u = u_\nu$ and $\overline{u}$ are velocity fields, while p and $\overline{p}$ are pressure (scalar) fields. The external forces, f, $\overline{f}$, are vector fields. (We adopt here the notation of Kato in [14].) We assume throughout that Ω is bounded and Γ has C^2 regularity, and write $\boldsymbol{n}$ for the outward unit normal vector.

The limit,

$$(VV) \qquad u \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)),$$

we refer to as the *classical vanishing viscosity limit*. Whether it holds in general, or fails in any one instance, is a major open problem in mathematical fluids mechanics.

In his seminal paper [14], Tosio Kato establishes a necessary and sufficient condition for (VV) to hold based upon the magnitude of ∇u in a layer near the boundary of width proportional to ν (see (1.6)). In this condition, ∇u can be replaced with the vorticity ($\operatorname{curl} u$) for both necessity and sufficiency as shown in [16]. Moreover, it is shown in [17] that (VV) is equivalent to the vorticity accumulating on the boundary to form a vortex sheet in the limit of vanishing viscosity. (This is in sympathy with Chorin's approximation of solutions to the Navier-Stokes equations by vortex sheets in [9]; see also [8, 7, 6].)

In the present work, we return to, and to an extent integrate, the themes explored in [14, 16, 17]: vorticity accumulation in the whole domain, in a boundary layer, and on the boundary itself.

We start in Section 1 with the notation and definitions we will need, and a summary of the pertinent results of [14, 16, 17].

Section 2 through Section 5 apply to all dimensions 2 and higher. In these sections we do not analyze the behavior of solutions in a boundary layer or on the boundary in detail. Rather, we obtain information concerning the global behavior of solutions given that (VV) holds or, conversely, obtain further information about the nature of the vanishing viscosity limit given that certain global conditions hold.

We re-express in a specifically 3D form the condition for vorticity accumulation on the boundary from [17] in Section 2. In Section 3, we show that if (VV) holds then the L^p norms of the vorticity for solutions to (NS) must blow up for all $p > 1$ as $\nu \to 0$ except in very special circumstances. This leaves only the possibility of control of the vorticity's L^1 norm. Assuming such control, we show in Section 4 that when (VV) holds we can characterize the accumulation of vorticity on the boundary as a convergence of Radon measures, which is a stronger convergence than that obtained in [17].

In Section 5, we show that the arguments in [17] lead to the conclusion that some kind of convergence of a subsequence of the solutions to (NS)

always occurs in the limit as $\nu \to 0$, but not necessarily to a solution to the Euler equations.

The remaining sections, Section 6 through Section 10, are concerned exclusively with 2D solutions. This simplifies the analysis substantially, in large part because the energy equality is known to hold for 2D weak solutions to the Navier-Stokes equations. This makes it easier to prove the necessity of conditions that imply (VV). It is also helpful that in 2D vorticity is transported by solutions to the Euler equations. In these sections we will consider in detail the behavior of solutions in a boundary layer and on the boundary. (Some of the results in these sections can, however, be extended, with greater difficulty, to higher dimension.)

In Section 6, we return to the theme of controlling the L^1 norm of the vorticity, showing that if we measure the width of the boundary layer by the size of the L^1-norm of the vorticity then the layer has to be wider than that of Kato if (VV) holds. We push this analysis further in Section 7 to obtain the theoretically optimal convergence rate when the initial vorticity has nonzero total mass, as is generic for non-compatible initial data. We turn a related observation into a conjecture concerning the connection between the vanishing viscosity limit and the applicability of the Prandtl theory.

We derive in Section 8 a condition on the solution to (NS) on the boundary that is equivalent in 2D to (VV), giving a number of examples to which this condition applies in Section 9.

In Section 10 we discuss some interesting recent results of Bardos and Titi that they developed using dissipative solutions to the Euler Equations. We show how weaker, though still useful, 2D versions of these results can be obtained using direct elementary methods.

1. Definitions and past results

We define the classical function spaces of incompressible fluids,

$$H = \left\{ u \in L^2(\Omega)^d : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}$$

with the L^2-norm and

$$V = \left\{ u \in (H_0^1(\Omega))^d : \operatorname{div} u = 0 \text{ in } \Omega \right\}$$

with the H^1-norm. For scalar functions, f, g, we define $(f, g) := \int_\Omega fg$. If v, w are vector fields then $(v, w) := (v^i, w^i)$, where we use here and below the common convention of summing over repeated indices. Similarly, if M, N are matrices of the same dimensions then $M \cdot N := M^{ij} N^{ij}$ and

$$(M, N) = (M^{ij}, N^{ij}) = \int_\Omega M \cdot N.$$

In all that follows, we assume that u and $\overline{u}$ satisfy the same initial conditions,

$$u(0) = u_0, \quad \overline{u}(0) = u_0, \quad u_0 \in C^{k+\epsilon}(\Omega) \cap H$$

for some $\epsilon \in (0, 1)$, where $k = 1$ for two dimensions and $k = 2$ for 3 and higher dimensions, and that

$$f = \overline{f} \in C^1_{loc}(\mathbb{R}; C^1(\Omega)).$$

Then as shown in [19] (Theorem 1 and the remarks on p. 508-509), there is some $T > 0$ for which there exists a unique solution,

$$\overline{u} \text{ in } C^1([0, T]; C^{k+\epsilon}(\Omega)), \tag{1.1}$$

to (EE). In two dimensions, T can be arbitrarily large, though it is only known that some positive T exists in three and higher dimensions.

With such initial velocities, we are assured that there are classical solutions to (NS) for all time in 2D. In 3D, however, the known lower bounds on the time of existence vanish with the viscosity (see, for instance, the remarks in the introduction to [24]); we will use only properties of weak solutions to (NS), so this does not represent any real difficulties. Such weak solutions are unique in 2D, but uniqueness is not known to hold in higher dimensions. So by $u = u_\nu$ we mean any of these solutions chosen arbitrarily.

We write

$$\mathcal{M}(\overline{\Omega}) \text{ for the space of Radon measures on } \overline{\Omega}. \tag{1.2}$$

That is, $\mathcal{M}(\overline{\Omega})$ is the dual space of $C(\overline{\Omega})$. We let μ in $\mathcal{M}(\overline{\Omega})$ be the measure supported on Γ for which $\mu|_\Gamma$ corresponds to Lebesgue measure on Γ (arc length for $d = 2$, area for $d = 3$). Then μ is also a member of $H^1(\Omega)^*$, the dual space of $H^1(\Omega)$.

We define the vorticity $\omega(u)$ to be the $d \times d$ antisymmetric matrix,

$$\omega(u) = \frac{1}{2}\left[\nabla u - (\nabla u)^T\right], \tag{1.3}$$

where ∇u is the Jacobian matrix for u: $(\nabla u)^{ij} = \partial_j u^i$. When working specifically in two dimensions, we alternately define the vorticity as the scalar curl of u:

$$\omega(u) = \partial_1 u^2 - \partial_2 u^1. \tag{1.4}$$

Letting $\omega = \omega(u)$ and $\overline{\omega} = \omega(\overline{u})$, we define the following conditions:

(A) $\quad u \to \overline{u}$ weakly in H uniformly on $[0, T]$,

(A') $\quad u \to \overline{u}$ weakly in $L^2(\Omega)^d$ uniformly on $[0, T]$,

(B) $\quad u \to \overline{u}$ in $L^\infty(0, T; H)$,

(C) $\quad \nabla u \to \nabla \overline{u} - ((\cdot) \cdot \mathbf{n}, \overline{u}\mu)$ in $(H^1(\Omega)^{d\times d})^*$ uniformly on $[0, T]$,

(D) $\quad \nabla u \to \nabla \overline{u}$ in $H^{-1}(\Omega)^{d\times d}$ uniformly on $[0, T]$,

(E) $\quad \omega \to \overline{\omega} - \frac{1}{2}\left((\cdot - \cdot^T) \cdot \boldsymbol{n}, \overline{u}\mu\right)$ in $(H^1(\Omega)^{d\times d})^*$ uniformly on $[0, T]$,

(F) $\quad \omega \to \overline{\omega}$ in $H^{-1}(\Omega)^{d\times d}$ uniformly on $[0, T]$.

We stress that $H^1(\Omega)^*$ is the dual space of $H^1(\Omega)$, in contrast to $H^{-1}(\Omega)$, which is the dual space of $H^1_0(\Omega)$.

The condition in (B) is the classical vanishing viscosity limit of (VV).

We will make the most use of condition (E), which more explicitly means that for any M in $H^1(\Omega)^{d\times d}$,

$$(\omega(t), M) \to (\overline{\omega}(t), M) - \frac{1}{2}\int_\Gamma ((M - M^T)\cdot \mathbf{n})\cdot \overline{u}(t) \text{ in } L^\infty([0,T]). \quad (1.5)$$

The condition (C) has a similar interpretation.

In two dimensions, defining the vorticity as in Equation (1.4), we also define the following two conditions:

$$(E_2) \qquad \omega \to \overline{\omega} - (\overline{u}\cdot\boldsymbol{\tau})\mu \text{ in } H^1(\Omega)^* \text{ uniformly on } [0,T],$$
$$(F_2) \qquad \omega \to \overline{\omega} \text{ in } H^{-1}(\Omega) \text{ uniformly on } [0,T].$$

Here, $\boldsymbol{\tau}$ is the unit tangent vector on Γ that is obtained by rotating the outward unit normal vector $\mathbf{n}$ counterclockwise by 90 degrees.

Theorem 1.1 is proved in [17] ($(A) \implies (B)$ having been proved in [14]), to which we refer the reader for more details.

Theorem 1.1 ([17])**.** *Conditions (A), (A′), (B), (C), (D), and (E) are equivalent (and each implies condition (F)). In two dimensions, condition (E_2) and, when Ω is simply connected, (F_2) are equivalent to the other conditions.*[1]

Theorem 1.1 remains silent about rates of convergence, but examining the proof of it in [17] easily yields the following:

Theorem 1.2. *Assume that (VV) holds with*

$$\|u - \overline{u}\|_{L^\infty(0,T;L^2(\Omega))} \le F(\nu)$$

for some fixed $T > 0$. Then

$$\|(u(t) - \overline{u}(t), v)\|_{L^\infty([0,T])} \le F(\nu)\,\|v\|_{L^2(\Omega)} \text{ for all } v \in L^2(\Omega)^d$$

and

$$\|(\omega(t) - \overline{\omega}(t), \varphi)\|_{L^\infty([0,T])} \le F(\nu)\,\|\nabla\varphi\|_{L^2} \text{ for all } \varphi \in H_0^1(\Omega).$$

Remark 1.3. Theorem 1.2 gives the rates of convergence for (A) and (F_2); the rates for (C), (D), (E), and (E_2) are like those given for (F_2) (though the test function, φ, will lie in different spaces).

In [14], Tosio Kato showed that (VV) is equivalent to

$$\nu \int_0^T \|\nabla u(s)\|^2_{L^2(\Omega)}\, dt \to 0 \text{ as } \nu \to 0$$

and to

$$\nu \int_0^T \|\nabla u(s)\|^2_{L^2(\Gamma_{c\nu})}\, dt \to 0 \text{ as } \nu \to 0. \quad (1.6)$$

[1]The restriction that Ω be simply connected for the equivalence of (F_2) was not, but should have been, in the published version of [17].

Here, and in what follows, Γ_δ is a boundary layer in Ω of width $\delta > 0$:

$$\Gamma_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

In [16] it is shown that in (1.6), the gradient can be replaced by the vorticity, so (VV) is equivalent to

$$\nu \int_0^T \|\omega(s)\|^2_{L^2(\Gamma_{c\nu})}\, dt \to 0 \text{ as } \nu \to 0. \tag{1.7}$$

Note that the necessity of (1.7) follows immediately from (1.6), but the sufficiency does not, since on the inner boundary of $\Gamma_{c\nu}$ there is no boundary condition of any kind.

We also mention the works [29, 30], which together establish conditions equivalent to Equation (1.6), with a boundary layer slightly larger than that of Kato, yet only involving the tangential derivatives of either the normal or tangential components of u rather than the full gradient. These conditions will not be used in the present work, however.

2. A 3D version of vorticity accumulation on the boundary

In Theorem 1.1, the vorticity is defined to be the antisymmetric gradient, as in (1.3). When working in 3D, it is usually more convenient to use the language of three-vectors in condition (E). This leads us to the condition (E') in Proposition 2.1.

Proposition 2.1. *In 3D, the condition (E) in Theorem 1.1 is equivalent to*

$$(E') \qquad \operatorname{curl} u \to \operatorname{curl} \overline{u} + (\overline{u} \times \boldsymbol{n})\mu \textit{ in } (H^1(\Omega)^3)^* \textit{ uniformly on } [0, T],$$

by which we mean that for any φ in $H^1(\Omega)^3$,

$$(\operatorname{curl} u(t), \varphi) \to (\operatorname{curl} \overline{u}(t), \varphi) + \int_\Gamma (\overline{u}(t) \times \mathbf{n}) \cdot \varphi \textit{ in } L^\infty([0, T]).$$

Proof. If A is an antisymmetric 3×3 matrix then

$$A \cdot M = A \cdot \frac{M - M^T}{2}.$$

Thus, since ω and $\overline{\omega}$ are antisymmetric, referring to Equation (1.5), we see that (E) is equivalent to

$$(\omega(t), M) \to (\overline{\omega}(t), M) - \int_\Gamma (M\boldsymbol{n}) \cdot \overline{u}(t) \text{ in } L^\infty([0, T])$$

for all *antisymmetric* matrices $M \in (H^1(\Omega))^{3 \times 3}$.

Now, for any three vector φ define

$$P(\varphi) = \begin{pmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{pmatrix}.$$

Then P is a bijection from the vector space of three-vectors to the space of antisymmetric 3×3 matrices. Straightforward calculations show that

$$P(\varphi) \cdot P(\psi) = 2\varphi \cdot \psi, \qquad P(\varphi)v = \varphi \times v$$

for any three-vectors, φ, ψ, v. Also, $P(\operatorname{curl} u) = 2\omega$ and $P(\operatorname{curl} \overline{u}) = 2\overline{\omega}$.

For any $\varphi \in (H^1(\Omega))^3$ let $M = P(\varphi)$. Then

$$\begin{aligned}(\omega, M) &= \frac{1}{2}\left(P(\operatorname{curl} u), P(\varphi)\right) = (\operatorname{curl} u, \varphi), \\ (\overline{\omega}, M) &= \frac{1}{2}\left(P(\operatorname{curl} \overline{u}), P(\varphi)\right) = (\operatorname{curl} \overline{u}, \varphi), \\ (M\boldsymbol{n}) \cdot \overline{u} &= (P(\varphi)\boldsymbol{n}) \cdot \overline{u} = (\varphi \times \boldsymbol{n}) \cdot \overline{u} = -(\overline{u} \times \boldsymbol{n}) \cdot \varphi.\end{aligned}$$

Because P is a bijection, this gives the equivalence of (E) and (E'). □

3. L^p-norms of the vorticity blow up for $p > 1$

Theorem 3.1. *Assume that $\overline{u}$ is not identically zero on $[0, T] \times \Gamma$. If any of the equivalent conditions of Theorem 1.1 holds then for all $p \in (1, \infty]$,*

$$\limsup_{\nu \to 0^+} \|\omega\|_{L^\infty(0,T;L^p)} \to \infty. \tag{3.1}$$

Proof. We prove the contrapositive. Assume that the conclusion is not true. Then for some $q' \in (1, \infty]$ it must be that for some $C_0 > 0$ and $\nu_0 > 0$,

$$\|\omega\|_{L^\infty(0,T;L^{q'})} \le C_0 \text{ for all } 0 < \nu \le \nu_0. \tag{3.2}$$

Since Ω is a bounded domain, if (3.2) holds for some $q' \in (1, \infty]$ it holds for all lower values of q' in $(1, \infty]$, so we can assume without loss of generality that $q' \in (1, \infty)$.

Let $q = q'/(q' - 1) \in (1, \infty)$ be Hölder conjugate to q' and $p = 2/q + 1 \in (1, 3)$. Then p, q, q' satisfy the conditions of Corollary A.3 with $(p - 1)q = 2$.

Applying Corollary A.3 gives, for almost all $t \in [0, T]$,

$$\begin{aligned}\|u(t) - \overline{u}(t)\|_{L^p(\Gamma)} &\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \|\nabla u(t) - \nabla \overline{u}(t)\|_{L^{q'}(\Omega)}^{\frac{1}{p}} \\ &\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \left(\|\nabla u(t)\|_{L^{q'}} + \|\nabla \overline{u}(t)\|_{L^{q'}}\right)^{\frac{1}{p}} \\ &\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \left(C(q') \|\omega(t)\|_{L^{q'}} + \|\nabla \overline{u}(t)\|_{L^{q'}}\right)^{\frac{1}{p}} \\ &\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}}\end{aligned}$$

for all $0 < \nu \le \nu_0$. Here we used (3.2) and the inequality, $\|\nabla u\|_{L^{q'}(\Omega)} \le C(q') \|\omega\|_{L^{q'}(\Omega)}$ for all $q' \in (1, \infty)$ of Yudovich [31]. Hence,

$$\|u - \overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} \le C \|u - \overline{u}\|_{L^\infty(0,T;L^2(\Omega))}^{1-\frac{1}{p}} \to 0$$

as $\nu \to 0$. But,

$$\|u - \overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} = \|\overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} \ne 0,$$

so condition (B) cannot hold and so neither can any of the equivalent conditions in Theorem 1.1. ☐

Observe that we used nothing about u being a solution to (NS) or to any other equation in the proof of Theorem 3.1, only that u converges in $L^\infty(0, T; L^2)$ to some vector field, $\overline{u}$, with $\nabla \overline{u} \in L^\infty([0, T] \times \Omega)$.

Remark 3.2. It was shown in [26] (remark at the end of Appendix A) that if u is a solution to $\partial_t u = \nu \Delta u$ with $u(t) = 0$ on the boundary for $t > 0$ then ∇u cannot be bounded in L^p for any $p > 1$ if $u(0) \not\equiv 0$ on the boundary.

4. Improved convergence when vorticity bounded in L^1

In Section 3 we showed that if the classical vanishing viscosity limit holds then the L^p norms of ω must blow up as $\nu \to 0$ for all $p \in (1, \infty]$—unless the Eulerian velocity vanishes identically on the boundary. This leaves open the possibility that the L^1 norm of ω could remain bounded, however, and still have the classical vanishing viscosity limit. This happens, for instance, for radially symmetric vorticity in a disk (Examples 1a and 3 in Section 9), as shown in [21].

In fact, as we show in Corollary 4.1, when (VV) holds and the L^1 norm of ω remains bounded in ν, the convergence in condition (E) is stronger; namely, *weak** in measure uniformly over time (as in [21]). (See (1.2) and the comments after it for the definitions of $\mathcal{M}(\overline{\Omega})$ and μ.)

Corollary 4.1. *Suppose that $u \to \overline{u}$ in $L^\infty(0, T; H)$ and $\operatorname{curl} u$ is bounded in $L^\infty(0, T; L^1(\Omega))$ uniformly in ν. Then in 3D,*

$$\operatorname{curl} u \to \operatorname{curl} \overline{u} + (\overline{u} \times \boldsymbol{n})\mu \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}) \text{ uniformly on } [0, T], \tag{4.1}$$

by which we mean that for any φ in $C(\overline{\Omega})^3$,

$$(\operatorname{curl} u(t), \varphi) \to (\operatorname{curl} \overline{u}(t), \varphi) + \int_\Gamma (\overline{u}(t) \times \mathbf{n}) \cdot \varphi \text{ in } L^\infty([0, T]).$$

Similarly, (C), (E), and (E_2) hold with weak* *convergence in $\mathcal{M}(\overline{\Omega})$ rather than in $H^1(\Omega)^*$.*

Proof. We prove (4.1) explicitly for 3D solutions, the results for (C), (E), and (E_2) following in the same way.

Let $\psi \in C(\overline{\Omega})^3$. What we must show is that

$$(\operatorname{curl} u(t) - \operatorname{curl} \overline{u}(t), \psi) \to \int_\Gamma (\overline{u}(t) \times \boldsymbol{n}) \cdot \psi \text{ in } L^\infty([0, T]).$$

So let $\epsilon > 0$ and choose $\varphi \in H^1(\Omega)^d$ with $\|\psi - \varphi\|_{C(\overline{\Omega})} < \epsilon$. We can always find such a φ because $H^1(\Omega)$ is dense in $C(\overline{\Omega})$. Let

$$M = \max \left\{ \|\operatorname{curl} u - \operatorname{curl} \overline{u}\|_{L^\infty(0,T;L^1(\Omega))}, \|\overline{u}\|_{L^\infty([0,T]\times\Omega)} \right\},$$

which we note is finite since $\|\operatorname{curl} u\|_{L^\infty(0,T;L^1(\Omega))}$ and $\|\operatorname{curl} \overline{u}\|_{L^\infty(0,T;L^1(\Omega))}$ are both finite. Then

$$\begin{aligned}
&\left|(\operatorname{curl} u(t) - \operatorname{curl} \overline{u}(t), \psi) - \int_\Gamma (\overline{u}(t) \times \boldsymbol{n}) \cdot \psi\right| \\
&\qquad \leq \left|(\operatorname{curl} u(t) - \operatorname{curl} \overline{u}(t), \psi - \varphi) - \int_\Gamma (\overline{u}(t) \times \boldsymbol{n}) \cdot (\psi - \varphi)\right| \\
&\qquad\qquad + \left|(\operatorname{curl} u(t) - \operatorname{curl} \overline{u}(t), \varphi) - \int_\Gamma (\overline{u}(t) \times \boldsymbol{n}) \cdot \varphi\right| \\
&\qquad \leq 2M\epsilon + \left|(\operatorname{curl} u(t) - \operatorname{curl} \overline{u}(t), \varphi) - \int_\Gamma (\overline{u}(t) \times \boldsymbol{n}) \cdot \varphi\right|.
\end{aligned}$$

By Theorem 1.1 and Proposition 2.1, we can make the last term above smaller than, say, ϵ, uniformly over t in $[0, T]$ by choosing ν sufficiently small, which is sufficient to give the result. $\square$

Remark 4.2. Suppose that we have the slightly stronger condition that ∇u is bounded in $L^\infty(0, T; L^1(\Omega))$ uniformly in ν. If we are in 2D, $W^{1,1}(\Omega)$ is compactly embedded in $L^2(\Omega)$. This is sufficient to conclude that (VV) holds, as shown in [13].

5. Some kind of convergence always happens

Assume that v is a vector field lying in $L^\infty(0, T; H^1(\Omega))$. An examination of the proof given in [17] of the chain of implications in Theorem 1.1 shows that all of the conditions except (B) are still equivalent with $\overline{u}$ replaced by v. That is, defining,

(A_v) $\quad u \to v$ weakly in H uniformly on $[0, T]$,

(A'_v) $\quad u \to v$ weakly in $L^2(\Omega)^d$ uniformly on $[0, T]$,

(B_v) $\quad u \to v$ in $L^\infty(0, T; H)$,

(C_v) $\quad \nabla u \to \nabla v - ((\cdot) \cdot \boldsymbol{n}, v\mu)$ in $(H^1(\Omega)^{d\times d})^*$ uniformly on $[0, T]$,

(D_v) $\quad \nabla u \to \nabla v$ in $H^{-1}(\Omega)^{d\times d}$ uniformly on $[0, T]$,

(E_v) $\quad \omega \to \omega(v) - \dfrac{1}{2}\left((\cdot - \cdot^T) \cdot \boldsymbol{n}, v\mu\right)$ in $(H^1(\Omega)^{d\times d})^*$ uniformly on $[0, T]$,

$(E_{2,v})$ $\quad \omega \to \omega(v) - (v \cdot \boldsymbol{\tau})\mu$ in $H^1(\Omega)^*$ uniformly on $[0, T]$,

$(F_{2,v})$ $\quad \omega \to \omega(v)$ in $H^{-1}(\Omega)$ uniformly on $[0, T]$,

we have the following theorem:

Theorem 5.1. *Conditions (A_v), (A'_v), (C_v), (D_v), and (E_v) are equivalent. In 2D, conditions $(E_{2,v})$ and, when Ω is simply connected, $(F_{2,v})$ are equivalent to the other conditions. Also, (B_v) implies all of the other conditions. Finally, the same equivalences hold if we replace each convergence above with the convergence of a subsequence.*

But we also have the following:

Theorem 5.2. *There exists v in $L^\infty(0,T;H)$ such that a subsequence (u_ν) converges weakly to v in $L^\infty(0,T;H)$.*

Proof. The argument for a simply connected domain in 2D is slightly simpler so we give it first. The sequence (u_ν) is bounded in $L^\infty(0,T;H)$ by the basic energy inequality for the Navier-Stokes equations. Letting ψ_ν be the stream function for u_ν vanishing on Γ, it follows by the Poincare inequality that (ψ_ν) is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Hence, there exists a subsequence, which we relabel as (ψ_ν), converging strongly in $L^\infty(0,T;L^2(\Omega))$ to some ψ lying in $L^\infty(0,T;H_0^1(\Omega))$. Let $v=\nabla^\perp\psi$.

Let g be any element of $L^\infty(0,T;H)$. Then

$$\begin{aligned}(u_\nu,g)&=(\nabla^\perp\psi_\nu,g)=-(\nabla\psi_\nu,g^\perp)=(\psi_\nu,-\operatorname{div}g^\perp)=(\psi_\nu,\omega(g))\\&\to(\psi,\omega(g))=(v,g).\end{aligned}$$

In the third equality we used the membership of ψ_v in $H_0^1(\Omega)$ and the last equality follows in the same way as the first four.

In dimension $d\geq 3$, let M_ν in $(H_0^1(\Omega))^d$ satisfy $u_\nu=\operatorname{div}M_\nu$; this is possible by Corollary 7.5 of [17]. Arguing as before it follows that there exists a subsequence, which we relabel as (M_ν), converging strongly in $L^\infty(0,T;L^2(\Omega))$ to some M that lies in $L^\infty(0,T;(H_0^1(\Omega))^{d\times d})$. Let $v=\operatorname{div}M$.

Let g be any element of $L^\infty(0,T;H)$. Then

$$(u_\nu,g)=(\operatorname{div}M_\nu,g)=-(M_\nu,\nabla g)\to-(M,\nabla g)=(v,g),$$

establishing convergence as before. □

It follows from Theorems 5.1 and 5.2 that all of the convergences in Theorem 1.1 hold except for (B), but for a subseqence of solutions and the convergence is to some velocity field v lying only in $L^\infty(0,T;H)$ and not necessarily in $L^\infty(0,T;H\cap H^1(\Omega))$. In particular, we do not know if v is a solution to the Euler equations, and, in fact, there is no reason to expect that it is.

6. Width of the boundary layer: 2D

Working in two dimensions, make the assumptions on the initial velocity and on the forcing in Theorem 1.1, and assume in addition that the total mass of the initial vorticity does not vanish; that is,

$$m:=\int_\Omega\omega_0=(\omega_0,1)\neq 0. \tag{6.1}$$

(In particular, this means that u_0 is not in V.) The total mass of the vorticity of the Euler solution is conserved so

$$(\overline{\omega}(t),1)=m \text{ for all } t\in\mathbb{R}. \tag{6.2}$$

The Navier-Stokes velocity, however, is in V for all positive time, so its total mass is zero; that is,

$$(\omega(t), 1) = 0 \text{ for all } t > 0. \tag{6.3}$$

Let us suppose that the vanishing viscosity limit holds. Fix $\delta > 0$ and let φ_δ be a smooth cutoff function equal to 1 on Γ_δ and equal to 0 on $\Omega \setminus \Gamma_{2\delta}$. Then by (F_2) of Theorem 1.1 and using (6.2),

$$|(\omega, 1 - \varphi_\delta) - m| \to |(\overline{\omega}, 1 - \varphi_\delta) - m| = |m - (\overline{\omega}, \varphi_\delta) - m| \leq C\delta,$$

the convergence being uniform on $[0, T]$. Thus, for all sufficiently small ν,

$$|(\omega, 1 - \varphi_\delta) - m| \leq C\delta. \tag{6.4}$$

In (6.4) we must hold δ fixed as we let $\nu \to 0$, for that is all we can obtain from the weak convergence in (F_2). Rather, this is all we can obtain without making some assumptions about the rates of convergence, a matter we will return to in the next section.

We can, however, show that the total mass of the vorticity (in fact, its L^1-norm) in any layer smaller than that of Kato goes to zero and, if the vanishing visocity limit holds, then the same holds for Kato's layer. This is the content of the following theorem.

Theorem 6.1. *For any positive function $\delta = \delta(\nu)$,*

$$\|\omega\|_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \leq C \left(\frac{\delta(\nu)}{\nu}\right)^{1/2}. \tag{6.5}$$

If the vanishing viscosity limit holds and

$$\limsup_{\nu \to 0^+} \frac{\delta(\nu)}{\nu} < \infty$$

then

$$\|\omega\|_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \to 0 \text{ as } \nu \to 0. \tag{6.6}$$

Proof. By the Cauchy-Schwarz inequality,

$$\|\omega\|_{L^1(\Gamma_{\delta(\nu)})} \leq \|1\|_{L^2(\Gamma_{\delta(\nu)})} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})} \leq C\delta^{1/2} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})}$$

so

$$\frac{C}{\delta} \|\omega\|^2_{L^1(\Gamma_{\delta(\nu)})} \leq \|\omega\|^2_{L^2(\Gamma_{\delta(\nu)})}$$

and

$$\frac{C\nu}{\delta} \|\omega\|^2_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \leq \nu \|\omega\|^2_{L^2(0,T;L^2(\Gamma_{\delta(\nu)}))}.$$

By the basic energy inequality for the Navier-Stokes equations, the right-hand side is bounded, giving Equation (6.5), and if the vanishing viscosity limit holds, the right-hand side goes to zero by (1.7), giving Equation (6.6). □

Remark 6.2. In Theorem 6.1, we do not need the assumption in Equation (6.1) nor do we need to assume that we are in dimension two. The result is of most interest, however, when one makes these two assumptions.

Remark 6.3. Equation (6.6) also follows from condition (iii") in [16] that $\nu \int_0^T \|\omega(u)\|^2_{L^2(\Gamma_{c\nu})} \to 0$ by applying the Cauchy-Schwarz inequality in the manner above, but that is using a sledgehammer to prove a simple inequality. Note that Equation (6.6) is necessary for the vanishing viscosity limit to hold, but is not (as far as we can show) sufficient.

7. Optimal convergence rate: 2D

Still working in two dimensions, let us return to (6.4), assuming as in the previous section that the vanishing viscosity limit holds, but bringing the rate of convergence function, F, of Theorem 1.2 into the analysis. We will now make $\delta = \delta(\nu) \to 0$ as $\nu \to 0$, and choose φ_δ slightly differently, requiring that it equal 1 on Γ_{δ^*} and vanish outside of Γ_δ for some $0 < \delta^* = \delta^*(\nu) < \delta$. We can see from the argument that led to (6.4), incorporating the convergence rate for (F_2) given by Theorem 1.2, that

$$|(\omega, 1 - \varphi_\delta) - m| \le C\delta + \|\nabla\varphi_\delta\|_{L^2(\Omega)} F(\nu).$$

Because $\partial\Omega$ is C^2, we can always choose φ_δ so that $|\nabla\varphi_\delta| \le C(\delta - \delta^*)^{-1}$. Then for all sufficiently small δ,

$$\|\nabla\varphi_\delta\|_{L^2(\Omega)} \le \left(\int_{\Gamma_\delta \setminus \Gamma_{\delta^*}} \left(\frac{C}{\delta - \delta^*}\right)^2\right)^{\frac{1}{2}} = C\frac{(\delta - \delta^*)^{\frac{1}{2}}}{\delta - \delta^*} = C(\delta - \delta^*)^{-\frac{1}{2}}.$$

We then have

$$|(\omega, 1 - \varphi_\delta) - m| \le C\left[\delta + (\delta - \delta^*)^{-\frac{1}{2}} F(\nu)\right]. \tag{7.1}$$

For any measurable subset Ω' of Ω, define

$$\mathbf{M}(\Omega') = \int_{\Omega'} \omega,$$

the total mass of vorticity on Ω'. Then

$$\mathbf{M}(\Gamma_\delta^C) = (\omega, 1 - \varphi_\delta) + \int_{\Gamma_\delta \setminus \Gamma_{\delta^*}} \varphi_\delta \omega$$

where

$$\Gamma_\delta^C := \Omega \setminus \Gamma_\delta$$

so

$$\begin{aligned}\left|(\omega, 1 - \varphi_\delta) - \mathbf{M}(\Gamma_\delta^C)\right| &\le \|\omega\|_{L^2(\Gamma_\delta \setminus \Gamma_{\delta^*})} \|\varphi_\delta\|_{L^2(\Gamma_\delta \setminus \Gamma_{\delta^*})} \\ &\le C(\delta - \delta^*)^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\delta)}.\end{aligned} \tag{7.2}$$

From these observations and those in the previous section, we have the following:

Theorem 7.1. *Assume that the classical vanishing viscosity limit in (VV) holds with a rate of convergence, $F(\nu) = o(\nu^{1/2})$. Then in 2D the initial mass of the vorticity must be zero.*

Proof. From (7.1) and (7.2),

$$\begin{aligned} M_\delta := \left|m - \mathbf{M}(\Gamma_\delta^C)\right| &\le |m - (\omega, 1 - \varphi_\delta)| + \left|(\omega, 1 - \varphi_\delta) - \mathbf{M}(\Gamma_\delta^C)\right| \\ &\le C\left[\delta + (\delta - \delta^*)^{-\frac{1}{2}} F(\nu)\right] + C(\delta - \delta^*)^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\delta)}. \end{aligned}$$

Choosing $\delta(\nu) = \nu$, $\delta^*(\nu) = \nu/2$, we have

$$M_\nu \le C\left[\nu + \nu^{-\frac{1}{2}} o(\nu^{\frac{1}{2}})\right] + C\nu^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\nu)},$$

uniformly over $[0, T]$. Squaring, integrating in time, and applying Young's inequality gives

$$\|M_\nu\|^2_{L^2([0,T])} = \int_0^T M_\nu^2 \le CT(\nu^2 + o(1)) + C\nu \int_0^T \|\omega\|^2_{L^2(0,T;L^2(\Gamma_\nu))} \to 0$$

as $\nu \to 0$ by (1.7). Then,

$$\begin{aligned} \|m - M(\Omega)\|_{L^2([0,T])} &\le \left\|m - M(\Gamma_\nu^C)\right\|_{L^2([0,T])} + \|M(\Gamma_\nu)\|_{L^2([0,T])} \\ &\le \|M_\nu\|_{L^2([0,T])} + \|\omega\|_{L^2(0,T;L^1(\Gamma_\nu))} \to 0 \end{aligned}$$

as $\nu \to 0$ by Theorem 6.1. But $u(t)$ lies in V so $M(\Omega) = 0$ for all $t > 0$. Hence, the limit above is possible only if $m = 0$. □

For non-compatible initial data, that is for $u_0 \notin V$, the total mass of vorticity will generically not be zero, so $C\sqrt{\nu}$ should be considered a bound on the rate of convergence for non-compatible initial data. As we will see in Remark 8.2, however, a rate of convergence as good as $C\sqrt{\nu}$ is almost impossible unless the initial data is fairly smooth, and even then it would only occur in special circumstances.

Now let us assume that the rate of convergence in (VV) is $F(\nu) = C\nu^\alpha$ for some $\alpha \le 1/4$. As we will see in Section 9, this is a more typical rate of convergence for the simple examples for which (VV) is known to hold. Letting $\delta = \nu^\beta$, $\delta^* = (1/2)\nu^\beta$ for $\beta > 0$, (7.1) gives

$$|(\omega, 1 - \varphi_{\nu^\beta}) - m| \le C\left[\nu^\beta + \nu^{\alpha - \frac{\beta}{2}}\right].$$

Thus,

$$\begin{aligned} \left|\mathbf{M}(\Gamma_{\nu^\beta}^C) - m\right| &\le |(\omega, 1 - \varphi_{\nu^\beta}) - m| + \left|\int_{\Gamma_\delta \setminus \Gamma_{\delta^*}} \varphi_{\nu^\beta}\omega\right| \\ &\le C\left[\nu^\beta + \nu^{\alpha - \frac{\beta}{2}}\right] + \left|\int_{\Gamma_\delta \setminus \Gamma_{\delta^*}} \varphi_{\nu^\beta}\omega\right|. \end{aligned}$$

This suggests that as long as $\beta < 2\alpha$, the total mass of vorticity converges to m outside a layer of width ν^β (we cannot conclude this rigorously, since we cannot bound the last term above). On the other hand, the total mass of vorticity for all positive time is zero, and the total mass in the Kato Layer, Γ_ν, goes to zero in the manner indicated by Theorem 6.1.

If $\alpha = 1/4$ so that $\beta < 1/2$, this suggests that there is a sharp transition in the behavior of the vorticity in the region between the Kato layer and a layer just outside the Prandtl layer, a region in which the total mass of vorticity would need to be $-m$. If $\alpha < 1/4$ so that $\beta = 1/2 + \epsilon$ for some $\epsilon > 0$, it suggests that the transition in vorticity extends well outside the Prandtl layer. This does not directly contradict any tenet of the Prandtl theory, but it suggests that for small viscosity the solution to the Navier-Stokes equations matches the solution to the Euler equations only well outside the Prandtl layer. This leads us to the following conjecture:

Conjecture 7.2. *If the vanishing viscosity limit in (VV) holds at a rate slower than* $C\nu^{\frac{1}{4}}$ *in 2D then the Prandtl theory fails.*

We conjecture no further, however, as to whether the Prandtl equations become ill-posed or whether the formal asymptotics fail to hold rigorously.

8. A condition on the boundary equivalent to (VV): 2D

Theorem 8.1. *For (VV) to hold in 2D it is necessary and sufficient that*

$$\nu \int_0^t \int_\Gamma \omega \, \overline{u} \cdot \boldsymbol{\tau} \to 0 \text{ as } \nu \to 0 \text{ uniformly over } [0, T]. \tag{8.1}$$

Proof. Since the solution is in 2D and $f \in L^2(0, T; H) \supseteq C^1_{loc}(\mathbb{R}; C^1(\Omega))$, Theorem III.3.10 of [28] gives

$$\begin{aligned} &\sqrt{t} u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V), \\ &\sqrt{t} \partial_t u \in L^2(0, T; H), \end{aligned} \tag{8.2}$$

so $\omega(t)$ is defined in the sense of a trace on the boundary. This shows that the condition in (8.1) is well-defined.

For simplicity we give the argument with $f = 0$. We perform the calculations using the d-dimensional form of the vorticity in (1.3), specializing to 2D only at the end. (The argument applies formally in higher dimensions; see Remark 8.3.)

Subtracting (EE) from (NS), multiplying by $w = u - \overline{u}$, integrating over Ω, using Lemma 8.4 for the time derivative, and using $u(t) \in H^2(\Omega)$, $t > 0$, for the spatial integrations by parts, leads to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 \\ &\qquad = -(w \cdot \nabla \overline{u}, w) + \nu (\nabla u, \nabla \overline{u}) - \nu \int_\Gamma (\nabla u \cdot \mathbf{n}) \cdot \overline{u}. \end{aligned} \tag{8.3}$$

Now,

$$\begin{aligned} (\nabla u \cdot \mathbf{n}) \cdot \overline{u} &= 2 \left(\frac{\nabla u - (\nabla u)^T}{2} \cdot \mathbf{n} \right) \cdot \overline{u} + ((\nabla u)^T \cdot \boldsymbol{n}) \cdot \overline{u} \\ &= 2 (\omega(u) \cdot \mathbf{n}) \cdot \overline{u} + ((\nabla u)^T \cdot \boldsymbol{n}) \cdot \overline{u}. \end{aligned}$$

But,

$$\begin{aligned}\int_\Gamma ((\nabla u)^T \cdot \boldsymbol{n}) \cdot \overline{u} &= \int_\Gamma \partial_i u^j n^j \overline{u}^i = \frac{1}{2} \int_\Gamma \partial_i (u \cdot \boldsymbol{n}) \overline{u}^i \\ &= \frac{1}{2} \int_\Gamma \nabla (u \cdot \boldsymbol{n}) \cdot \overline{u} = 0,\end{aligned}$$

since $u \cdot \boldsymbol{n} = 0$ on Γ and $\overline{u}$ is tangent to Γ. Hence,

$$\int_\Gamma (\nabla u \cdot \mathbf{n}) \cdot \overline{u} = 2(\omega(u) \cdot \mathbf{n}) \cdot \overline{u} \tag{8.4}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \|w\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 \\ &\qquad = -(w \cdot \nabla \overline{u}, w) + \nu (\nabla u, \nabla \overline{u}) - 2\nu \int_\Gamma (\omega(u) \cdot \mathbf{n}) \cdot \overline{u}.\end{aligned}$$

By virtue of Lemma 8.4, we can integrate over time to give

$$\begin{aligned}\|w(T)\|_{L^2}^2 + 2\nu \int_0^T \|\nabla u\|_{L^2}^2 &= -2 \int_0^T (w \cdot \nabla \overline{u}, w) + 2\nu \int_0^T (\nabla u, \nabla \overline{u}) \\ &\quad - 2\nu \int_0^T \int_\Gamma (\omega(u) \cdot \mathbf{n}) \cdot \overline{u}.\end{aligned} \tag{8.5}$$

In two dimensions, we have (see (4.2) of [15])

$$(\nabla u \cdot \mathbf{n}) \cdot \overline{u} = ((\nabla u \cdot \mathbf{n}) \cdot \boldsymbol{\tau})(\overline{u} \cdot \boldsymbol{\tau}) = \omega(u)\, \overline{u} \cdot \boldsymbol{\tau}, \tag{8.6}$$

and (8.5) can be written

$$\begin{aligned}\|w(T)\|_{L^2}^2 + 2\nu \int_0^T \|\nabla u\|_{L^2}^2 &= -2 \int_0^T (w \cdot \nabla \overline{u}, w) + 2\nu \int_0^T (\nabla u, \nabla \overline{u}) \\ &\quad - \nu \int_0^T \int_\Gamma \omega(u)\, \overline{u} \cdot \boldsymbol{\tau}.\end{aligned} \tag{8.7}$$

The sufficiency of Equation (8.1) for the vanishing viscosity limit (VV) to hold (and hence for the other conditions in Theorem 1.1 to hold) follows from the bounds,

$$\begin{aligned}|(w \cdot \nabla \overline{u}, w)| &\le \|\nabla \overline{u}\|_{L^\infty([0,T]\times\Omega)} \|w\|_{L^2}^2 \le C \|w\|_{L^2}^2, \\ \nu \int_0^T |(\nabla u, \nabla \overline{u})| &\le \sqrt{\nu}\, \|\nabla \overline{u}\|_{L^2([0,T]\times\Omega)} \sqrt{\nu}\, \|\nabla u\|_{L^2([0,T]\times\Omega)} \le C\sqrt{\nu},\end{aligned}$$

and Gronwall's inequality.

Proving the necessity of Equation (8.1) is just as easy. Assume that (VV) holds, so that $\|w\|_{L^\infty(0,T;L^2(\Omega))} \to 0$. Then by the two inequalities above, the first two terms on the right-hand side of Equation (8.7) vanish with the viscosity as does the first term on the left-hand side. The second term on the left-hand side vanishes as proven in [14] (it follows from a simple argument using the energy equalities for (NS) and (E)). It follows that, of necessity, Equation (8.1) holds. □

Remark 8.2. It follows from the proof of Theorem 8.1 that in 2D,

$$\|u(t) - \overline{u}(t)\|_{L^2} \le C \left[\nu^{\frac{1}{4}} + \nu^{\frac{1}{2}} \left| \int_0^T \int_\Gamma \omega \, \overline{u} \cdot \boldsymbol{\tau} \right|^{\frac{1}{2}} \right] e^{Ct}.$$

Suppose that $\overline{u}_0$ is smooth enough that $\Delta \overline{u} \in L^\infty([0,T] \times \Omega)$. Then before integrating to obtain (8.3) we can replace the term $\nu(\Delta u, w)$ with $\nu(\Delta w, w) + \nu(\Delta \overline{u}, w)$. Integrating by parts gives

$$\nu(\Delta w, w) = \nu \|\nabla w\|_{L^2}^2 \, ,$$

and we also have,

$$\nu(\Delta \overline{u}, w) \le \nu \|\Delta \overline{u}\|_{L^2} \|w\|_{L^2} \le \frac{\nu^2}{2} \|\Delta \overline{u}\|_{L^2}^2 + \frac{1}{2} \|w\|_{L^2}^2 \, .$$

This leads to the bound,

$$\|u(t) - \overline{u}(t)\|_{L^2} \le C \left[\nu + \nu^{\frac{1}{2}} \left| \int_0^T \int_\Gamma \omega \, \overline{u} \cdot \boldsymbol{\tau} \right|^{\frac{1}{2}} \right] e^{Ct}$$

(and also $\|u - \overline{u}\|_{L^2(0,T;H^1)} \le C\nu^{1/2} e^{Ct}$). Thus, the bound we obtain on the rate of convergence in ν is never better than $O(\nu^{1/4})$ unless the initial data is smooth enough and the boundary integral vanishes (as in Example 1 in Section 9).

Remark 8.3. Formally, the argument in the proof of Theorem 8.1 would give in any dimension the condition

$$\nu \int_0^T \int_\Gamma (\omega(u) \cdot \mathbf{n}) \cdot \overline{u} \to 0 \text{ as } \nu \to 0 \text{ uniformly over } [0,T].$$

In 3D, one has $\omega(u) \cdot \boldsymbol{n} = (1/2)\vec{\omega} \times \boldsymbol{n}$, so the condition could be written

$$\nu \int_0^T \int_\Gamma (\vec{\omega} \times \boldsymbol{n}) \cdot \overline{u} = \nu \int_0^T \int_\Gamma \vec{\omega} \cdot (\overline{u} \times \boldsymbol{n}) \to 0 \text{ as } \nu \to 0,$$

uniformly over $[0,T]$, where $\vec{\omega}$ is the 3-vector form of the curl of u. We can only be assured, however, that $u(t) \in V$ for all $t > 0$, which is insufficient to define $\vec{\omega}$ on the boundary. (The normal component could be defined, though, since both $\vec{\omega}(t)$ and $\operatorname{div} \vec{\omega}(t) = 0$ lie in L^2.) Even assuming more compatible initial data in 3D, such as $u_0 \in V$, we can only conclude that $u(t) \in H^2$ for a short time, with that time decreasing to 0 as $\nu \to 0$ (in the presence of forcing; see, for instance, Theorem 9.9.4 of [10]).

The condition in Equation (8.1) indicates that there are two mechanisms by which the vanishing viscosity limit can hold: Either the blowup of ω on the boundary happens slowly enough that

$$\nu \int_0^T \|\omega\|_{L^1(\Gamma)} \to 0 \text{ as } \nu \to 0 \tag{8.8}$$

or the vorticity for (NS) is generated on the boundary in such a way as to oppose the sign of $\overline{u} \cdot \boldsymbol{\tau}$. (This latter line of reasoning is followed in [11], leading to a new condition in a boundary layer slightly thicker than that of Kato.) In the second case, it could well be that vorticity for (NS) blows up fast enough that Equation (8.8) does not hold, but cancellation in the integral in Equation (8.1) allows that condition to hold.

We used Lemma 8.4 in the proof of Theorem 8.1.

Lemma 8.4. *Assume that* $v \in L^\infty(0, T; V)$ *with* $\partial_t v \in L^2(0, T; V')$ *as well as* $\sqrt{t}\partial_t v \in L^2(0, T; H)$. *Then* $v \in C([0, T]; H)$,

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 = (\partial_t v, v) \text{ in } \mathcal{D}'((0, T)) \text{ with } \sqrt{t}(\partial_t v, v) \in L^1(0, T),$$

and

$$\int_0^T \frac{d}{dt}\|v(t)\|_{L^2}^2 \, dt = \|v(T)\|_{L^2}^2 - \|v(0)\|_{L^2}^2 .$$

Proof. Follows from the results of Sections III.1.1 and III.1.4 of [28] (or see Section 5.9 of [12]). ☐

9. Examples where condition on the boundary holds: 2D

All 2D examples where the vanishing viscosity limit is known to hold have some kind of symmetry—in geometry of the domain or the initial data—or have some degree of analyticity.

Since Equation (8.1) is a necessary condition, it holds for all of these examples. But though it is also a sufficient condition, it is not always practicable to apply it to establish the limit. We give here examples in which it is practicable. This includes all known 2D examples having symmetry. In all explicit cases, the initial data is a steady solution to the Euler equations.

Example 1: Let $\overline{u}$ be any solution to the Euler equations for which $\overline{u} = 0$ on the boundary. The integral in Equation (8.1) then vanishes for all ν. From Remark 8.2, the rate of convergence (here, and below, in ν) is $C\nu^{1/4}$ or, for smoother initial data, $C\nu$.

Example 1a: Example 1 is not explicit, in that it is not given by a formula in closed form. As a first example of an explicit solution, let D be the disk of radius $R > 0$ centered at the origin and let $\omega_0 \in L^\infty(D)$ be radially symmetric. Then the associated velocity field, u_0, is given by the Biot-Savart law. By exploiting the radial symmetry, u_0 can be written,

$$u_0(x) = \frac{x^\perp}{|x|^2}\int_0^{|x|} \omega_0(r) r \, dr, \tag{9.1}$$

where we abuse notation a bit in writing $\omega_0(r)$. Since u_0 is perpendicular to $\nabla\omega_0$ it follows from the vorticity form of the Euler equations that $\overline{u} \equiv u_0$ is a steady solution to the Euler equations.

Now assume that the total mass of vorticity,

$$m := \int_{\mathbb{R}^2} \omega_0, \tag{9.2}$$

is zero. We see from Equation (9.1) that on Γ, $u_0 = 0$, giving a steady solution to the Euler equations with velocity vanishing on the boundary.

(Note that $m = 0$ is equivalent to u_0 lying in the space V of divergence-free vector fields vanishing on the boundary.)

Example 1b: Let $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ be a compactly supported radially symmetric initial vorticity for which the total mass of vorticity vanishes; that is, $m = 0$. Then the expression for u_0 in Equation (9.1), which continues to hold throughout all of $\mathbb{R}^2$, shows that u_0 vanishes outside of the support of its vorticity.

If we now restrict such a radially symmetric ω_0 so that its support lies inside a domain (even allowing the support of ω_0 to touch the boundary of the domain) then the velocity u_0 will vanish on the boundary. In particular, $u_0 \cdot \boldsymbol{n} = 0$ so, in fact, u_0 is a steady solution to the Euler equations in the domain, being already one in the whole plane. In fact, one can use a superposition of such radially symmetric vorticities, as long as their supports do not overlap, and one will still have a steady solution to the Euler equations whose velocity vanishes on the boundary.

Such a superposition is called a *superposition of confined eddies* in [22], where their properties in the full plane, for lower regularity than we are considering, are analyzed. These superpositions provide a fairly wide variety of examples in which the vanishing viscosity limit holds.

In [23], Maekawa considers initial vorticity supported away from the boundary in a half-plane. We note that the analogous result in a disk, even were it shown to hold, would not cover this Example 1b when the support of the vorticity touches the boundary.

Example 2 [2D shear flow]: Let ϕ solve the heat equation,

$$\begin{cases} \partial_t \phi(t, z) = \nu \partial_{zz} \phi(t, z) & \text{on } [0, \infty) \times [0, \infty), \\ \phi(t, 0) = 0 & \text{for all } t > 0, \\ \phi(0) = \phi_0. \end{cases} \tag{9.3}$$

Assume for simplicity that $\phi_0 \in W^{1,\infty}((0, \infty))$. Let $u_0 = (\phi_0, 0)$ and $u(t, x) = (\phi(t, x_2), 0)$.

Let $\Omega = [-L, L] \times (0, \infty)$ be periodic in the x_1-direction. Then $u_0 \cdot \boldsymbol{n} = 0$ and $u(t) = 0$ for all $t > 0$ on $\partial\Omega$ and

$$\begin{aligned} \partial_t u(t, x) &= \nu(\partial_{x_2 x_2} \phi(t, x_2), 0) = \nu \Delta u(t, x), \\ (u \cdot \nabla u)(t, x) &= \begin{pmatrix} \partial_1 u^1 & \partial_1 u^2 \\ \partial_2 u^1 & \partial_2 u^2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \partial_2 \phi(t, x_2) & 0 \end{pmatrix} \begin{pmatrix} \phi(t, x_2) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \partial_2 \phi(t, x_2) \phi(t, x_2) \end{pmatrix} = \frac{1}{2} \nabla \phi(t, x_2). \end{aligned}$$

It follows that u solves the Navier-Stokes equations on Ω with pressure, $p = -\frac{1}{2}\phi(t, x_2)$.

Similarly, letting $\overline{u} \equiv u_0$, we have $\partial_t \overline{u} = 0$, $\overline{u} \cdot \nabla \overline{u} = \frac{1}{2}\nabla\phi_0$ so $\overline{u} \equiv u_0$ is a steady solution to the Euler equations.

Now, $\omega = \partial_1 u^2 - \partial_2 u^1 = -\partial_2 \phi(t, x_2)$ so

$$\begin{aligned}\int_\Gamma \omega\, \overline{u} \cdot \boldsymbol{\tau} &= -\int_\Gamma \partial_2 \phi(t, x_2)|_{x_2=0} \phi_0(0) = -\phi_0(0) \int_{-L}^{L} \partial_{x_2} \phi(t, x_2)|_{x_2=0}\, dx_1 \\ &= -L\phi_0(0)\partial_{x_2}\phi(t, x_2)|_{x_2=0}.\end{aligned}$$

The explicit solution to Equation (9.3) is

$$\phi(t, z) = \frac{1}{\sqrt{4\pi\nu t}} \int_0^\infty \left[e^{-\frac{(z-y)^2}{4\nu t}} - e^{-\frac{(z+y)^2}{4\nu t}} \right] \phi_0(y)\, dy$$

(see, for instance, Section 3.1 of [27]). Thus,

$$\begin{aligned}\partial_z \phi(t, z)|_{z=0} &= -\frac{2}{4\nu t\sqrt{4\pi\nu t}} \int_0^\infty y \left[e^{-\frac{y^2}{4\nu t}} + e^{-\frac{y^2}{4\nu t}} \right] \phi_0(y)\, dy \\ &= -\frac{1}{\nu t\sqrt{4\pi\nu t}} \int_0^\infty y e^{-\frac{y^2}{4\nu t}} \phi_0(y)\, dy \\ &= -\frac{1}{\nu t\sqrt{4\pi\nu t}} \int_0^\infty (-2\nu t) \frac{d}{dy} e^{-\frac{y^2}{4\nu t}} \phi_0(y)\, dy \\ &= -\frac{1}{\sqrt{\pi\nu t}} \int_0^\infty \frac{d}{dy} e^{-\frac{y^2}{4\nu t}}\, \phi_0(y)\, dy \\ &= \frac{1}{\sqrt{\pi\nu t}} \left[-\phi_0(0) + \int_0^\infty e^{-\frac{y^2}{4\nu t}} \phi_0'(y)\, dy \right]\end{aligned}$$

so that

$$|\partial_{x_2}\phi(t, x_2)|_{x_2=0}| \le \frac{C}{\sqrt{\nu t}}.$$

We conclude that

$$\nu \left| \int_0^T \int_\Gamma \omega\, \overline{u} \cdot \boldsymbol{\tau} \right| \le C\sqrt{\nu} \int_0^T t^{-1/2}\, dt = C\sqrt{\nu T}.$$

The condition in Equation (8.1) thus holds (as does (8.8)). From Remark 8.2, the rate of convergence is $C\nu^{\frac{1}{4}}$ (even for smoother initial data).

Example 3: Consider Example 1a of radially symmetric vorticity in the unit disk, but without the assumption that m given by Equation (9.2) vanishes. This example goes back at least to Matsui in [25]. The convergence also follows from the sufficiency of the Kato-like conditions established in [29], as pointed out in [30]. A more general convergence result in which the disk is allowed to impulsively rotate for all time appears in [21]. A simple argument to show that the vanishing viscosity limit holds is given in Theorem 6.1 [18], though without a rate of convergence. Here we prove it with a rate of convergence by showing that the condition in Equation (8.1) holds.

Because the nonlinear term disappears, the vorticity satisfies the heat equation, though with Dirichlet boundary conditions not on the vorticity but on the velocity:

$$\begin{cases} \partial_t \omega = \nu \Delta \omega & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{9.4}$$

Unless $u_0 \in V$, however, $\omega \notin C([0,T]; L^2)$, so we cannot easily make sense of the initial condition this way.

An orthonormal basis of eigenfunctions satisfying these boundary conditions is

$$u_k(r,\theta) = \frac{J_1(j_{1k} r)}{\pi^{1/2} \left| J_0(j_{1k}) \right|} \widehat{e}_\theta, \quad \omega_k(r,\theta) = \frac{j_{1k} J_0(j_{1k} r)}{\pi^{1/2} \left| J_0(j_{1k}) \right|},$$

where J_0, J_1 are Bessel functions of the first kind and j_{1k} is the k-th positive root of $J_1(x) = 0$. (See [18] or [20].) We normalize the (u_k) so that[2]

$$\|u_k\|_H = 1, \quad \|\omega_k\|_{L^2} = j_{1k}.$$

We can write,

$$u_0 = \sum_{k=1}^\infty a_k u_k, \quad \|u_0\|_H^2 = \sum_{k=1}^\infty a_k^2 < \infty.$$

We claim that

$$u(t) = \sum_{k=1}^\infty a_k e^{-\nu j_{1k}^2 t} u_k$$

provides a solution to the Navier-Stokes equations, (NS). To see this, first observe that $u \in C([0,T]; H)$, so $u(0) = u_0$ makes sense as an initial condition. Also, $u(t) \in V$ for all $t > 0$. Next observe that

$$\omega(t) := \omega(u(t)) = \sum_{k=1}^\infty a_k e^{-\nu j_{1k}^2 t} \omega_k$$

for all $t > 0$, this sum converging in H^n for all $n \ge 0$. Since each term satisfies (9.4) so does the sum. Taken together, this shows that ω satisfies (9.4) and thus u solves (NS).

The condition in Equation (8.1) becomes

$$\begin{aligned} \nu \int_0^T \int_\Gamma \omega \, \overline{u} \cdot \boldsymbol{\tau} &= \nu \sum_{k=1}^\infty \int_0^T \int_\Gamma a_k e^{-\nu j_{1k}^2 t} \omega_k \, \overline{u} \cdot \boldsymbol{\tau} \, dt \\ &= \nu \sum_{k=1}^\infty \int_0^T a_k e^{-\nu j_{1k}^2 t} \omega_k|_{r=1} \int_\Gamma \overline{u} \cdot \boldsymbol{\tau} \, dt \\ &= m\nu \sum_{k=1}^\infty a_k \frac{j_{1k} J_0(j_{1k})}{\pi^{1/2} \left| J_0(j_{1k}) \right|} \int_0^T e^{-\nu j_{1k}^2 t} \, dt. \end{aligned}$$

[2] This differs from the normalization in [18], where $\|u_k\|_H = j_{1k}^{-1}$, $\|\omega_k\|_{L^2} = 1$.

In the final equality, we used

$$\int_\Gamma \overline{u}\cdot\boldsymbol{\tau} = -\int_\Gamma \overline{u}^\perp\cdot\boldsymbol{n} = -\int_\Omega \operatorname{div}\overline{u}^\perp = \int_\Omega \overline{\omega} = m.$$

(Because vorticity is transported by the Eulerian flow, m is constant in time.)

Then,

$$\begin{aligned}
\nu\left|\int_0^T\int_\Gamma \omega\,\overline{u}\cdot\boldsymbol{\tau}\right| &\le |m|\,\nu\sum_{k=1}^\infty \frac{|a_k|}{\pi^{1/2}}j_{1k}\int_0^T e^{-\nu j_{1k}^2 t}\,dt \\
&= |m|\,\nu\sum_{k=1}^\infty \frac{|a_k|}{\pi^{1/2}}j_{1k}\frac{1-e^{-\nu j_{1k}^2 T}}{\nu j_{1k}^2} \\
&\le \frac{|m|}{\pi^{\frac12}}\left(\sum_{k=1}^\infty a_k^2\right)^{\frac12}\left(\sum_{k=1}^\infty \frac{(1-e^{-\nu j_{1k}^2 T})^2}{j_{1k}^2}\right)^{\frac12} \\
&= \frac{|m|}{\pi^{\frac12}}\|u_0\|_H\left(\sum_{k=1}^\infty \frac{(1-e^{-\nu j_{1k}^2 T})^2}{j_{1k}^2}\right)^{\frac12}.
\end{aligned}$$

Classical bounds on the zeros of Bessel functions give $1+k < j_{1k} \le \pi(\frac12+k)$ (see, for instance, Lemma A.3 of [18]). Hence, with $M=(\nu T)^{-\alpha}$, $\alpha>0$ to be determined, we have

$$\begin{aligned}
\sum_{k=1}^\infty \frac{(1-e^{-\nu j_{1k}^2 T})^2}{j_{1k}^2} &\le C\sum_{k=1}^\infty \frac{(1-e^{-\nu k^2 T})^2}{k^2} \\
&\le (1-e^{-\nu T})^2 + \int_{k=1}^M \frac{(1-e^{-\nu x^2 T})^2}{x^2}\,dx + \int_{k=M+1}^\infty \frac{(1-e^{-\nu x^2 T})^2}{x^2}\,dx \\
&\le \nu^2T^2 + \nu^2T^2\int_{k=1}^M \frac{x^4}{x^2}\,dx + \int_{k=M+1}^\infty \frac{1}{x^2}\,dx \\
&\le \nu^2T^2 + \nu^2T^2\frac13\left(M^3-1\right) + \frac1M \le \nu^2T^2+\nu^2T^2M^3+\frac1M \\
&= \nu^2T^2+\nu^2T^2\nu^{-3\alpha}T^{-3\alpha}+(\nu T)^\alpha = \nu^2T^2+(\nu T)^{2-3\alpha}+(\nu T)^\alpha
\end{aligned}$$

as long as $\nu M^2 T\le 1$ (used in the third inequality); that is, as long as

$$(\nu T)^{1-2\alpha}\le 1. \tag{9.5}$$

Thus (8.1) holds (as does (8.8)), so (VV) holds.

The rate of convergence in (VV) is optimized when $(\nu T)^{2-3\alpha}=(\nu T)^\alpha$, which occurs when $\alpha=\frac12$. The condition in Equation (9.5) is then satisfied with equality. Remark 8.2 then gives a rate of convergence in the vanishing viscosity limit of $C\nu^{\frac14}$ (even for smoother initial data), except in the special case $m=0$, which we note reduces to Example 1a.

Return to Example 1a: Let us apply our analysis of Example 3 to the special case of Example 1a, in which $u_0 \in V$. Now, on the boundary,

$$\nabla p \cdot \boldsymbol{\tau} = (\partial_t u + u \cdot \nabla u + \nabla p) \cdot \boldsymbol{\tau} = \nu \Delta u \cdot \boldsymbol{\tau} = \nu \Delta u^\perp \cdot (-\boldsymbol{n}) = -\nu \nabla^\perp \omega \cdot \boldsymbol{n}.$$

But $\nabla p \equiv 0$ so the left-hand side vanishes. Hence, the vorticity satisfies homogeneous Neumann boundary conditions for positive time. (This is an instance of Lighthill's formula.) Since the nonlinear term vanishes, in fact, ω satisfies the heat equation, $\partial_t \omega = \nu \Delta \omega$ with homogeneous Neumann boundary conditions and hence $\omega \in C([0,T]; L^2(\Omega))$.

Moreover, multiplying $\partial_t \omega = \nu \Delta \omega$ by ω and integrating gives

$$\|\omega(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \omega(s)\|_{L^2}^2 \, ds = \|\nabla \omega_0\|_{L^2}^2 .$$

We conclude that the L^2-norm of ω, and so the L^p-norms for all $p \in [1,2]$, are bounded in time uniformly in ν. (In fact, this holds for all $p \in [1,\infty]$.) This conclusion is not incompatible with Theorem 3.1, since $\overline{u} \equiv 0$ on Γ.

This argument for bounding the L^p-norms of the vorticity fails for Example 3 because the vorticity is no longer continuous in L^2 down to time zero unless $u_0 \in V$. It is shown in [21] (and see [13]) that such control is nonetheless obtained for the L^1 norm.

10. On a result of Bardos and Titi: 2D

Bardos and Titi in [5, 1] (and see [3]), also starting from, essentially, Equation (8.5) make the observation that for the vanishing viscosity limit to hold, it is necessary and sufficient that $\nu\omega$ (or, equivalently, $\nu[\partial_{\boldsymbol{n}} u]_{\boldsymbol{\tau}}$) converge to zero on the boundary in a weak sense. (They do this in dimension ≥ 2.) In their result, the boundary is assumed to be C^∞, but the initial velocity is assumed to only lie in H. Hence, the sufficiency condition does not follow immediately from Equation (8.5).

Their proof of sufficiency involves the use of dissipative solutions to the Euler equations. (The use of dissipative solutions for the Euler equations in a domain with boundaries was initiated in [2]. See also [4].) We present here the weaker version of their results in 2D that can be obtained without employing dissipative solutions. The simple and elegant proof of necessity is as in [1], simplified further because of the higher regularity of our initial data.

Theorem 10.1 (Bardos and Titi [5, 1])**.** *Working in 2D, assume that $\partial\Omega$ is C^2 and that $\overline{u} \in C^1([0,T] \times \Omega)$. Then for $u \to \overline{u}$ in $L^\infty(0,T;H)$ to hold it is necessary and sufficient that*

$$\nu \int_0^t \int_\Gamma \omega \, \varphi \to 0 \text{ as } \nu \to 0 \text{ uniformly over } [0,T] \quad \text{for any } \varphi \in C^1([0,T] \times \Gamma). \tag{10.1}$$

Proof. Sufficiency of the condition follows immediately from setting $\varphi = (\overline{u} \cdot \boldsymbol{\tau})|_\Gamma$ in Theorem 8.1.

To prove necessity, let $\varphi \in C^1([0,T] \times \Gamma)$. We will need divergence-free vector fields $v_\delta \in C^1([0,T]; H \cap C^\infty(\Omega))$ such that $v_\delta \cdot \boldsymbol{\tau} = \varphi$. Moreover, we require of v_δ that it satisfy the same bounds as the boundary layer corrector of Kato in [14]; in particular,

$$\|\partial_t v_\delta\|_{L^1(0,T;L^2(\Omega))} \le C\delta^{1/2}, \qquad \|\nabla v_\delta\|_{L^\infty(0,T;L^2(\Omega))} \le C\delta^{-1/2}. \tag{10.2}$$

These vector fields can be constructed along the lines given in [14, 16].

The proof now proceeds very simply. We multiply the Navier-Stokes equations by v_δ and integrate over space and time to obtain, for any $t \in [0,T]$,

$$\begin{aligned} \int_0^t (\partial_t u, v_\delta) + \int_0^t (u \cdot \nabla u, v_\delta) + \nu \int_0^t (\nabla u, \nabla v_\delta) \\ = \nu \int_0^t \int_\Gamma (\nabla u \cdot \boldsymbol{n}) \cdot v_\delta = \nu \int_0^t \int_\Gamma \omega\, v_\delta \cdot \boldsymbol{\tau} = \nu \int_0^t \int_\Gamma \omega\, \varphi. \end{aligned} \tag{10.3}$$

Here, we used Equation (8.6) with v_δ in place of $\overline{u}$, and we note that no integrations by parts were involved.

Now, assuming that the vanishing viscosity limit holds, Kato shows in [14] that setting $\delta = c\nu$—and using the bounds in Equation (10.2)—each of the terms on the left hand side of Equation (10.3) vanishes as $\nu \to 0$, uniformly in t. By necessity, then, so does the right hand side, giving the necessity of the condition in Equation (10.1). $\square$

To establish the necessity of the stronger condition in Theorem 10.1, we used (based on Bardos's [1]) a vector field supported in a boundary layer of width $c\nu$, as in [14]. We used it, however, to extend to the whole domain an arbitrary cutoff function defined on the boundary, rather than to correct the Eulerian velocity as in [14].

Remark 10.2. In this proof of Theorem 10.1 the time regularity in the test functions could be weakened slightly to assuming that $\partial_t \varphi \in L^1(0,T; C(\Gamma))$, for this would still allow the first bound in Equation (10.2) to be obtained.

Remark 10.3. Using the results of [5, 4] it is possible to change the condition in Equation (10.1) to apply to test functions φ in $C^1([0,T]; C^\infty(\Gamma))$ ([1]). Moreover, this can be done without assuming special time or spatial regularity of the solution to the Euler equations, but only that the initial velocity lies in H.

Appendix A. A Trace Lemma

Corollary A.3, which we used in the proof of Theorem 3.1, follows from Lemma A.1.

Lemma A.1 (Trace lemma)**.** *Let $p \in (1,\infty)$ and $q \in [1,\infty]$ be chosen arbitrarily, and let q' be Hölder conjugate to q. There exists a constant $C = C(\Omega)$ such that for all $f \in W^{1,p}(\Omega) \cap W^{1,q'}(\Omega)$,*

$$\|f\|_{L^p(\Gamma)} \le C \|f\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \|f\|_{W^{1,q'}(\Omega)}^{\frac{1}{p}}.$$

If $f \in W^{1,p}(\Omega)$ *has mean zero or* $f \in W_0^{1,p}(\Omega)$ *then*

$$\|f\|_{L^p(\Gamma)} \le C \|f\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \|\nabla f\|_{L^{q'}(\Omega)}^{\frac{1}{p}}.$$

Proof. We prove this for $f \in C^\infty(\Omega)$, the result following from the density of $C^\infty(\Omega)$ in $W^{1,p}(\Omega) \cap W^{1,q'}(\Omega)$. We also prove it explicitly in two dimensions, though the proof extends easily to any dimension greater than two.

Let Σ be a tubular neighborhood of Γ of uniform width δ, where δ is half of the maximum possible width. Place coordinates (s, r) on Σ where s is arc length along Γ and r is the distance of a point in Σ from Γ, with negative distances being inside of Ω. Then r ranges from $-\delta$ to δ, with points $(s, 0)$ lying on Γ. Also, because Σ is only half the maximum possible width, $|J|$ is bounded from below, where

$$J = \det \frac{\partial(x, y)}{\partial(s, r)}$$

is the Jacobian of the transformation from (x, y) coordinates to (s, r) coordinates.

Let $\varphi \in C^\infty(\Omega)$ equal 1 on Γ and equal 0 on $\Omega \setminus \Sigma$. Then

$$\nabla |\varphi f|^p = p \operatorname{sgn}(\varphi f) |\varphi f|^{p-1} \nabla(\varphi f),$$

since $p > 1$ and φf is smooth. This shows that $\nabla |\varphi f|^p$ is continuous. Hence, letting a be the arc length of Γ, we can calculate

$$\begin{aligned}
\|f\|_{L^p(\Gamma)}^p &= \int_0^a \int_{-\delta}^0 \frac{\partial}{\partial r} |(\varphi f)(s, r)|^p \, dr \, ds \\
&\le \int_0^a \int_{-\delta}^0 \left| \frac{\partial}{\partial r} |(\varphi f)(s, r)|^p \right| dr \, ds \\
&\le \int_0^a \int_{-\delta}^0 |\nabla |(\varphi f)(s, r)|^p| \, dr \, ds \\
&= \left(\inf_{\operatorname{supp} \varphi} |J| \right)^{-1} \int_0^a \int_{-\delta}^0 |\nabla |(\varphi f)(s, r)|^p| \inf_{\operatorname{supp} \varphi} |J| \, dr \, ds \\
&\le \left(\inf_{\operatorname{supp} \varphi} |J| \right)^{-1} \int_0^a \int_{-\delta}^0 |\nabla |(\varphi f)(s, r)|^p| \, |J| \, dr \, ds \\
&= C \int_{\Sigma \cap \Omega} |\nabla |(\varphi f)(x, y)|^p| \, dx \, dy \\
&\le C \|\nabla |\varphi f|^p\|_{L^1(\Omega)} \\
&= Cp \left\| |\varphi f|^{p-1} \nabla(\varphi f) \right\|_{L^1(\Omega)} \\
&\le Cp \left\| |\varphi f|^{p-1} \right\|_{L^q} \|\nabla(\varphi f)\|_{L^{q'}(\Omega)} \\
&= Cp \left[\int_\Omega |\varphi f|^{(p-1)q} \right]^{\frac{1}{q}} \|\nabla(\varphi f)\|_{L^{q'}(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&= Cp \left\| \varphi f \right\|_{L^{(p-1)q}(\Omega)}^{p-1} \left\| \varphi \nabla f + f \nabla \varphi \right\|_{L^{q'}(\Omega)} \\
&\le Cp \left\| f \right\|_{L^{(p-1)q}(\Omega)}^{p-1} \left\| f \right\|_{W^{1,q'}(\Omega)}.
\end{aligned}
$$

The first inequality then follows from raising both sides to the $\frac{1}{p}$ power and using $p^{1/p} \le e^{1/e}$. The second inequality follows from Poincare's inequality. □

Remark A.2. The trace inequality in Lemma A.1 is a folklore result, most commonly referenced in the special case where $p = q = q' = 2$. We proved it for completeness, since we could not find a proof (or even clear statement) in the literature. We also note that a simple, but incorrect, proof of it (for $p = q = q' = 2$) is to apply the *invalid* trace inequality from $H^{\frac{1}{2}}(\Omega)$ to $L^2(\Gamma)$ then use Sobolev interpolation.

Note that in Lemma A.1 it could be that $(p-1)q \in (0,1)$, though in our application of it in Section 3, via Corollary A.3, we have $(p-1)q = 2$.

Corollary A.3. *Let p, q, q' be as in Lemma A.1. For any $v \in H$,*

$$
\left\| v \right\|_{L^p(\Gamma)} \le C \left\| v \right\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \left\| \nabla v \right\|_{L^{q'}(\Omega)}^{\frac{1}{p}}
$$

and for any $v \in V \cap H^2(\Omega)$,

$$
\left\| \operatorname{curl} v \right\|_{L^p(\Gamma)} \le C \left\| \operatorname{curl} v \right\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \left\| \nabla \operatorname{curl} v \right\|_{L^{q'}(\Omega)}^{\frac{1}{p}}.
$$

Proof. If $v \in H$, then

$$
\int_\Omega v^i = \int_\Omega v \cdot \nabla x_i = -\int_\Omega \operatorname{div} v \, x_i + \int_\Gamma (v \cdot \boldsymbol{n}) x_i = 0.
$$

If $v \in V$ then

$$
\int_\Omega \operatorname{curl} v = -\int_\Omega \operatorname{div} v^\perp = -\int_{\partial\Omega} v^\perp \cdot \boldsymbol{n} = 0.
$$

Thus, Lemma A.1 can be applied to v_1, v_2, and $\operatorname{curl} v$, giving the result. □

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References

[1] Claude Bardos. *Private communication.* 22, 23

[2] Claude Bardos, François Golse, and Lionel Paillard. The incompressible Euler limit of the Boltzmann equation with accommodation boundary condition. *Commun. Math. Sci.*, 10(1):159–190, 2012. 22

[3] Claude Bardos and Toan Nguyen. Non-uniqueness for the Euler equations: the effect of the boundary. *arXiv:1410.4952 [math.AP]*, 2014. 22

[4] Claude Bardos, László Székelyhidi Jr., and Emil Wiedemann. Non-uniqueness for the Euler equations: the effect of the boundary. *arXiv:1305.0773 [math.AP]*, 2014. 22, 23

[5] Claude W. Bardos and Edriss S. Titi. Mathematics and turbulence: where do we stand? *J. Turbul.*, 14(3):42–76, 2013. 22, 23

[6] Alexandre J. Chorin and Jerrold E. Marsden. *A mathematical introduction to fluid mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 1993. 2

[7] Alexandre J. Chorin, Marjorie F. McCracken, Thomas J. R. Hughes, and Jerrold E. Marsden. Product formulas and numerical algorithms. *Comm. Pure Appl. Math.*, 31(2):205–256, 1978. 2

[8] Alexandre Joel Chorin. Numerical study of slightly viscous flow. *J. Fluid Mech.*, 57(4):785–796, 1973. 2

[9] Alexandre Joel Chorin. Vortex sheet approximation of boundary layers. *Computational Physics*, 27:428–442, 1978. 2

[10] Peter Constantin and Ciprian Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988. 16

[11] Peter Constantin, Igor Kukavica, and Vlad Vicol. On the inviscid limit of the Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 143(7):3075–3090, 2015. 17

[12] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998. 17

[13] Gung-Min Gie, M. C. Lopes Filho, James P Kelliher, A. L. Mazzucato, and Nussenzveig Lopes. Vanishing viscosity limit of some symmetric flows. *Preprint*, 2014. 9, 22

[14] Tosio Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 85–98. Springer, New York, 1984. 2, 5, 15, 23

[15] James P. Kelliher. Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane. *SIAM Math Analysis*, 38(1):210–232, 2006. 15

[16] James P. Kelliher. On Kato's conditions for vanishing viscosity. *Indiana University Mathematics Journal*, 56(4):1711–1721, 2007. 2, 6, 12, 23

[17] James P. Kelliher. Vanishing viscosity and the accumulation of vorticity on the boundary. *Commun. Math. Sci.*, 6(4):869–880, 2008. 2, 5, 9, 10, 25

[18] James P. Kelliher. On the vanishing viscosity limit in a disk. *Math. Ann.*, 343(3):701–726, 2009. 19, 20, 21

[19] Herbert Koch. Transport and instability for perfect fluids. *Math. Ann.*, 323(3):491–523, 2002. 4

[20] D.-S. Lee and B. Rummler. The eigenfunctions of the Stokes operator in special domains. III. *ZAMM Z. Angew. Math. Mech.*, 82(6):399–407, 2002. 20

[21] M. C. Lopes Filho, A. L. Mazzucato, H. J. Nussenzveig Lopes, and Michael Taylor. Vanishing viscosity limits and boundary layers for circularly symmetric 2D flows. *Bull. Braz. Math. Soc. (N.S.)*, 39(4):471–513, 2008. 8, 19, 22

[22] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Yuxi Zheng. Convergence of the vanishing viscosity approximation for superpositions of confined eddies. *Comm. Math. Phys.*, 201(2):291–304, 1999. 18

[23] Yasunori Maekawa. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *CPAM*, 67(7):1045–1128, 2014. 18

[24] Nader Masmoudi and Frédéric Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.*, 203(2):529–575, 2012. 4

[25] Shin'ya Matsui. Example of zero viscosity limit for two-dimensional nonstationary Navier-Stokes flows with boundary. *Japan J. Indust. Appl. Math.*, 11(1):155–170, 1994. 19

[26] Anna Mazzucato and Michael Taylor. Vanishing viscosity limits for a class of circular pipe flows. *Comm. Partial Differential Equations*, 36(2):328–361, 2011. 8

[27] Walter A. Strauss. *Partial differential equations*. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction. 19

[28] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. 14, 17

[29] Roger Temam and Xiaoming Wang. On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(3-4):807–828 (1998), 1997. Dedicated to Ennio De Giorgi. 6, 19

[30] Xiaoming Wang. A Kato type theorem on zero viscosity limit of Navier-Stokes flows. *Indiana Univ. Math. J.*, 50(Special Issue):223–241, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000). 6, 19

[31] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066 (Russian), 1963. 7

Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521

Current address: Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521

E-mail address: `kelliher@math.ucr.edu`