

ON THE VANISHING VISCOSITY LIMIT IN A DISK

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ABSTRACT. We say that a solution of the Navier-Stokes equations converges in the vanishing viscosity limit to a solution of the Euler equations if their velocities converge in the energy (L^2) norm uniformly in time as the viscosity ν vanishes. We show that a necessary and sufficient condition for the vanishing viscosity limit to hold in a disk is that the space-time energy density of the solution to the Navier-Stokes equations in a boundary layer of width proportional to ν vanish with ν , and that one need only consider spatial variations whose frequencies in the radial or tangential direction lie in a band centered around $1/\nu$.

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1. INTRODUCTION

In the presence of a boundary, the question of whether solutions of the Navier-Stokes equations with no-slip boundary conditions converge to a solution of the Euler equations as the viscosity vanishes—the so-called vanishing viscosity limit—is very difficult. The convergence of most general interest is of the velocities, uniformly over finite time and L^2 in space. Except in very special cases, such as radially symmetric initial vorticity in a disk, where convergence is known to hold (see Theorem 6.1), the question of convergence or the lack thereof is unresolved for nonzero initial velocity in a bounded domain.

Tosio Kato in [6] gave necessary and sufficient conditions on the velocity u of the Navier-Stokes equations for the vanishing viscosity limit to hold. The most interesting of these is that

$$\nu \int_0^T \|\nabla u(t)\|_{L^2(\Gamma_{c\nu})}^2 dt \rightarrow 0 \text{ as } \nu \rightarrow 0,$$

where $\Gamma_{c\nu}$ is the boundary strip of width $c\nu$ with $c > 0$ fixed but arbitrary. Making only a small change to Kato's proof, it is possible to replace ∇u with the vorticity,

$$\omega = \omega(u) = \partial_1 u^2 - \partial_2 u^1, \tag{1.1}$$

giving Equation (2.3) (see [7]). (The necessity of Equation (2.3) is immediate from Kato's condition, but because we do not have a boundary condition on the inner boundary of $\Gamma_{c\nu}$ the sufficiency of the condition requires proof.)

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Other necessary and sufficient conditions were established by Teman and Wang in [14] and [15]. These are the conditions in Equation (2.5) and Equation (2.6) of Theorem 2.3, and involve only the derivatives in the directions tangential to the boundary of either the tangential or normal components of the velocity, though for a slightly larger boundary layer. Finally, a condition that requires that the average energy density in the boundary layer of the same width as Kato's vanish with viscosity, Equation (2.7), is proven in [7]. All these conditions (which apply to a bounded domain in dimensions 2 and higher) are summarized in Theorem 2.3.

We consider the issue of vanishing viscosity in the (unit) disk and look for weaker necessary and sufficient conditions for the limit to hold. Working in the disk allows us to make quite explicit calculations involving the eigenfunctions of the Stokes operator, which are composed of Bessel functions of the first kind. In a sense, this connects the energy method with the geometry. We will find that we need only consider certain ranges of frequencies (or equivalently, length scales) in the various conditions: this is Theorem 2.4. Although Theorem 2.4 is specific to the disk, there is no hydrodynamical reason to expect the disk to be special as regards the vanishing viscosity limit, so one would expect a version of the theorem to apply to all sufficiently smooth bounded domains in \mathbb{R}^2 , and probably in higher dimensions as well. We discuss this issue more fully in Remark 2.1, and in Section 8 compare our results for the disk for what one obtains for a two-dimensional channel.

In [3], Cheng and Wang obtain a result regarding vanishing viscosity in two dimensions analogous to Equation (2.15) and Equation (2.16). Their result applies to an approximating sequence to a solution of the Navier-Stokes equations as the viscosity vanishes, whereas our result applies to the necessary and sufficient condition for the vanishing viscosity limit to hold. While for the other conditions in Theorem 2.4 we use very different techniques than those in [3], our proof of the necessity and sufficiency of Equation (2.15) and Equation (2.16) uses the key inequality in their paper. Section 7 contains a brief comparison of the two results.

In [11], the authors consider the Stokes problem (linearized Navier-Stokes equations) *external* to a disk with time-varying Dirichlet boundary conditions, showing that the vanishing viscosity limit holds. In fact, they do much more than this, giving an explicit construction of the solution to the Stokes problem and showing that it can be decomposed into the sum of the solution to the linearized Euler equations, the solution to the associated Prandtl equations, and a small correction term. The symmetry of the geometry allows the authors of [11] to construct the solutions in an explicit form (involving Bessel functions of the first and second kind). The nonlinear term in the Navier-Stokes equations makes an explicit solution impossible for us; however, we can expand the solution in terms of eigenfunctions of the Stokes operator for which we have an explicit form (in terms of Bessel functions of the first kind) which we can use to obtain finer estimates on the

behavior of the Navier-Stokes equations in the boundary layer than would be possible for a general domain.

A word on notation: We use C to represent an unspecified constant that always has the same value on both sides of an equality but may have a different value on each side of an inequality.

2. DEFINITIONS AND KATO-TYPE CONDITIONS

We now give definitions of the Euler and Navier-Stokes equations, and state the results from [6], [7], [14], and [15] that we will need.

In Section 4 we will specialize to the unit disk, but for now we assume only that Ω is a bounded domain in \mathbb{R}^2 with C^2 -boundary Γ , and we let \mathbf{n} be the outward normal vector to Γ .

A classical solution (\bar{u}, \bar{p}) to the Euler equations satisfies, for fixed $T > 0$,

$$(E) \quad \begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} = \bar{f} \text{ and } \operatorname{div} \bar{u} = 0 \text{ on } [0, T] \times \Omega, \\ \bar{u} \cdot \mathbf{n} = 0 \text{ on } [0, T] \times \Gamma, \text{ and } \bar{u} = \bar{u}^0 \text{ on } \{0\} \times \Omega, \end{cases}$$

where $\operatorname{div} \bar{u}^0 = 0$. These equations describe the motion of an incompressible fluid of constant density and zero viscosity.

We assume that \bar{u}^0 is in $C^{k+\epsilon}(\Omega) \cap H$, $\epsilon > 0$, and \bar{f} is in $C^k([0, t] \times \Omega)$ for all $t > 0$, where $k = 1$ or 2 . Then as shown in [8] (Theorem 1 and the remarks on p. 508-509), there exists a unique solution \bar{u} in $C_{loc}^1([0, \infty); C^{k+\epsilon}(\Omega))$.

The Navier-Stokes equations describe the motion of an incompressible fluid of constant density and positive viscosity ν . A classical solution to the Navier-Stokes equations can be defined in analogy with (E) by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f \text{ and } \operatorname{div} u = 0 \text{ on } [0, T] \times \Omega, \\ u = 0 \text{ on } [0, T] \times \Gamma, \text{ and } u = u_\nu^0 \text{ on } \{0\} \times \Omega. \end{cases}$$

We will work, however, with weak solutions to the Navier-Stokes equations.

Definition 2.1 (Weak Navier-Stokes Solutions). Given $T > 0$, viscosity $\nu > 0$, and initial velocity u_ν^0 in H , u in $L^2([0, T]; V)$ with $\partial_t u$ in $L^2([0, T]; V')$ is a weak solution to the Navier-Stokes equations if $u(0) = u_\nu^0$ and

$$(NS) \quad \int_{\Omega} \partial_t u \cdot v + \int_{\Omega} (u \cdot \nabla u) \cdot v + \nu \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v$$

for all v in V . (The spaces H and V are defined in Section 3.)

Definition 2.2. We say that the *vanishing viscosity limit* holds if

$$u \rightarrow \bar{u} \text{ in } L^\infty([0, T]; L^2(\Omega)) \text{ as } \nu \rightarrow 0. \quad (2.1)$$

Theorem 2.3 applies to a bounded domain with C^2 -boundary in \mathbb{R}^d , $d \geq 2$. The conditions in Equation (2.2) and Equation (2.4) are due to Kato ([6]), the conditions in Equation (2.3) and Equation (2.7) appear in [7], and the conditions in Equation (2.5) and Equation (2.6) are due to Temam and Wang ([14], [15]).

Theorem 2.3. *Let $T > 0$ and assume that u_ν^0 is in H and that \bar{u}^0 is in $C^{k+\epsilon}(\Omega) \cap H$, $\epsilon > 0$ with $k = 1$ or 2 . In addition, assume that*

- (a) $u_\nu^0 \rightarrow \bar{u}^0$ in $L^2(\Omega)$ as $\nu \rightarrow 0$,
- (b) f is in $L^1([0, T]; L^2(\Omega))$,
- (c) $\|f - \bar{f}\|_{L^1([0, T]; L^2(\Omega))} \rightarrow 0$ as $\nu \rightarrow 0$.

Let $\delta : [0, \infty) \rightarrow [0, \infty)$ be such that $\delta(\nu)$ converges to 0 while $\delta(\nu)/\nu$ diverges to ∞ as $\nu \rightarrow 0$. Define ω as in Equation (1.1). Then the vanishing viscosity limit (Definition 2.2) holds if and only if any of the following conditions holds:

$$\nu \int_0^T \|\omega(s)\|_{L^2(\Omega)}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.2)$$

$$\nu \int_0^T \|\omega(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.3)$$

$$\nu \int_0^T \|\nabla u(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.4)$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.5)$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (2.6)$$

Here $\nabla_{\boldsymbol{\tau}}$ represents the derivatives in the boundary layer in the directions tangential to the boundary, $u_{\boldsymbol{\tau}}$ is the projection of u in the direction tangential to the boundary, and $u_{\mathbf{n}}$ is the projection of u in the direction normal to the boundary.

When $k = 2$, these conditions are also equivalent to

$$\frac{1}{\nu} \int_0^T \|u(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (2.7)$$

The quantity in Equation (2.7) is proportional to the space-time average of the energy in the boundary layer.

We show (see Remark 5.3) that in Equation (2.2), Equation (2.4), and Equation (2.7), contributions from the high frequency modes can be ignored. This result applies to an arbitrary bounded domain in \mathbb{R}^d , $d \geq 2$, with a C^2 -boundary.

Our main result is Theorem 2.4, which is an improvement of Theorem 2.3 in the special case of the unit disk. In what follows we decompose the solution u in the form

$$u(t, x) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} g_{mj}(t) u_{mj}(x),$$

where (u_{mj}) are the eigenfunctions of the Stokes operator described in Section 4, and let

$$u^N(t, x) = \sum_{m=-N}^N \sum_{j=1}^N g_{mj}(t) u_{mj}(x) \quad (2.8)$$

and

$$\tilde{u}^N(t, x) = \sum_{m=-N}^N \sum_{j=1}^{\infty} g_{mj}(t) u_{mj}(x) \quad (2.9)$$

with vorticities $\omega^N(t, x) = \omega(u^N(t, x))$ and $\tilde{\omega}^N(t, x) = \omega(\tilde{u}^N(t, x))$.

As we will see in Section 4, the frequency of u_{mk} in the tangential direction is m and the radial frequency of u_{mk} is, in effect, k . Thus, u^N includes the contributions from all modes with both frequencies less than N , while \tilde{u}^N includes the contributions from all modes with tangential frequency less than N .

Theorem 2.4. *Assume that Ω is the unit disk and make the same assumptions on the initial data, forcing, and the function δ as in Theorem 2.3. Let L and M be any functions mapping $(0, \infty)$ to \mathbb{Z}^+ with*

$$\nu L(\nu) \rightarrow 0, \nu M(\nu) \rightarrow \infty \text{ as } \nu \rightarrow 0. \quad (2.10)$$

Then the the vanishing viscosity limit (Definition 2.2) holds if and only if any of the following conditions holds:

$$\nu \int_0^T \|\omega(s)^{M(\nu)} - \omega^{L(\nu)}(s)\|_{L^2(\Omega)}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.11)$$

$$\nu \int_0^T \|\omega(s) - \tilde{\omega}^{L(\nu)}(s)\|_{L^2(\Omega)}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.12)$$

$$\nu \int_0^T \|\omega(s) - \omega^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.13)$$

$$\nu \int_0^T \|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.14)$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}(s) - \nabla_{\boldsymbol{\tau}} \tilde{u}_{\boldsymbol{\tau}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0, \quad (2.15)$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}(s) - \nabla_{\boldsymbol{\tau}} \tilde{u}_{\mathbf{n}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (2.16)$$

When $k = 2$, these conditions are also equivalent to

$$\frac{1}{\nu} \int_0^T \|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (2.17)$$

Remark 2.1. By Lemma A.3 and Equation (4.2), u^N is essentially the contributions of all the modes with eigenvalues less than CN^2 . In fact, suppose that we replace the definition of u^N in Equation (2.8) with

$$u^N(t, x) = \sum_{\{j: \lambda_j < N^2\}} g_j(t) u_j(x), \quad (2.18)$$

the single subscripts in Equation (2.18) referring to the eigenfunctions and eigenvalues of the Stokes operator on a general domain in \mathbb{R}^d , $d \geq 2$, defined in Section 3. It follows easily from Theorem 2.4 that the conditions in Equation (2.11), Equation (2.13), Equation (2.14), and Equation (2.17) continue to be equivalent to the vanishing viscosity limit. It is in this form that we would expect Theorem 2.4 to generalize to fairly arbitrary smooth domains in \mathbb{R}^2 and—with N^2 in Equation (2.18) replaced by N raised to some other power—to domains in \mathbb{R}^d , $d \geq 3$. The obstacle to establishing this generalization is the difficulty of obtaining the equivalents of Lemma A.8 and Lemma A.9—along with an approximate form of Lemma A.10—for high frequencies.

3. THE STOKES OPERATOR IN A BOUNDED DOMAIN

Before specializing to the case of a disk, we discuss first some general properties related to the Stokes operator.

We define the function spaces H and V as follows (see Section I.1.4 of [13] for more details). First let

$$\mathcal{V} = \{u \in (\mathcal{D}(\Omega))^2 : \operatorname{div} u = 0\}$$

be the space of vector-valued divergence-free distributions on Ω . We let H be the closure of \mathcal{V} in $L^2(\Omega)$ and V be the closure of \mathcal{V} in $H_0^1(\Omega)$. Alternate characterizations of H and V are

$$\begin{aligned} H &= \{u \in (L^2(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \\ V &= \{u \in (L^2(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma\}, \end{aligned}$$

the boundary conditions applying in terms of a trace.

By $\langle \cdot, \cdot \rangle$ we mean the inner product in $L^2(\Omega)$: $\langle f, g \rangle = \int_{\Omega} f \bar{g}$. (It will be convenient to use complex-valued eigenfunctions, so the complex conjugate is required in this definition. Our velocity fields and vorticities, however, are real, so conjugation will not always appear in our calculations.) Then $\langle u, v \rangle_H = \langle u, v \rangle$ and $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle$.

Although \mathcal{V} is dense in H it is not dense in $H \cap H^1(\Omega)$ (with the H^1 -norm). This is because if v is a vector in $H \cap H^1(\Omega)$ that does not vanish on the boundary it cannot be approximated in the $H^{1/2}(\Gamma)$ -norm by a sequence of vectors in \mathcal{V} . By the continuity of the trace operator from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$, then, v cannot be approximated in the $H^1(\Omega)$ -norm by any sequence of vectors in \mathcal{V} . Thus, we have proved:

Lemma 3.1. *The space \mathcal{V} is not dense in $H \cap H^1(\Omega)$.*

We now briefly describe the properties we will need of the Stokes operator A on Ω , referring the reader, for instance, to Section I.2 of [13] for more details. One way to define A is that given u in $V \cap H^2(\Omega)$, Au in H satisfies $Au = -\Delta u + \nabla p$ for some harmonic scalar field p . We have $D(A) = V \cap H^2(\Omega)$ with A mapping $D(A)$ onto H , and there exists a set of eigenfunctions $\{u_j\}$ for A , complete in H and in V , with corresponding eigenvalues $\{\lambda_j\}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and each u_j is in $H^2(\Omega)$ since we are assuming that Γ is C^2 . (When we specialize to the disk, the eigenfunctions will be in $C^\infty(\Omega)$.) An eigenfunction u_j of A satisfies $Au_j = \lambda_j u_j$ or, equivalently,

$$\begin{cases} \Delta u_j + \lambda_j u_j = \nabla p_j, \Delta p_j = 0, \operatorname{div} u_j = 0 \text{ on } \Omega, \\ u_j = 0 \text{ on } \Gamma. \end{cases} \quad (3.1)$$

The eigenfunctions are orthogonal in both H and V . The usual convention is to make the eigenvectors orthonormal in H , but we will find it more convenient to normalize them to be orthonormal in V so that $\|\nabla u_j\|_{L^2(\Omega)}^2 = \|\omega_j\|_{L^2(\Omega)}^2 = 1$ and

$$\|u_j\|_{L^2(\Omega)}^2 = \langle u_j, u_j \rangle = \frac{1}{\lambda_j} \langle u_j, Au_j \rangle = \frac{1}{\lambda_j} \langle \nabla u_j, \nabla u_j \rangle = \frac{1}{\lambda_j}. \quad (3.2)$$

Moreover, we have Lemma 3.2.

Lemma 3.2. *If u is in V with $\omega = \omega(u)$ then*

$$u = \sum_{j=1}^{\infty} \langle \omega, \omega_j \rangle u_j, \quad (3.3)$$

with the sum converging in both V and H .

Proof. Let u be in V and let $u^n = \sum_{j=1}^n (\langle u, u_j \rangle_H / \langle u_j, u_j \rangle_H) u_j$. Then u^n converges in H to u because $\{u_j\}$ is complete in H . But,

$$\begin{aligned} \frac{\langle u, u_j \rangle_H}{\langle u_j, u_j \rangle_H} &= \frac{\lambda_j \langle u, u_j \rangle}{\lambda_j \langle u_j, u_j \rangle} = \frac{\langle u, Au_j \rangle}{\langle u_j, Au_j \rangle} = \frac{\langle \nabla u, \nabla u_j \rangle}{\langle \nabla u_j, \nabla u_j \rangle} \\ &= \langle \nabla u, \nabla u_j \rangle = \langle \omega, \omega_j \rangle, \end{aligned}$$

so the expansion of u in V in terms of the eigenfunctions of A is the same as the expansion of u in H (and the coefficients are as given in Equation (3.3)), meaning that u^n converges in V to u as well. \square

Corollary 3.3. *If u is in V then*

$$\nabla u = \sum_{j=1}^{\infty} \langle \omega, \omega_j \rangle \nabla u_j \text{ and } \omega = \sum_{j=1}^{\infty} \langle \omega, \omega_j \rangle \omega_j,$$

with the sums converging in $L^2(\Omega)$.

Since the solution u to (NS) lies in V for all positive time, we can write

$$\begin{aligned} \omega(t) &= \sum_{j=1}^{\infty} g_j(t)\omega_j, & u(t) &= \sum_{j=1}^{\infty} g_j(t)u_j, \\ \|\omega(t)\|_{L^2(\Omega)}^2 &= \sum_{j=1}^{\infty} |g_j(t)|^2, & \|u(t)\|_{L^2(\Omega)}^2 &= \sum_{j=1}^{\infty} \frac{|g_j(t)|^2}{\lambda_j}, \end{aligned} \quad (3.4)$$

where g_j are functions of time. The expansion of u will converge for all $t \geq 0$ and that of ω for $t > 0$ —and also for $t = 0$ if and only if the initial velocity is in V ; in general, we only assume that it is H . Because $u(t) \rightarrow u_v^0$ in $L^2(\Omega)$ as $t \rightarrow 0$, each $g_j(t)$ is continuous at $t = 0$, though this does not mean that $\omega(t)$ is continuous in $L^2(\Omega)$ at $t = 0$. Also, note that $g_j(t)$ is complex-valued since the eigenvectors are complex-valued, but $u(t)$ and $\omega(t)$ are real-valued.

4. EIGENFUNCTIONS OF THE STOKES OPERATOR IN THE UNIT DISK

We now fix Ω to be the unit disk in \mathbb{R}^2 centered at the origin.

In [10], a complete set of eigenfunctions for the annulus is derived in terms of Bessel functions of the first and second kind, J_n and Y_n . By ignoring the terms involving Y_n and modifying somewhat the calculation of the eigenvalues, one can easily obtain the eigenvalues and eigenfunctions for the unit disk.

Letting

$$j_{nk} = \text{the } k\text{-th positive root of } J_{n+1}(x) = 0, \quad (4.1)$$

the eigenvalues are

$$\lambda_{nk} = j_{n+1,k}^2, \quad (4.2)$$

doubly indexed by $n \in \mathbb{Z}$ and $k \in \mathbb{N} := \{1, 2, \dots\}$. We can write the eigenfunctions in the form

$$\begin{aligned} u_{nk}(r, \theta) &= \frac{J_n(\lambda_{nk}^{1/2} r) - J_n(\lambda_{nk}^{1/2}) r^n}{\pi^{1/2} \lambda_{nk} |J_n(\lambda_{nk}^{1/2})| r} i n e^{in\theta} \hat{e}_r \\ &+ \frac{\lambda_{nk}^{1/2} \left(J_{n+1}(\lambda_{nk}^{1/2} r) - J_{n-1}(\lambda_{nk}^{1/2} r) \right) + 2n J_n(\lambda_{nk}^{1/2}) r^{n-1}}{2\pi^{1/2} \lambda_{nk} |J_n(\lambda_{nk}^{1/2})|} e^{in\theta} \hat{e}_\theta. \end{aligned} \quad (4.3)$$

The vorticity of u_{nk} (Equation (1.1)) is given by

$$\omega_{nk}(r, \theta) = C_{nk} J_n(\lambda_{nk}^{1/2} r) e^{in\theta},$$

with the constant

$$C_{nk} = \frac{1}{\pi^{1/2} |J_n(j_{|n|+1,k})|},$$

which normalizes the eigenfunctions so that $\|\omega_{nk}\|_{L^2(\Omega)} = 1$.

The eigenvalues for $n = 0$ are simple while for all other n the eigenvalues are double. (Higher order eigenvalues do not occur because J_0, J_1, \dots share no zeros.)

5. PROOF OF THEOREM 2.4

From the fundamental energy equality for (NS) we have for all t in $[0, T]$,

$$\nu \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 = \nu \int_0^t \|\omega\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|u_\nu^0\|_H^2 + 4 \|f\|_{L^1([0, T]; L^2(\Omega))}^2.$$

It follows from Equation (3.4) and assumptions (a) and (b) of Theorem 2.3 that for all sufficiently small $\nu > 0$,

$$\nu \int_0^t \|\omega\|_{L^2(\Omega)}^2 = \nu \int_0^t \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} |g_{mj}(s)|^2 ds \leq C. \quad (5.1)$$

Remark 5.1. In Equation (5.1) and most of what follows we use the doubly indexed notation for the eigenfunctions and eigenvalues of Section 4, though we will occasionally find it convenient to use the singly indexed notation of Section 3 instead.

Theorem 5.1. *With the assumptions of Theorem 2.4,*

$$\lim_{\nu \rightarrow 0} \nu \int_0^t \|\omega^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2 = 0. \quad (5.2)$$

Proof. Using Lemma A.10,

$$\begin{aligned} & \nu \int_0^t \|\omega^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2 \\ &= \nu \int_0^t \sum_{m=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{n=-L(\nu)}^{L(\nu)} \sum_{k=1}^{L(\nu)} g_{mj}(s) \overline{g_{nk}(s)} ds \langle \omega_{mj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} \\ &= \nu \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{k=1}^{L(\nu)} g_{nj}(s) \overline{g_{nk}(s)} ds \langle \omega_{nj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} \\ &\leq \nu \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{k=1}^{L(\nu)} |g_{nj}(s)| |g_{nk}(s)| ds \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})} \|\omega_{nk}\|_{L^2(\Gamma_{c\nu})} \\ &= \nu \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \left(\sum_{j=1}^{L(\nu)} |g_{nj}(s)| \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})} \right)^2 ds \\ &\leq \nu \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 \sum_{j=1}^{L(\nu)} \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})}^2 ds, \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step.

By Lemma A.3,

$$1/L(\nu) < C/(L(\nu) + 2) \leq C/j_{L(\nu)+1,1} = C\lambda_{L(\nu),1}^{-1/2}.$$

Since $\nu L(\nu) \rightarrow 0$ as $\nu \rightarrow 0$, for all sufficiently small ν we have $c\nu < C/L(\nu) \leq (2\pi)^{-1}\lambda_{L(\nu),1}^{-1/2} \leq (2\pi)^{-1}\lambda_{j,1}^{-1/2}$ for all $j \leq L(\nu)$, so by Lemma A.8,

$$\sum_{j=1}^{L(\nu)} \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})}^2 \leq 2\nu L(\nu). \quad (5.3)$$

Then using Equation (5.1),

$$\nu \int_0^t \|\omega^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2 \leq C\nu L(\nu) \left(\nu \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 ds \right) \leq C\nu L(\nu),$$

which vanishes with ν by the condition in Equation (2.10), and Equation (5.2) therefore holds. \square

Remark 5.2. We could try to improve Theorem 5.1 by using $\omega(\tilde{u}^N)$ of Equation (2.9) in place of ω^N , thereby incorporating all of the frequencies in the radial direction for a given angular frequency. Unfortunately, the best bound that one can achieve on $\|\omega_{nj}\|_{L^2(\Gamma_\delta)}^2$ for $j > n$ is the extension of Lemma A.8 described in Remark A.1, and this is very much insufficient to bound the terms with $j > n$.

Corollary 5.2. *The conditions in Equation (2.1) and Equation (2.13) of Theorem 2.4 are equivalent.*

Proof. That Equation (2.1) implies Equation (2.13) follows directly from Theorem 2.3. So assume that Equation (2.13) holds. Because $\|A + B\|^2 \leq 2\|A\|^2 + 2\|B\|^2$ for any norm,

$$\nu \int_0^t \|\omega\|_{L^2(\Gamma_{c\nu})}^2 \leq 2\nu \int_0^t \|\omega^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2 + 2\nu \int_0^t \|\omega - \omega^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2.$$

This vanishes with ν by Theorem 5.1 and Equation (2.13), showing that Equation (2.3) holds and hence by Theorem 2.3 that Equation (2.1) holds. \square

Theorem 5.3. *With the assumptions of Theorem 2.4,*

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu} \int_0^t \|u(s) - u^{M(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds = 0 \quad (5.4)$$

and

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu} \int_0^t \|u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds = 0. \quad (5.5)$$

Proof. We can write $u(t) - u^{M(\nu)}(t) = A(t) + B(t)$, where

$$A(t) = \sum_{m=-M(\nu)}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} g_{mj}(t) u_{mj}(x), \quad B(t) = \sum_{|m|>M(\nu)} \sum_{j=1}^{\infty} g_{mj}(t) u_{mj}(x)$$

and

$$\|u(t) - u^{M(\nu)}(t)\|_{L^2(\Gamma_{c\nu})}^2 \leq 2 \|A(t)\|_{L^2(\Gamma_{c\nu})}^2 + 2 \|B(t)\|_{L^2(\Gamma_{c\nu})}^2.$$

Now,

$$\begin{aligned} \|A(t)\|_{L^2(\Gamma_{c\nu})}^2 &\leq \|A(t)\|_{L^2(\Omega)}^2 = \sum_{m=-M(\nu)}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} |g_{mj}(t)|^2 \|u_{mj}\|_{L^2(\Omega)}^2 \\ &= \sum_{m=-M(\nu)}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} \frac{|g_{mj}(t)|^2}{\lambda_{mj}} \leq \frac{1}{\lambda_{1M(\nu)}} \sum_{m=-M(\nu)}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} |g_{mj}(t)|^2 \\ &\leq \frac{1}{\lambda_{1M(\nu)}} \|\omega(t) - \omega^{M(\nu)}(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

where we used Equation (3.2). Similarly,

$$\|B(t)\|_{L^2(\Gamma_{c\nu})}^2 \leq \|B(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_{M(\nu)1}} \|\omega(t) - \omega^{M(\nu)}(t)\|_{L^2(\Omega)}^2.$$

By Equation (4.2) and Lemma A.3, $\lambda_{M(\nu)1}$ and $\lambda_{1M(\nu)}$ are both bounded below (and above) by $CM(\nu)^2$, so

$$\|u(t) - u^{M(\nu)}(t)\|_{L^2(\Gamma_{c\nu})}^2 \leq \frac{C}{M(\nu)^2} \|\omega(t) - \omega^{M(\nu)}(t)\|_{L^2(\Omega)}^2.$$

Then,

$$\begin{aligned} &\frac{1}{\nu} \int_0^t \|u(s) - u^{M(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds \\ &\leq \frac{C}{\nu M(\nu)^2} \int_0^t \|\omega(s) - \omega^{M(\nu)}(s)\|_{L^2(\Omega)}^2 ds \\ &\leq \frac{C}{\nu M(\nu)^2} \int_0^t \|\omega(s)\|_{L^2(\Omega)}^2 ds \\ &= \frac{C}{\nu^2 M(\nu)^2} \nu \int_0^t \|\omega(s)\|_{L^2(\Omega)}^2 ds \leq \frac{C}{\nu^2 M(\nu)^2}, \end{aligned}$$

where in the last inequality we used Equation (5.1). This vanishes with ν by the assumption on M in Equation (2.10) giving Equation (5.4).

Arguing as in the proof of Theorem 5.1,

$$\begin{aligned} \frac{1}{\nu} \int_0^t \|u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 ds &\leq \frac{1}{\nu} \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 \sum_{j=1}^{L(\nu)} \|u_{nj}\|_{L^2(\Gamma_{c\nu})}^2 ds \\ &\leq \frac{CL(\nu)\nu^3}{\nu} \int_0^t \sum_{n=-L(\nu)}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 ds \leq CL(\nu)\nu \end{aligned}$$

for all sufficiently small ν . In the second inequality we used Lemma A.9 and in the last inequality we used Equation (5.1). This integral also vanishes with ν by the assumption on L in Equation (2.10), giving Equation (5.5). \square

Corollary 5.4. *The conditions in Equation (2.1), Equation (2.11), Equation (2.14), and Equation (2.17) of Theorem 2.4 are equivalent.*

Proof. For sufficiently large ν , $L(\nu) \leq M(\nu)$, and we have

$$\begin{aligned} \|u(s)\|_{L^2(\Gamma_{c\nu})}^2 &\leq 3\|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 \\ &\quad + 3\|u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 + 3\|u(s) - u^{M(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2. \end{aligned}$$

It follows from Theorem 5.3 that

$$\limsup_{\nu \rightarrow 0} \frac{1}{\nu} \int_0^t \|u(s)\|_{L^2(\Gamma_{c\nu})}^2 \leq 3 \limsup_{\nu \rightarrow 0} \frac{1}{\nu} \int_0^t \|u^{M(\nu)} - u^{L(\nu)}\|_{L^2(\Gamma_{c\nu})}^2.$$

In particular, the first limsup is zero if and only if the second limsup is zero (the reverse inequality without the factor of 3 being trivial). Then Equation (2.7) of Theorem 2.3 shows that Equation (2.17) holds if and only if Equation (2.1) holds. The sufficiency of Equation (2.11) and Equation (2.14) for Equation (2.1) to hold then follows from Poincaré's inequality in the form

$$\begin{aligned} \|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 &\leq C\nu^2 \|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|_{L^2(\Gamma_{c\nu})}^2 \\ &\leq C\nu^2 \|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|_{L^2(\Omega)}^2 \\ &= C\nu^2 \|\omega^{M(\nu)}(s) - \omega^{L(\nu)}(s)\|_{L^2(\Omega)}^2. \end{aligned}$$

The necessity of Equation (2.11) and Equation (2.14) follow immediately from Theorem 2.3. \square

Remark 5.3. If we replace the definition of u^N in Equation (2.8) with that in Equation (2.18), then it is clear that Equation (5.4) continues to hold in any bounded domain in \mathbb{R}^2 with a C^2 -boundary. It follows as in Corollary 5.4 that the vanishing viscosity limit of Definition 2.2 holds if and only if the condition in Equation (2.11), Equation (2.14), or (when $k = 2$) Equation (2.17) holds with the term involving $u^{L(\nu)}$ in each of these conditions removed. A similar result would hold in any dimension for an arbitrary bounded domain with a C^2 -boundary.

Theorem 5.5. *With the assumptions of Theorem 2.4,*

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \tilde{u}_{\boldsymbol{\tau}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0 \quad (5.6)$$

and

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \tilde{u}_{\mathbf{n}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (5.7)$$

Proof. In the unit disk, $u_{\boldsymbol{\tau}} = u^\theta$ and $\nabla_{\boldsymbol{\tau}} = \partial_\sigma$, where σ is arc length along the circle of radius r , in which r is held constant. Thus,

$$\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}} = \frac{\partial u^\theta}{\partial \sigma} = \frac{1}{r} \frac{\partial u^\theta}{\partial \theta}$$

and for any positive integer N it follows from Poincaré's inequality that

$$\begin{aligned} \|\nabla_{\boldsymbol{\tau}} \tilde{u}_{\boldsymbol{\tau}}^N(s)\|_{L^2(\Gamma_\delta)}^2 &= \left\| \frac{1}{r} \frac{\partial}{\partial \theta} (\tilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \leq \frac{1}{(1-\delta)^2} \left\| \frac{\partial}{\partial \theta} (\tilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \\ &\leq \frac{C\delta^2}{(1-\delta)^2} \left\| \frac{\partial^2}{\partial r \partial \theta} (\tilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \leq \frac{C\delta^2}{(1-\delta)^2} \left\| \frac{\partial^2}{\partial r \partial \theta} (\tilde{u}^N(s))^\theta \right\|_{L^2(\Omega)}^2. \end{aligned}$$

But,

$$\begin{aligned} \frac{\partial^2}{\partial r \partial \theta} (\tilde{u}^N(s))^\theta &= \sum_{m=-N}^N \sum_{j=1}^{\infty} g_{mj}(s) \frac{\partial^2}{\partial r \partial \theta} u_{mj}(r, \theta) \\ &= i \sum_{m=-N}^N m \sum_{j=1}^{\infty} g_{mj}(s) \frac{\partial}{\partial r} u_{mj}(r, \theta), \end{aligned}$$

the last equality following from the simple dependence of u_{mj} on θ in Equation (4.3). Thus,

$$\begin{aligned} \left\| \frac{\partial^2}{\partial r \partial \theta} (\tilde{u}^N(s))^\theta \right\|_{L^2(\Omega)}^2 &\leq \left\| i \sum_{m=-N}^N m \sum_{j=1}^{\infty} g_{mj}(s) \nabla u_{mj}(r, \theta) \right\|_{L^2(\Omega)}^2 \\ &= \sum_{m=-N}^N m^2 \sum_{j=1}^{\infty} |g_{mj}(s)|^2 \leq (2N+1)^2 \sum_{m=-N}^N \sum_{j=1}^{\infty} |g_{mj}(s)|^2 \\ &\leq CN^2 \|\nabla u\|_{L^2(\Omega)}^2, \end{aligned}$$

where we used the orthonormality of the eigenfunctions in V .

Combining these two inequalities gives

$$\|\nabla_{\boldsymbol{\tau}} \tilde{u}_{\boldsymbol{\tau}}^N(s)\|_{L^2(\Gamma_\delta)}^2 \leq \frac{CN^2\delta^2}{(1-\delta)^2} \|\nabla u\|_{L^2(\Omega)}^2.$$

Then using Equation (5.1),

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \tilde{u}_{\boldsymbol{\tau}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 ds \leq \frac{CL(\delta)^2\delta^2}{(1-\delta)^2} \nu \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 ds \leq \frac{CL(\delta)^2\delta^2}{(1-\delta)^2}.$$

This vanishes with δ by the assumption on L in Equation (2.10) and hence vanishes with ν since δ vanishes with ν , giving Equation (5.6). The proof of Equation (5.7) is entirely analogous. \square

The technique used in the proof of Theorem 5.5 comes from the key inequality following Equation (49) in [3].

Corollary 5.6. *The conditions in Equation (2.1), Equation (2.12), Equation (2.15), and Equation (2.16) of Theorem 2.4 are equivalent.*

Proof. This corollary can be proved much along the lines of the proofs of Corollary 5.2 and Corollary 5.4. (It is here that we use the assumption that $\delta(\nu)/\nu$ diverges to ∞ as $\nu \rightarrow 0$, which is needed in applying Theorem 2.3.) \square

Together, Corollaries 5.2, 5.4, and 5.6 establish Theorem 2.4.

6. RADially SYMMETRIC INITIAL VORTICITY

When the initial vorticity is radially symmetric, much more can be said. The nonlinear terms in (NS) and (E) disappear and the solution to (E) is steady state. Assuming that the initial velocity is in H , convergence of the velocity as in Equation (2.1) holds. This follows immediately from the conditions in Equation (2.12), Equation (2.15), or Equation (2.16) of Theorem 2.4. The convergence also follows from the sufficiency of the conditions in Equation (2.5) and Equation (2.6) as established in [14], since both conditions are satisfied (the gradients in the tangential direction being zero) as pointed out in [15]. When the forcing is zero, however, there is a simple proof that uses only Kato's original conditions.

Theorem 6.1. *Assume that u_ν^0 and \bar{u}^0 are as in Theorem 2.3 with (for simplicity) $u_\nu^0 = \bar{u}^0$, that $f = \bar{f} = 0$, and that $\omega^0 = \omega(u^0)$ is radially symmetric. Then the vanishing viscosity limit of Equation (2.1) holds.*

Proof. Because ω^0 is radially symmetric, ω remains radially symmetric for all time, so $\omega(u \cdot \nabla u) = u \cdot \nabla \omega = 0$. Then because Ω is simply connected, $u \cdot \nabla u = \nabla q$ for some scalar field q , and the nonlinear term in (NS) disappears. Thus, (NS) reduces to $u_\nu(0) = \bar{u}^0$ and

$$\int_{\Omega} \partial_t u_\nu \cdot v + \nu \int_{\Omega} \nabla u_\nu \cdot \nabla v = 0 \quad (6.1)$$

for all v in V . This is Stokes problem in weak form, which is invariant under the transformation $(\nu, t, x) \mapsto (1, \nu t, x)$. That is, if u_1 is a solution to Equation (6.1) with $\nu = 1$, then $u_\nu(t, x) = u_1(\nu t, x)$ is a solution to Equation (6.1) because $u_\nu(0) = u_1(0) = \bar{u}^0$ and

$$\begin{aligned} & \int_{\Omega} \frac{\partial}{\partial t} u_1(\nu t, x) \cdot v(x) dx + \nu \int_{\Omega} \nabla u_1(\nu t, x) \cdot \nabla v(x) dx \\ &= \nu \left[\int_{\Omega} (\partial_t u_1)(\nu t, x) \cdot v(x) dx + \int_{\Omega} \nabla u_1(\nu t, x) \cdot \nabla v(x) dx \right] = 0. \end{aligned}$$

It follows that

$$\nu \int_0^t \|\omega(s)\|_{L^2(\Omega)}^2 ds = \nu \int_0^t \|\omega_1(\nu s)\|_{L^2(\Omega)}^2 ds = \int_0^{\nu t} \|\omega_1(\tau)\|_{L^2(\Omega)}^2 d\tau.$$

This vanishes as $\nu \rightarrow 0$ by the continuity of the integral, because u_1 is in $L^2([0, T]; V)$. The limit in Equation (2.1) then follows from the condition in Equation (2.2) of Theorem 2.3. \square

7. INTERPRETATION IN TERMS OF LENGTH SCALES

In [3], Cheng and Wang consider the vanishing viscosity limit in the setting of a two-dimensional rectangular channel R , periodic in the x direction with period L and with characteristic boundary conditions (which include no-slip boundary conditions as a special case). They decompose any vector u on R of sufficient regularity as $u = \sum_{j=0}^{\infty} e^{2\pi i j x/L} u^j$ and define the projection $P_k u = \sum_{j=0}^k e^{2\pi i j x/L} u^j$ onto the space spanned by the first k modes. This in effect allows one to isolate successively finer-scale spatial variations in the direction tangential to the boundary. They then construct an approximation sequence $\{v^L\}$ to u by letting v^L be the solution to the equation that results after projecting each term in (NS) using P_N . (We have changed their notation somewhat.) Their v^L is the approximate-solution analog of the exact solution truncation represented by \tilde{u}^L in Equation (2.9).

The main result in [3] is that $v^{L(\nu)}$ converges to \bar{u} in $L^\infty([0, T]; L^2(\Omega))$ as $\nu \rightarrow 0$. The requirement on $L(\nu)$ is the same as our condition on L in Equation (2.10) (with the additional condition that $L(\nu) \rightarrow \infty$ as $\nu \rightarrow 0$ as one would expect), so convergence of $v^{L(\nu)}$ to \bar{u} occurs when only tangential length scales of order larger than ν are included in the approximations. (All length scales in the normal direction, however, are included. See Remark 5.2 concerning this issue in regards to the vorticity.)

The result in [3] is an important observation about the difficulty of determining numerically whether or not the vanishing viscosity limit holds. Our method of decomposing the solution using the eigenfunctions of the Stokes operator, on the other hand, says little about computation, since approximating this decomposition numerically is probably as least as hard as approximating the solution itself. Nonetheless, it more directly characterizes the properties of the solution itself at different length scales.

The analog to the result in [3] is Theorem 5.5, which shows that Temam and Wang's conditions in Equation (2.5) and Equation (2.6), when applied only to the modes with tangential wavelengths of $CL(\nu)$ or higher, holds as long as the condition on N in Equation (2.10) hold. This does not, however, imply that $u^{L(\nu)}$ converges to \bar{u} in the vanishing viscosity limit, only that if the vanishing viscosity limit fails to hold, the failure originates in the behavior of the tangential component of the gradient projected into the space spanned by the modes with tangential frequencies of order $L(\nu)$ or higher; that is, at length scales of order ν or lower.

The other conditions in Theorem 2.4 give alternative ways to measure the behavior of the solution at different length scales or frequencies. They show that we cannot simply say if the vanishing viscosity fails to hold that the failure must lie in the behavior of the solution at any particular range of length scales, but rather that the pertinent range of length scales varies with the measure of behavior. Whether any of these conditions brings us any closer to proving that the vanishing viscosity limit holds in general for smooth initial data in a bounded domain or to proving that it fails to hold in at least one instance remains completely unclear.

8. A TWO-DIMENSIONAL CHANNEL

It is instructive to compare Theorem 2.4, which we established for a disk, to the equivalent theorem for a channel. A complete analysis of a channel would be nearly as involved as our analysis for the disk, so we limit our discussion to those results that can be easily obtained analytically or for which there is strong numerical evidence.

Let R be the rectangular channel $[0, 2\pi] \times [-1, 1]$, with the velocity periodic in the x -direction and vanishing on the boundary, $y = \pm 1$. Analytic expressions for the eigenfunctions of the Stokes operator for R were calculated in [1] (see also [12], restricted to the case $k_3 = 0$, which clarifies some misprints in [1]). We can write them as

$$u_{nk}^e = C_{nk}^e \left(i \frac{\cos(\sigma_{nk}y)}{\cos \sigma_{nk}} - i \frac{\cosh(ny)}{\cosh n}, \frac{n \sin(\sigma_{nk}y)}{\sigma_{nk} \cos \sigma_{nk}} - \frac{\sinh(ny)}{\cosh n} \right) e^{inx}$$

when σ_{nk} is the k -th positive value satisfying $\sigma_{nk} \cot \sigma_{nk} = n \coth n$, or

$$u_{n,k}^o = C_{nk}^o \left(i \frac{\sin(\sigma_{nk}y)}{\sin \sigma_{nk}} - i \frac{\sinh(ny)}{\sinh n}, -\frac{n \cos(\sigma_{nk}y)}{\sigma_{nk} \sin \sigma_{nk}} - \frac{\cosh(ny)}{\sinh n} \right) e^{inx}$$

when σ_{nk} is the k -th positive value satisfying $\sigma_{nk} \tan \sigma_{nk} = -n \tanh n$. In both cases, n ranges over the integers and the corresponding eigenvalue is $\lambda_{nk} = \sigma_{nk}^2 + n^2$. Choosing the normalizing constants

$$C_{nk}^e = \frac{\sigma_{n,k}^{3/2} \cos \sigma_{n,k}}{\sqrt{2\pi} \lambda_{n,k} \sqrt{\sigma_{n,k} - \cos \sigma_{n,k} \sin \sigma_{n,k}}},$$

$$C_{nk}^o = \frac{\sigma_{n,k}^{3/2} \sin \sigma_{n,k}}{\sqrt{2\pi} \lambda_{n,k} \sqrt{\sigma_{n,k} + \cos \sigma_{n,k} \sin \sigma_{n,k}}},$$

the corresponding vorticities,

$$\omega_{nk}^e = \frac{\sigma_{n,k}^{1/2} \sin(\sigma_{n,k}y)}{\sqrt{2\pi} \sqrt{\sigma_{n,k} - \cos \sigma_{n,k} \sin \sigma_{n,k}}} i e^{inx},$$

$$\omega_{n,k}^o = \frac{\sigma_{n,k}^{1/2} \cos(\sigma_{n,k}y)}{\sqrt{2\pi} \sqrt{\sigma_{n,k} + \cos \sigma_{n,k} \sin \sigma_{n,k}}} i e^{inx},$$

each have L^2 -norm equal to 1.

Suppose first that $\sigma_{n,k}$ were an integer or half-integer. Then simply from the fact that the integral over a multiple of a quarter-period of \sin^2 or \cos^2 is equal to the quarter-period itself, and because ω_{nk}^e has L^2 -norm equal to 1, it would follow that $\|\omega_{nk}^e\|_{L^2(\Gamma_\delta)}^2 \leq \delta/2$, where Γ_δ is a boundary layer of width $\delta/2$, the same bound holding for ω_{nk}^o . A bound that holds for all periods is $3\delta/2$.

Moreover, this bound is independent of both n and k . Thus if we replace the sums on m and n in the proof of Theorem 5.1 with sums over \mathbb{Z} , the critical last inequality in that proof will still hold. (It does not help at all with the sums over j and k , however: see Remark 5.2.) Thus, for a channel, we can replace $\omega^{L(\nu)}$ in Equation (2.13) with the vorticity of

$$\sum_{m=-\infty}^{\infty} \sum_{j=1}^{L(\nu)} (g_{mj}^e(t)u_{mj}^e(x) + g_{mj}^o(t)u_{mj}^o(x)). \quad (8.1)$$

The inequality $\|\omega_{nk}\|_{L^2(\Gamma_\delta)}^2 \leq C\delta$ demonstrably does not hold for a disk. This is because for a fixed value of k , as n gets large, the vorticity is concentrated more and more tightly near the boundary, an effect that is due to J_n being exponentially small until shortly before its first zero. This prevents the bound in Lemma A.8 from being materially improved.

The improvement for a channel in the analysis of the vorticity comes about because the vorticity of each eigenfunction is nearly uniformly distributed across the channel. The same is true, empirically, of the Euclidean norm of the gradient of the velocity. If this norm were constant on Ω , then applying Poincaré's inequality would give

$$\|u_{nk}^e\|_{L^2(\Gamma_\delta)}^2 \leq C\delta^2 \|\nabla u_{nk}^e\|_{L^2(\Gamma_\delta)}^2 \leq C\delta^3 \|\nabla u_{nk}^e\|_{L^2(\Omega)}^2 = C\delta^3,$$

since ∇u_{nk}^e and ω_{nk}^e have the same L^2 -norm of 1 on Ω . In fact we find numerically that $\|u_{nk}^e\|_{L^2(\Gamma_\delta)}^2 \leq (1/3)\delta^3$ and similarly for u_0 for all δ independently of n and k . If we accept this numeric evidence (which is quite strong), it follows that $u^{L(\nu)}$ in Equation (2.11), Equation (2.14), and Equation (2.17) can be replaced by the sum in Equation (8.1).

For both a channel and a disk, the L^2 -norm of the velocity of the eigenfunctions increases almost precisely cubically in a boundary layer of width proportional to $\lambda^{-1/2}$, where λ is the associated eigenvalue. For a disk, however, ∇u_{nk} is concentrated near the boundary for n large relative to k , as it is for the vorticity, so one cannot make the simple argument using Poincaré's inequality above. Unlike the situation for the vorticity, this does not itself rule out the possibility of improving the result for the velocity in the disk. But an obstacle to achieving this improvement is that experimenting numerically with the velocity for a disk, and even more so with its gradient, requires fairly resource-intensive calculations, even for quite low values of n .

and k , so it is hard to gain insight this way. This is in contrast to the channel, where all the computations execute quickly even for quite large values of n and k .

APPENDIX A. BOUNDS ON THE EIGENFUNCTIONS

The Bessel function, J_n , of the first kind of order n is a solution to

$$\frac{d^2 J_n(s)}{ds^2} + \frac{1}{s} \frac{dJ_n(s)}{ds} - \left(1 - \frac{n^2}{s^2}\right) J_n(s) = 0. \quad (\text{A.1})$$

In Lemma A.1 we state the basic identities involving the Bessel functions that we use. We then give a series of lemmas that lead to the bounds on the velocity and vorticity of the eigenfunctions in the boundary layer that we used in the proof of Theorem 2.4.

It is perhaps important to note that in the proofs that follow we avoid the use of asymptotic formulas for the Bessel functions, even when such formulas might appear to be useful. This is because we need to deal with the relative values of Bessel functions of different orders near a zero of one of the Bessel functions, and it is precisely in these situations that the errors in the asymptotic formulas dominate. Also, most of the following lemmas apply without change to their proofs with n being any nonnegative real value.

Lemma A.1. *For all nonnegative real numbers n and x ,*

$$2nJ_n(x) - xJ_{n-1}(x) = xJ_{n+1}(x), \quad (\text{A.2})$$

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x), \quad (\text{A.3})$$

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J'_n(x), \quad (\text{A.4})$$

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J'_n(x), \quad (\text{A.5})$$

$$\frac{x^n J_n(\alpha x)}{\alpha} = \int x^n J_{n-1}(\alpha x) dx, \quad (\text{A.6})$$

$$J_n(\alpha x)x^{-n} = -\alpha \int J_{n+1}(\alpha x)x^{-n} dx, \quad (\text{A.7})$$

$$(\beta^2 - \alpha^2) \int x J_n(\alpha x) J_n(\beta x) dx = x [\alpha J'_n(\alpha x) J_n(\beta x) - \beta J'_n(\beta x) J_n(\alpha x)], \quad (\text{A.8})$$

$$\int x J_n(ax)^2 dx = \frac{1}{2} \left[x^2 J'_n(ax)^2 + \left(x^2 - \frac{n^2}{a^2}\right) J_n(ax)^2 \right], \quad (\text{A.9})$$

$$\int x J_n(ax)^2 dx = \frac{x^2}{2} [J_n(ax)^2 - J_{n-1}(ax) J_{n+1}(ax)]. \quad (\text{A.10})$$

Proof. These are standard identities for Bessel functions. For instance, see Equations (6.28), (6.29), (6.30), (6.31), (6.38), (6.39), (6.51), (6.52), and (6.53) of [2]. \square

Lemma A.2. *For all nonnegative integers n and all positive integers k ,*

$$1 < j_{n+1,k} - j_{nk} < \frac{\pi}{2},$$

where j_{nk} is defined in Equation (4.1).

Proof. Let $j_{\nu k}$ be the k -th positive zero of J_{ν} , where we now allow ν to be a real number in the interval $[0, \infty)$. It is shown in [4] and [5] that for all $k \geq 1$, $j_{\nu k}$ is strictly concave as a function of ν and that $dj_{\nu k}/d\nu > 1$ (see also [9]). Thus, the function $n \mapsto j_{n+1,k} - j_{nk}$ is strictly decreasing as a function of n . But by Equation (2.9) of [5], $j_{n+1,k} - j_{nk} \rightarrow 1$ as $n \rightarrow \infty$, so $j_{n+1,k} - j_{nk} > 1$.

The positive zeros of J_0 lie in the intervals $(m\pi + \frac{3}{4}\pi, m\pi + \frac{7}{8}\pi)$, $m = 0, 1, \dots$, and the positive zeros of J_1 lie in the intervals $(m'\pi + \frac{1}{8}\pi, m'\pi + \frac{1}{4}\pi)$, $m' = 1, 2, \dots$. That the zeros lie in only these intervals is shown in Section 15.32 p. 489 and Section 15.34 p. 491 of [16] using an approach of Schafheitlin's. That each of these intervals contains at least one zero is shown on p. 104 of [2]. But $j_{1k} - j_{0k} > 1$ as we showed above so each interval contains precisely one zero. Because the zeros of J_0 and J_1 are interleaved (see p. 106 of [2], for instance) we can then conclude that $j_{1k} - j_{0k} < \frac{\pi}{2}$. But as we observed above, the function $n \mapsto j_{n+1,k} - j_{nk}$ is strictly decreasing as a function of n , so $j_{n+1,k} - j_{nk} < \frac{\pi}{2}$ holds for all $n \geq 0$. \square

Lemma A.3. *For all $n = 0, 1, \dots$ and $k = 1, 2, \dots$,*

$$n + k < j_{nk} < \pi(n/2 + k) \leq \pi(n + k).$$

Proof. By Lemma A.2, for all n and j ,

$$j_{nk} = j_{0k} + \sum_{m=1}^n (j_{mk} - j_{m-1,k}) \geq j_{0k} + n > n + k,$$

because $j_{0k} > k$ (which follows directly from Equation (A.1); see p. 485-486 of [16], for instance). By an observation in the proof of Lemma A.2 it follows that $j_{0k} < \pi k$, and a similar argument using the inequality $j_{n+1,k} - j_{nk} < \frac{\pi}{2}$ from Lemma A.2 gives the upper bound on j_{nk} . \square

Lemma A.4. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. For $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots$,*

$$\left| \frac{J_n(\alpha x)}{J_n(\alpha)} \right| \leq 1 \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Let $g(x) = J_n(\alpha x)/|J_n(\alpha)|$. From Equation (A.5), $J'_n(\alpha) = (n/\alpha)J_n(\alpha)$, so $J'_n(\alpha)$ has the same sign as $J_n(\alpha)$. From this we conclude that $|g|$ is increasing in a left-neighborhood N of 1.

Between each zero of J_n there is exactly one zero of J_{n+1} (see p. 106 of [2], for instance). Between each zero of J_n there is also exactly one zero of J'_n , because the maximum values of J_n are all positive and the minimum values are all negative (see, for instance, p. 107 of [2]) and J'_n has no repeated positive roots (this follows from the defining equation Equation (A.1)). Thus, the neighborhood N includes all x such that $\beta < \alpha x < \alpha$. Since $|g(1)| = 1$ it follows that $|g(x)| \leq 1$ for all such x . \square

Lemma A.5. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. There exists a constant C such that for all $n = 0, 1, \dots$ and $k = 1, 2, \dots, n$,*

$$\left| \frac{J_{n+1}(\alpha x)}{J_n(\alpha)} \right| \leq Cn(1-x) \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Since $J_{n+1}(\alpha) = 0$, Equation (A.6) with $n+1$ in place of n gives

$$J_{n+1}(\alpha x) = -\frac{\alpha}{x^{n+1}} \int_x^1 t^{n+1} J_n(\alpha t) dt.$$

As long as $\beta < \alpha x < \alpha$, $J_n(\alpha t)$ does not change sign on the interval $(x, 1]$ and has its maximum value on this interval at 1, as observed in the proof of Lemma A.4. Thus,

$$|J_{n+1}(\alpha x)| \leq \frac{\alpha}{x^{n+1}} |J_n(\alpha)| \int_x^1 t^{n+1} dt \leq \frac{\alpha |J_n(\alpha)|}{(\beta/\alpha)^{n+1}} (1-x).$$

But by Lemma A.2 and Lemma A.3,

$$1 - \frac{\beta}{\alpha} = \frac{\alpha - \beta}{\alpha} \leq \frac{\pi/2}{n+2} \implies \frac{\beta}{\alpha} \geq 1 - \frac{\pi}{2n+4}$$

so

$$(\beta/\alpha)^{-(n+1)} \leq \left(1 - \frac{\pi}{2n+4}\right)^{-(n+1)} \leq e^{\pi/2},$$

the last inequality following from elementary calculus. We conclude that

$$|J_{n+1}(\alpha x)| \leq Cn |J_n(\alpha)| (1-x), \quad (\text{A.11})$$

which completes the proof. \square

Lemma A.6. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. There exists a constant C such that for all $n = 0, 1, \dots$ and $k = 1, 2, \dots, n$,*

$$\left| \frac{J_{n-1}(\alpha x)}{J_n(\alpha)} \right| \leq C \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Because the positive zeros of J_{n-1} are interlaced with those of J_n , J_{n-1} does not change sign on the interval $[\beta, \alpha]$. From Equation (A.5) with $n-1$ in place of n , $J'_{n-1}(\beta) = ((n-1)/\beta)J_{n-1}(\beta)$, so $J'_{n-1}(\beta)$ has the same sign as $J_{n-1}(\beta)$, and we conclude that J_{n-1} reaches its maximum value on the interval $[\beta, \alpha]$ at β . Therefore, for $\beta/\alpha < x < 1$,

$$\left| \frac{J_{n-1}(\alpha x)}{J_n(\alpha)} \right| \leq \left| \frac{J_{n-1}(\beta)}{J_n(\alpha)} \right|.$$

But, by Equation (A.2), $J_{n+1}(\beta) = 2(n/\beta)J_n(\beta) - J_{n-1}(\beta) = -J_{n-1}(\beta)$, so

$$\left| \frac{J_{n-1}(\alpha x)}{J_n(\alpha)} \right| \leq \left| \frac{J_{n+1}(\beta)}{J_n(\alpha)} \right| \leq Cn(1 - \beta/\alpha) \leq C,$$

where we used Lemma A.5 and Lemma A.3. \square

Lemma A.7. *For all j in \mathbb{N} , $\|\omega_j\|_{L^2(\Gamma_\delta)}^2 \leq 2\delta$ when $\delta \leq \lambda_j^{-1/2}$.*

Proof. Let $\omega_j = \omega_{nk}$ and $\alpha = j_{n+1,k} = \lambda_j^{1/2}$, where without loss of generality we assume that $n \geq 0$. Then

$$\|\omega_j\|_{L^2(\Gamma_\delta)}^2 = 2\pi C_{nk}^2 \int_{1-\delta}^1 r J_n(\alpha r)^2 dr = 2 \int_{1-\delta}^1 r \frac{J_n(\alpha r)^2}{J_n(\alpha)^2} dr.$$

In the integrals above, with $\beta = j_{nk}$,

$$\beta/\alpha = 1 - (\beta - \alpha)/\alpha \leq 1 - 1/\alpha = 1 - \lambda_j^{-1/2} \leq 1 - \delta < r < 1,$$

where we used Lemma A.2, and the lemma follows from Lemma A.4. \square

Employing Lemma A.7, we can extend its range of applicability, though with a higher bound on the width of the boundary layer.

Lemma A.8. *For all n in \mathbb{Z} , $k = 1, \dots, n$, and all $\delta < (2\pi)^{-1}\lambda_{n1}^{-1/2}$,*

$$\|\omega_{nk}\|_{L^2(\Gamma_\delta)}^2 \leq 2\delta.$$

Proof. Without loss of generality we assume that $n \geq 0$. It follows from Lemma A.3 that $\lambda_{nk}^{1/2}/\lambda_{n1}^{1/2} = j_{n+1,k}/j_{n+1,1} \leq \pi(n+1+k)/(n+2) \leq 2\pi$ for $k = 1, \dots, n$; the lemma follows from this inequality and Lemma A.7. \square

Remark A.1. It is possible to extend Lemma A.8 to include all values of k . The idea of the proof is that for $k > n$, ω_{nk} passes through k complete half-periods (annuli in the unit disk lying between successive nonnegative zeroes of $J_n(j_{n+1,k}r)$) and ends with a partial period. Since $J_n(x)$ decays like $x^{1/2}$ and the spacing between consecutive zeros of J_n approaches a constant, the L^2 -norms of ω_{nk} on each of those half-periods converges to a constant, and since the L^2 -norm of ω_{nk} on the entire unit disk is 1, the square of the L^2 -norm of ω_{nk} on the last half-period is less than C/k (with C near 1). But the last half-period has a width that is greater than C/k . Extending this argument to m periods, what we have shown is that

$$\|\omega_{nk}\|_{L^2(\Gamma_{Cm/k})}^2 \leq m/k.$$

With the assumed bound on δ , we choose m so that m/k is of the same order as δ , and the proof is essentially complete.

Lemma A.9. *There exist positive constants C_1 and C_2 with $C_2 < 1$ such that for all n in \mathbb{Z} and all $k = 1, \dots, n$,*

$$\|u_{nk}\|_{L^2(\Gamma_\delta)}^2 \leq C_1\delta^3$$

when $\delta < C_2\lambda_{n1}^{-1/2}$.

Proof. In the proof that follows, we will often use Lemma A.3 without explicit mention. Also, without loss of generality we assume that $n \geq 0$.

Let $\alpha = j_{n+1,k}$. We bound first the radial component of u_{nk} . We have,

$$\frac{J_n(\alpha r) - J_n(\alpha)r^n}{J_n(\alpha)r} = r^{n-1}g_n(r),$$

where

$$g_n(r) = \frac{J_n(\alpha r)r^{-n}}{J_n(\alpha)} - 1 = -\frac{\alpha}{J_n(\alpha)} \int_r^1 \frac{J_{n+1}(\alpha x)}{x^n} dx = -\alpha \int_r^1 \frac{B_{nk}(x)}{x^n} dx,$$

and

$$B_{nk}(x) = \frac{J_{n+1}(\alpha x)}{J_n(\alpha)}.$$

To verify the second equality we use the identity in Equation (A.7), from which it follows that the second expression for g_n is

$$\frac{J_n(\alpha r)r^{-n} + C}{J_n(\alpha)}$$

for some constant C . But all three expressions for g_n are zero at $r = 1$, so we have the correct limits of integration in the second expression. It follows from Lemma A.5 and our third expression for g_n that

$$|g_n(r)| \leq Cn\alpha \int_r^1 x^{-n}(1-x) dx \leq \frac{Cn^2}{r^n}(1-r)^2$$

for all $1-r \leq 1/\alpha$.

From Equation (4.3),

$$u_{nk}^r(r, \theta) = \frac{g(r)r^{n-1}}{\alpha^2 \pi^{1/2}} in e^{in\theta} \widehat{e}_r,$$

so when $\delta \leq 1/\alpha$ we have

$$\begin{aligned} \|u_{nk}^r\|_{L^2(\Gamma_\delta)}^2 &= 2\pi \int_{1-\delta}^1 r |u_{nk}^r|^2 dr \leq \frac{Cn^6}{\alpha^4} \int_{1-\delta}^1 \frac{(1-r)^4}{r^{2n-1}} dr \\ &\leq Cn^2(1-\delta)^{1-2n}\delta^5 \leq C\delta^3. \end{aligned}$$

In the last inequality we used

$$\delta < C_2 \lambda_{n1}^{-1/2} = \frac{C_2}{j_{n+1,k}} \leq \frac{C_2}{n+1}$$

so

$$\begin{aligned} (1-\delta)^{2n-1} &\geq \left(1 - \frac{C_2}{n+1}\right)^{2n-1} = (G(C_2, n+1))^2 \left(1 - \frac{C_2}{n+1}\right)^{-3} \\ &\geq (1-C_2)^2 (1-C_2)^{-3} = (1-C_2)^{-1} = C > 0, \end{aligned}$$

where G is the function of Lemma A.11.

For the angular component of u_{nk} , we write

$$\begin{aligned}
& \alpha J_{n+1}(\alpha r) - \alpha J_{n-1}(\alpha r) + 2nJ_n(\alpha)r^{n-1} \\
&= [2nJ_n(\alpha r) - \alpha r J_{n-1}(\alpha r)] + \alpha r J_{n-1}(\alpha r) - 2nJ_n(\alpha r) \\
&\quad + \alpha J_{n+1}(\alpha r) - \alpha J_{n-1}(\alpha r) + 2nJ_n(\alpha)r^{n-1} \\
&= \alpha(r+1)J_{n+1}(\alpha r) + 2n[J_n(\alpha)r^{n-1} - J_n(\alpha r)] \\
&\quad + \alpha J_{n-1}(\alpha r)(r-1).
\end{aligned}$$

From Equation (4.3) we then have

$$\begin{aligned}
|u_{nk}^\theta|^2 &\leq C \frac{\alpha^2(r+1)^2}{\alpha^4} \frac{J_{n+1}(\alpha r)^2}{J_n(\alpha)^2} + C \frac{n^2}{\alpha^4} \left(\frac{J_n(\alpha)r^{n-1} - J_n(\alpha r)}{J_n(\alpha)} \right)^2 \\
&\quad + C \frac{\alpha^2 J_{n-1}(\alpha r)^2 (1-r)^2}{\alpha^4 J_n(\alpha)^2} \\
&\leq C(1-r)^2 + \frac{C}{n^2} (r^{n-2} g_{n-1}(r))^2,
\end{aligned} \tag{A.12}$$

where we applied both Lemma A.5 and Lemma A.6.

The first term in Equation (A.12) contributes no more than

$$C \int_{1-\delta}^1 r(1-r)^2 dr \leq C \int_{1-\delta}^1 (1-r)^2 dr \leq C\delta^3,$$

and the same is true of the second term in Equation (A.12) arguing as for u_{nk}^r , and this completes the proof. \square

Lemma A.10. *When $m \neq n$, $\langle u_{mj}, u_{nk} \rangle_{L^2(\Gamma_{c\nu})} = \langle \omega_{mj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} = 0$.*

Proof. We have,

$$\langle \omega_{mj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} = \int_{1-c\nu}^1 r f(r) \int_0^{2\pi} e^{i(m-n)\theta} d\theta dr,$$

where $f(r)$ is a product of two Bessel functions. When $m \neq n$, the inner integral is zero. A similar argument gives $\langle u_{mj}, u_{nk} \rangle_{L^2(\Gamma_{c\nu})} = 0$. \square

Lemma A.11. *Let α be in $(0, 1)$ and define $G_\alpha : [1, \infty) \rightarrow [0, \infty)$ by*

$$G_\alpha(x) = \left(1 - \frac{\alpha}{x}\right)^x.$$

Then for all $x > 1$,

$$1 - \alpha \leq G_\alpha(x) < e^{-\alpha}.$$

Proof. The proof is elementary. \square

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