SOME FURTHER OBSERVATIONS ON THE VANISHING VISCOSITY LIMIT AND THE ACCUMULATION OF VORTICITY ON THE BOUNDARY

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ABSTRACT. These are some further, simple observations that serve as a companion to the paper, "Vanishing viscosity and the accumulation of vorticity on the boundary" ([4]).

CONTENTS

1. L^2-NORM OF THE VORTICITY BLOWS UP

Theorem 1.1. *Assume that $\overline{u}$ is not identically zero on $[0,T] \times \Gamma$. If any of the equivalent conditions of Theorem 2.1 of [4] holds then*

$$\limsup_{\nu \to 0^+} \|\omega\|_{L^\infty([0,T];L^2)} \to \infty. \tag{1.1}$$

Proof. We prove the contrapositive. Suppose that Equation (1.1) does not hold. This means that for some $C_0 > 0$ and $\nu_0 > 0$,

$$\|\omega\|_{L^\infty([0,T];L^2)} \le C_0 \text{ for all } 0 < \nu \le \nu_0. \tag{1.2}$$

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Then

$$\begin{aligned}
\|u(t) - \overline{u}(t)\|_{L^2(\Gamma)} &\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1/2} \|\nabla u(t) - \nabla \overline{u}(t)\|_{L^2(\Omega)}^{1/2} \\
&\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1/2} (\|\nabla u(t)\|_{L^2} + \|\nabla \overline{u}(t)\|_{L^2})^{1/2} \\
&= C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1/2} (\|\omega(t)\|_{L^2} + \|\nabla \overline{u}(t)\|_{L^2})^{1/2} \\
&\le C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1/2}
\end{aligned}$$

for all $0 < \nu \le \nu_0$. The first inequality follows from a standard trace theorem for velocity fields in $H \cap H^1(\Omega)$, the third inequality from Equation (1.2). Hence,

$$\|u - \overline{u}\|_{L^\infty([0,T];L^2(\Gamma))} \le C \|u - \overline{u}\|_{L^\infty([0,T];L^2(\Omega))}^{1/2}$$

for all $0 < \nu \le \nu_0$. But,

$$\|u - \overline{u}\|_{L^\infty([0,T];L^2(\Gamma))} = \|\overline{u}\|_{L^\infty([0,T];L^2(\Gamma))} \ne 0,$$

so condition (B) of [4] cannot hold. Hence, none of the equivalent conditions in Theorem 2.1 of [4] can hold. □

2. Navier boundary conditions in 2D

Theorem 1.1 says that if the vanishing viscosity limit holds, then there cannot be a uniform (in ν) bound on the L^2-norm of the vorticity. This is in stark contrast to the situation in the whole space, where such a bound holds, or for Navier boundary conditions in 2D, where such a bound holds for L^p, $p > 2$, as shown in [7] and [1]. For Navier boundary conditions in 2D, then, as long as the initial vorticity is in L^p for $p > 2$ there will be a uniform bound on the L^2-norm of the vorticity, since the domain is bounded.

In fact, for Navier boundary conditions in 2D the vanishing viscosity limit (condition (B) of [4]) does hold, even for much weaker regularity on the initial velocity than that considered here (see [5]). The argument in the proof of Theorem 1.1 then shows that

$$u \to \overline{u} \text{ in } L^\infty([0,T]; L^2(\Gamma)). \tag{2.1}$$

We also have weak* convergence of the vorticity in H^1:

Theorem 2.1. *Assume that the solutions to (NS) are with Navier boundary conditions in 2D, and that the initial vorticity $\omega^0 = \overline{\omega}^0$ is in L^∞ (slightly weaker assumptions as in [5] can be made). Then all of the conditions in Theorem 2.1 of [4] hold, but with the three conditions below replacing conditions (C), (E), and (E_2), respectively:*

$$\begin{aligned}
&(C^N) \qquad \nabla u \to \nabla \overline{u} \text{ in } ((H^1(\Omega))^{d\times d})' \text{ uniformly on } [0,T], \\
&(E^N) \qquad \omega \to \overline{\omega} \text{ in } ((H^1(\Omega))^{d\times d})' \text{ uniformly on } [0,T], \\
&(E_2^N) \qquad \omega \to \overline{\omega} \text{ in } (H^1(\Omega))' \text{ uniformly on } [0,T].
\end{aligned}$$

Proof. First observe that (E^N) is just a reformulation of (E_2^N) with vorticity viewed as a matrix.

It is shown in [5] that condition (B) holds, from which (A) and (A') follow immediately. Condition (D) is weaker than (C^N) and condition (F_2) is weaker than conditions (E_2^N), so it remains only to show that (C^N) and (E_2^N) hold. We show this by modifying slightly the argument in the proof of Theorem 2.1 of [4].

$(\mathbf{A}') \implies (\mathbf{C^N})$: Assume that (A') holds and let M be in $(H^1(\Omega))^{d\times d}$. Then

$$\begin{aligned}(\nabla u, M) &= -(u, \operatorname{div} M) + \int_\Gamma (M \cdot \mathbf{n}) \cdot u \\ &\to -(\overline{u}, \operatorname{div} M) + \int_\Gamma (M \cdot \mathbf{n}) \cdot \overline{u} \text{ in } L^\infty([0,T]).\end{aligned}$$

The convergence follows from condition (A') and Equation (2.1). But,

$$-(\overline{u}, \operatorname{div} M) = (\nabla \overline{u}, M) - \int_\Gamma (M \cdot \mathbf{n}) \cdot \overline{u},$$

giving (C^N).

$(\mathbf{A}') \implies (\mathbf{E_2^N})$: Assume that (A') holds and let f be in $H^1(\Omega)$. Then

$$\begin{aligned}(\omega, f) = -(\operatorname{div} u^\perp, f) = (u^\perp, \nabla f) - \int_\Gamma (u^\perp \cdot \mathbf{n}) f \\ = -(u, \nabla^\perp f) + \int_\Gamma (u \cdot \boldsymbol{\tau}) f \\ \to -(\overline{u}, \nabla^\perp f) + \int_\Gamma (\overline{u} \cdot \boldsymbol{\tau}) f \text{ in } L^\infty([0,T])\end{aligned}$$

where $u^\perp = -\left\langle u^2, u^1 \right\rangle$ and we used the identity $\omega(u) = -\operatorname{div} u^\perp$ and Equation (2.1). But,

$$\begin{aligned}-(\overline{u}, \nabla^\perp f) = (\overline{u}^\perp, \nabla f) = -(\operatorname{div} \overline{u}^\perp, f) + \int_\Gamma (\overline{u}^\perp \cdot \mathbf{n}) f \\ = -(\operatorname{div} \overline{u}^\perp, f) - \int_\Gamma (\overline{u} \cdot \boldsymbol{\tau}) f = (\overline{\omega}, f) - \int_\Gamma (\overline{u} \cdot \boldsymbol{\tau}) f,\end{aligned}$$

giving (E_2^N). □

Remark 2.2. If one could show that Equation (2.1) holds in dimension three then Theorem 2.1 would hold in dimension three as well for initial velocities in $H^{5/2}(\Omega)$. This is because by [2] the vanishing viscosity limit holds for such initial velocities, and the argument in the proof of Theorem 2.1 would then carry over to three dimensions by making adaptations similar to those we made to the 2D arguments in [4]. Note that Equation (2.1) would follow, just as in 2D, from a uniform (in ν) bound on the L^p-norm of the vorticity for some $p \geq 2$ if that could be shown to hold, though that seems unlikely.

3. Width of the boundary layer

Working in two dimensions, make the assumptions on the initial velocity and on the forcing in Theorem 2.1 of [4], and assume in addition that

$$m = \int_\Omega \overline{\omega}^0 \neq 0. \tag{3.1}$$

(In particular, this means that $\overline{u}^0$ is not in V.)

Let us suppose that the vanishing viscosity limit holds. Then by Theorem 2.1 of [4],

$$\omega \to \overline{\omega} - (\overline{u} \cdot \boldsymbol{\tau})\mu \text{ in } (H^1(\Omega))' \text{ uniformly on } [0, T]. \tag{3.2}$$

For $\delta > 0$ let φ_δ be a smooth cutoff function equal to 1 on Γ_δ and equal to 0 on $\Omega \setminus \Gamma_{2\delta}$. Then since $\varphi_\delta = 1$ on Γ,

$$\begin{aligned}
(\omega, \varphi_\delta) &\to (\overline{\omega}, \varphi_\delta) - \int_\Gamma \overline{u} \cdot \boldsymbol{\tau} = (\overline{\omega}, \varphi_\delta) + \int_\Gamma \overline{u}^\perp \cdot \mathbf{n} \\
&= (\overline{\omega}, \varphi_\delta) + \int_\Omega \operatorname{div} \overline{u}^\perp = (\overline{\omega}, \varphi_\delta) - \int_\Omega \overline{\omega} \\
&= (\overline{\omega}, \varphi_\delta) - \int_\Omega \overline{\omega}^0 = (\overline{\omega}, \varphi_\delta) - m.
\end{aligned}$$

The convergence here is uniform over $[0, T]$.

Now, since vorticity $\overline{\omega}$ is transported by the flow,

$$|(\overline{\omega}, \varphi_\delta)| \leq \|\overline{\omega}\|_{L^\infty} |\Gamma_{2\delta}| = \left\|\overline{\omega}^0\right\|_{L^\infty} |\Gamma_{2\delta}| \leq C\delta.$$

Thus, for all sufficiently small ν,

$$|(\omega, \varphi_\delta) + m| \leq C\delta. \tag{3.3}$$

For $t > 0$, u is in V so the total mass of ω is zero for all $t > 0$; that is,

$$\int_\Omega \omega = 0.$$

It follows that for all sufficiently small ν,

$$|(\omega, 1 - \varphi_\delta) - m| \leq C\delta. \tag{3.4}$$

This reflects one of the consequences of Theorem 2.1 of [4] that

$$\omega \to \overline{\omega} \text{ in } H^{-1}(\Omega) \text{ uniformly on } [0, T],$$

which represents a kind of weak internal convergence of the vorticity.

Notice that in the bounds such as those in Equations (3.3) and (3.4) we must hold δ fixed as we let $\nu \to 0$, for that is all we can obtain from the weak convergence in Equation (3.2). We could also make the cutoff function φ_δ decrease from 1 to 0 as rapidly as we wish and still obtain the same limit; however, we must always have φ_δ in $H^1(\Omega)$ and it does not follow that the total mass of ω in a boundary layer of width δ goes to zero. Nonetheless, this is a useful way to think of Equation (3.3).

Still, it is natural to ask whether we can set $\delta = c\nu$ in Equations (3.3) and (3.4), this being the width of the boundary layer in Kato's seminal paper

[3] on the subject. If this could be shown to hold it would say that outside of Kato's layer the vorticity for solutions to (NS) converges weakly to the vorticity for the solution to (E), but that the price for this convergence is a buildup of vorticity inside the layer to satisfy the constraint that the total mass of vorticity must be zero.

In fact, however, this is not the case, at least not by a closely related measure of vorticity buildup near the boundary. The total mass of the vorticity (in fact, its L^1-norm) in any layer smaller than that of Kato's goes to zero and, if the vanishing visocity limit holds, then the same holds for Kato's layer. Hence, if there is a layer in which vorticity accumulates, that layer is at least as wide as Kato's and is wider than Kato's if the vanishing viscosity limit holds. This is the content of the following theorem.

Theorem 3.1. *Make the assumptions on the initial velocity and on the forcing in Theorem 2.1 of* [4]*. For any positive function $\delta = \delta(\nu)$,*

$$\|\omega\|_{L^2([0,T];L^1(\Gamma_{\delta(\nu)}))} \le C \left(\frac{\delta(\nu)}{\nu}\right)^{1/2}. \tag{3.5}$$

If the vanishing viscosity limit holds and

$$\limsup_{\nu \to 0^+} \frac{\delta(\nu)}{\nu} < \infty$$

then

$$\|\omega\|_{L^2([0,T];L^1(\Gamma_{\delta(\nu)}))} \to 0 \text{ as } \nu \to 0. \tag{3.6}$$

Proof. By the Cauchy-Schwarz inequality,

$$\|\omega\|_{L^1(\Gamma_{\delta(\nu)})} \le \|1\|_{L^2(\Gamma_{\delta(\nu)})} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})} \le C\delta^{1/2} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})}$$

so

$$\|\omega\|^2_{L^2(\Gamma_{\delta(\nu)})} \ge \frac{C}{\delta} \|\omega\|^2_{L^1(\Gamma_{\delta(\nu)})}$$

and

$$\nu \|\omega\|^2_{L^2([0,T];L^2(\Gamma_{\delta(\nu)}))} \ge \frac{C\nu}{\delta} \|\omega\|^2_{L^2([0,T];L^1(\Gamma_{\delta(\nu)}))}.$$

By the basic energy inequality for the Navier-Stokes equations, the left-hand side is bounded, giving Equation (3.5), and if the vanishing viscosity limit holds, the left-hand side goes to zero, giving Equation (3.6). □

Remark 3.2. In Theorem 3.1, we do not need the assumption in Equation (3.1) nor do we need to assume that we are in dimension two. The result is of most interest, however, when one makes these two assumptions.

Remark 3.3. Equation (3.6) also follows from condition (iii") in [6] using the Cauchy-Schwarz inequality in the manner above, but that is using a sledge hammer to prove a simple inequality. Note that Equation (3.6) is necessary for the vanishing viscosity limit to hold, but is not (as far as we can show) sufficient.

4. Some kind of convergence always happens

Assume that v is a vector field lying in $L^\infty([0,T];H^1(\Omega))$. From the proof of the chain of implications in Theorem 3.1 we see that all of the conditions except (B) are still equivalent with $\overline{u}$ replaced by v. That is, defining

(A_v) $u \to v$ weakly in H uniformly on $[0,T]$,

(A'_v) $u \to v$ weakly in $(L^2(\Omega))^d$ uniformly on $[0,T]$,

(B_v) $u \to v$ in $L^\infty([0,T];H)$,

(C_v) $\nabla u \to \nabla v - \langle \gamma_{\mathbf{n}} \cdot, v\mu \rangle$ in $((H^1(\Omega))^{d\times d})'$ uniformly on $[0,T]$,

(D_v) $\nabla u \to \nabla v$ in $(H^{-1}(\Omega))^{d\times d}$ uniformly on $[0,T]$,

(E_v) $\omega \to \omega(v) - \dfrac{1}{2}\left\langle \gamma_{\mathbf{n}}(\cdot - \cdot^T), v\mu \right\rangle$ in $((H^1(\Omega))^{d\times d})'$ uniformly on $[0,T]$,

$(E_{2,v})$ $\omega \to \omega(v) - (v \cdot \boldsymbol{\tau})\mu$ in $(H^1(\Omega))'$ uniformly on $[0,T]$,

$(F_{2,v})$ $\omega \to \omega(v)$ in $H^{-1}(\Omega)$ uniformly on $[0,T]$,

we have the following theorem:

Theorem 4.1. *Conditions (A_v), (A'_v), (C_v), (D_v), and (E_v) are equivalent. In two dimensions, conditions $(E_{2,v})$ and $(F_{2,v})$ are equivalent to the other conditions. Also, (B_v) implies all of the other conditions. Finally, the same equivalences hold if we replace each convergence above with the convergence of a subsequence.*

But we also have the following:

Theorem 4.2. *There exists v in $L^\infty([0,T];H)$ such that a subsequence (u_ν) converges weakly to v in $L^\infty([0,T];H)$.*

Proof. The argument in 2D is slightly simpler so we give it first. The sequence (u_ν) is bounded in $L^\infty([0,T];H)$ by the basic energy inequality for the Navier-Stokes equations. Letting ψ_ν be the stream function for u_ν vanishing on Γ, it follows by the Poincare inequality that (ψ_ν) is bounded in $L^\infty([0,T];H_0^1(\Omega))$. Hence, there exists a subsequence, which we relabel as (ψ_ν), converging strongly in $L^\infty([0,T];L^2(\Omega))$ and weak-* in $L^\infty([0,T];H_0^1(\Omega))$ to some ψ lying in $L^\infty([0,T];H_0^1(\Omega))$. Let $v = \nabla^\perp \psi$.

Let g be any element of $L^\infty([0,T];H)$. Then

$$\begin{aligned}(u_\nu, g) &= (\nabla^\perp \psi_\nu, g) = -(\nabla \psi_\nu, g^\perp) = (\psi_\nu, -\operatorname{div} g^\perp) = (\psi_\nu, \omega(g)) \\ &\to (\psi, \omega(g)) = (v, g).\end{aligned}$$

In the third equality we used the membership of ψ_v in $H_0^1(\Omega)$ and the last equality follows in the same way as the first four. The convergence follows from the weak-* convergence of ψ_ν in in $L^\infty([0,T];H_0^1(\Omega))$ and the membership of $\omega(g)$ in $H^{-1}(\Omega)$.

In dimension $d \geq 3$, let M_ν in $(H_0^1(\Omega))^d$ satisfy $u_\nu = \operatorname{div} M_\nu$; this is possible by Corollary 7.5 of [4]. Arguing as before it follows that there exists a subsequence, which we relabel as (M_ν), converging strongly in $L^\infty([0,T];L^2(\Omega))$

and weak-* in $L^\infty([0,T];H_0^1(\Omega))$ to some M lying in $L^\infty([0,T];(H_0^1(\Omega))^{d\times d})$. Let $v = \operatorname{div} M$.

Let g be any element of $L^\infty([0,T];H)$. Then

$$\begin{aligned}(u_\nu, g) &= (\operatorname{div} M_\nu, g) = -(M_\nu, \nabla g)\\ &\to -(M, \nabla g) = (v, g),\end{aligned}$$

establishing convergence as before. □

It follows from Theorems 4.1 and 4.2 that all of the convergences in Theorem 3.1 of [4] hold except for (B), but for a subseqence of solutions and the convergence is to some velocity field v lying only in $L^\infty([0,T];H)$ and not necessarily in $L^\infty([0,T];H\cap H^1(\Omega))$. In particular, we do not know if v is a solution to the Euler equations, and, in fact, there is no reason to expect that it is.

I should point out that this section came about because of a question asked by Claude Bardos when I gave a talk on [4]; it was an attempt to clarify the issue.

5. Physical meaning of the vortex sheet on the boundary?

Calling the term $\omega^* := -(\overline{u}\cdot\boldsymbol{\tau})\mu$ (in 2D) a *vortex sheet* is misleading, and I regret referring to it that way in [4] without some words of explanation. The problem is that we cannot interpret ω^* as a distribution on Ω because applying it to any function in $\mathcal{D}(\Omega)$ gives zero. And how could we recover the velocity associated to ω^*?

One natural, if unjustified, way to try to interpret ω^* is to extend it to the whole space so that it is a measure supported along the curve Γ. To determine the associated velocity v, let $\Omega_- = \Omega$ and $\Omega_+ = \Omega^C$ with $v_\pm = v|_{\Omega_\pm}$, and let $[v] = v_+ - v_-$. Then as on page 364 of [8], we must have

$$[v]\cdot\mathbf{n} = 0, \quad [v]\cdot\boldsymbol{\tau} = -\overline{u}\cdot\boldsymbol{\tau}.$$

That is, the normal component of the velocity is continuous across the boundary while the jump in the tangential component is the strength of the vortex sheet.

Now, let us assume that the vanishing viscosity limit holds, so that the limiting vorticity is $\overline{\omega} - (\overline{u}\cdot\boldsymbol{\tau})\mu = \overline{\omega} - \omega^*$. Since $u \to \overline{u}$ strongly with $\omega(\overline{u}) = \overline{\omega}$, the term $\overline{\omega}$ has to account for all of the kinetic energy of the fluid. If the limit is to be physically meaningful, certainly energy cannot be *gained* (though it conceivably could be lost to diffusion, even in the limit). Thus, we would need to have the velocity v associated with ω^* vanish in Ω; in other words, $v_- \equiv 0$. This leads to $\omega(v_+) = \operatorname{div} v_+ = 0$ in Ω_+, $v_+\cdot\mathbf{n} = 0$ on Γ, $v_+\cdot\boldsymbol{\tau} = \overline{u}\cdot\boldsymbol{\tau}$ on Γ, with some conditions on v_+ at infinity. But this is an overdetermined set of equations. In fact, if Ω is simply connected then Ω_+ is an exterior domain, and if we ignore the last equation, then up to a multiplicative constant there is a unique solution vanishing at infinity. This cannot, in general, be reconciled with the need for the last equation to hold.

Actually, perhaps the correct physical interpretation of ω^* comes from the observation in the first paragraph of this section: that it has no physical effect at all since, as a distribution, it is zero. If the vanishing viscosity limit holds, it is reasonable to assume that if there is a boundary separation of the vorticity it weakens in magnitude as the viscosity vanishes and so contributes nothing in the limit.

Or, looked at another way, if in looking for the velocity v corresponding to the vortex sheet ω as we did above we assume that v is zero outside Ω, we would obtain

$$v \cdot \mathbf{n} = 0, \quad v \cdot \boldsymbol{\tau} = \overline{u} \cdot \boldsymbol{\tau}$$

on the boundary. For a very small viscosity, then, u has almost the same effect as $\overline{u}$ in the interior of Ω, while the vortex sheet that is forming on the boundary as the viscosity vanishes has nearly the same effect as $\overline{u}$ on the boundary.

6. An equivalent condition on the boundary in 2D

Theorem 6.1. *The vanishing viscosity limit holds over the finite time interval* $[0, T]$ *if and only if*

$$\nu \int_0^T \int_\Gamma \omega \, \overline{u} \cdot \boldsymbol{\tau} \to 0 \text{ as } \nu \to 0. \tag{6.1}$$

Proof. Subtracting (EE) from (NS), multiplying by $w = u - \overline{u}$, and integrating over Ω leads to

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 &= -(w \cdot \nabla \overline{u}, w) + \nu(\nabla u, \nabla \overline{u}) - \nu \int_\Gamma (\nabla u \cdot \mathbf{n}) \cdot \overline{u} \\ &= -(w \cdot \nabla \overline{u}, w) + \nu(\nabla u, \nabla \overline{u}) - \nu \int_\Gamma \omega(u) \, \overline{u} \cdot \boldsymbol{\tau}.\end{aligned}$$

Here we used Equation (4.2) of [5] to conclude that

$$(\nabla u \cdot \mathbf{n}) \cdot \overline{u} = ((\nabla u \cdot \mathbf{n}) \cdot \boldsymbol{\tau})(\overline{u} \cdot \boldsymbol{\tau}) = \omega(u) \, \overline{u} \cdot \boldsymbol{\tau}.$$

Integrating over time gives

$$\begin{aligned}\frac{1}{2} \|w(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u\|_{L^2}^2 = &- \int_0^t (w \cdot \nabla \overline{u}, w) + \nu \int_0^t (\nabla u, \nabla \overline{u}) \\ &- \nu \int_0^t \int_\Gamma \omega(u) \, \overline{u} \cdot \boldsymbol{\tau}.\end{aligned} \tag{6.2}$$

The sufficiency of Equation (6.1) for the vanishing viscosity limit (VV) to hold (and hence for the other conditions in Theorem 2.1 of [4] to hold) follows from the bounds,

$$|(w \cdot \nabla \overline{u}, w)| \le \|\nabla \overline{u}\|_{L^\infty([0,T] \times \Omega)} \|w\|_{L^2}^2 \le C \|w\|_{L^2}^2,$$

$$\nu \int_0^T |(\nabla u, \nabla \overline{u})| \le \sqrt{\nu} \|\nabla \overline{u}\|_{L^2([0,T] \times \Omega)} \sqrt{\nu} \|\nabla u\|_{L^2([0,T] \times \Omega)} \le C \sqrt{\nu},$$

and Gronwall's inequality.

Proving the necessity of Equation (6.1) is just as easy. Assume that (VV) holds, so that $\|w\|_{L^\infty([0,T];L^2(\Omega)} \to 0$. Then by the two inequalities above, the first two terms on the right-hand side of Equation (6.2) vanish with the viscosity as does the first term on the left-hand side. The second term on the left-hand side vanishes as proven in [3] (it follows from a simple argument using the energy equalities for (NS) and (E)). It follows that, of necessity, Equation (6.1) holds. $\square$

Remark 6.2. In higher dimensions, the same argument gives the necessary and sufficient condition that $\nu \int_0^T \int_\Gamma (\nabla u \cdot \mathbf{n}) \cdot \overline{u}$ vanish with viscosity. The argument becomes formal, however, as even if $\overline{u}$ were to vanish on the boundary, we should at best expect to obtain an inequality in the energy argument (which would, however, be sufficient). I need to think about whether one can justify Equation (6.2) with an inequality in three dimensions.

There is nothing deep about the condition in Equation (6.1), but what it says is that there are two mechanisms by which the vanishing viscosity limit can hold. First, the blowup of ω on the boundary can happen slowly enough that

$$\nu \int_0^T \|\omega\|_{L^1(\Gamma)} \to 0 \text{ as } \nu \to 0 \tag{6.3}$$

or, second, the vorticity for (NS) can be generated on the boundary in such a way as to oppose the sign of $\overline{u} \cdot \boldsymbol{\tau}$. In the second case, it could well be that vorticity for (NS) blows up fast enough that Equation (6.3) does not hold, but cancellation in the integral in Equation (6.1) allows that condition to hold.

A natural question to ask is whether the condition,

$$(G) \qquad \nu \int_0^T \|\omega\|_{L^1(\Gamma)} \to 0 \text{ as } \nu \to 0$$

is equivalent to the conditions in Theorem 2.1 of [4]. The sufficiency of this condition follows immediately, since it implies that Equation (6.1) holds.

To see why we might suspect that (G) is necessary for (VV) to hold, we start with the necessary and sufficient condition (iii') of Theorem 1.2 of [6] that

$$\nu \int_0^T \|\omega\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0,$$

where $\Gamma_\nu = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma) < \nu\}$. For sufficiently regular u^0_ν, for all $t > 0$, $\omega(t)$ will lie in $H^2(\Omega) \supseteq C(\overline{\Omega})$, one might expect to have

$$\nu \int_0^T \|\omega\|^2_{L^2(\Gamma_\nu)} \cong \nu \int_0^T \int_0^\nu \|\omega\|^2_{L^2(\Gamma)} = \nu^2 \int_0^T \|\omega\|^2_{L^2(\Gamma)}. \tag{6.4}$$

Then using Hölder's inequality followed by Jensen's inequality,

$$\left(\frac{\nu}{T^{3/2}} \int_0^T \|\omega\|_{L^1(\Gamma)}\right)^2 \le \left(\frac{\nu}{T} \int_0^T \|\omega\|_{L^2(\Gamma)}\right)^2 \le \frac{\nu^2}{T} \int_0^T \|\omega\|_{L^2(\Gamma)}^2. \tag{6.5}$$

But the left-hand side of Equation (6.4) must vanish, and so too must the left-hand side of Equation (6.5), implying that (G) holds.

The problem with this argument, however, is that the best we can say rigorously is that from Theorem 3.1 and the continuity of $\omega(t)$ for all $t > 0$,

$$\begin{aligned}\nu \int_0^T \|\omega\|_{L^1(\Gamma)}^2 &= \nu \int_0^T \lim_{\delta \to 0} \frac{1}{\delta^2} \|\omega\|_{L^1(\Gamma_\delta)}^2 \le \nu \liminf_{\delta \to 0} \frac{1}{\delta^2} \int_0^T \|\omega\|_{L^1(\Gamma_\delta)}^2 \\ &\le \nu \lim_{\delta \to 0} \frac{1}{\delta^2} \frac{C\delta}{\nu} \le \infty,\end{aligned}$$

where in the first inequality we used Fatou's lemma.

If we could improve this inequality to show that $\nu \int_0^T \|\omega\|_{L^1(\Gamma)}^2$ is $o(1/\nu)$, then using Hölder's inequality followed by Jensen's inequality,

$$\left(\frac{\nu}{T} \int_0^T \|\omega\|_{L^1(\Gamma)}\right)^2 \le \frac{\nu^2}{T} \int_0^T \|\omega\|_{L^1(\Gamma)}^2 \to 0 \text{ as } \nu \to 0.$$

7. An alternate derivation of Kato's conditions

The energy argument in [3], which started a whole sub-branch of analysis of the boundary layer, always mystified me a little, as it seems unmotivated. That is to say, one can follow the technical details easily enough, but it is hard to see what the plan is at the outset.

I give a different derivation below, which starts with Remark 6.2. Thus, in higher than two dimensions, this argument is formal, unless I can justify the result of Remark 6.2 more than formally in higher dimensions. So perhaps it would be better to view this entire argument as motivating why Kato's result should be true.

Let v be the boundary layer velocity defined by Kato in [3], where $\delta = c\nu$: so v is divergence-free, vanishes outside of $\Gamma_{c\nu}$, and $v = \overline{u}$ on Γ. (In all that follows, one can also refer to [6], which gives Kato's argument using (almost) his same notation.) Since $v = \overline{u}$ on Γ, by Remark 6.2, (VV) holds if and only if

$$\nu \int_0^T \int_\Gamma (\nabla u \cdot \mathbf{n}) \cdot v = \int_0^T (\nu \Delta u, v) + \nu \int_0^T (\nabla u, \nabla v) \to 0$$

as $\nu \to 0$. We note also that we could have included forcing in the proof of Remark 6.2 with little difficulty.

Using Lemma A.2 of [6],

$$
\begin{aligned}
\nu \int_0^T (\nabla u, \nabla v) &= 2\nu \int_0^T (\omega(u), \omega(v)) \le 2\nu \int_0^T \|\omega(u)\|_{L^2(\Gamma_{c\nu})} \|\omega(v)\|_{L^2} \\
&\le \sqrt{\nu}\, \|\nabla v\|_{L^2([0,T]\times\Omega)} \sqrt{\nu}\, \|\omega(u)\|_{L^2([0,T]\times\Gamma_{c\nu})} \\
&\le C \left(\nu \int_0^T \|\omega(u)\|^2_{L^2(\Gamma_{c\nu})} \right)^{1/2},
\end{aligned}
$$

since $\|\nabla v\|_{L^2([0,T]\times\Omega)} \le C\nu^{-1/2}$.

Also,

$$
\int_0^T (\nu \Delta u, v) = \int_0^T \left[(\partial_t u, v) + (u \cdot \nabla u, v) + (\nabla p, v) - (f, v) \right].
$$

The integral involving the pressure disappears, while

$$
\int_0^T |(f, v)| \le C\nu^{1/2} \int_0^T \|f\|_{L^2(\Gamma_{c\nu})},
$$

using the bound on $\|v\|_{L^\infty([0,T];L^2)}$ in [3] (Equation (3.1) of [6]). This vanishes with the viscosity since f lies in $L^1([0,T]; L^2)$.

The integral involving $(u \cdot \nabla u, v)$ we bound the same way as in [6]. Using Lemma A.4 of [6],

$$
\begin{aligned}
&\left| \int_0^t (u \cdot \nabla u, v) \right| = 2 \left| \int_0^t (v, u \cdot \omega(u)) \right| \\
&\qquad \le 2\, \|v\|_{L^\infty([0,T]\times\Omega)} \int_0^t \|u\|_{L^2(\Gamma_{c\nu})} \|\omega(u)\|_{L^2(\Gamma_{c\nu})} \\
&\qquad \le C\nu \int_0^t \|\nabla u\|_{L^2(\Gamma_{c\nu})} \|\omega(u)\|_{L^2(\Gamma_{c\nu})} \\
&\qquad \le C\nu^{1/2} \|\nabla u\|_{L^2([0,T];L^2(\Gamma_{c\nu}))} \nu^{1/2} \|\omega(u)\|_{L^2([0,T];L^2(\Gamma_{c\nu}))} \\
&\qquad \le C \left(\nu \int_0^t \|\omega(u)\|^2_{L^2(\Gamma_{c\nu})} \right)^{1/2}.
\end{aligned}
$$

Finally,

$$
\int_0^T (\partial_t u, v) = \int_0^T \int_\Omega \partial_t (uv) + \int_0^T (u, \partial_t v).
$$

As in [3],

$$
\left| \int_0^t (u, \partial_t v) \right| \le \int_0^t \|u\|_{L^2(\Omega)} \|\partial_t v\|_{L^2(\Omega)} \le C\nu^{1/2}.
$$

Also,

$$\begin{aligned}\int_0^T \int_\Omega \partial_t(uv) &= \int_0^T \frac{d}{dt}(u,v) = (u(T),v(T)) - (u_\nu^0, v(0)) \\ &\le \|u(T)\|_{L^2}\,\|v\|_{L^2} + \|u_\nu^0\|_{L^2}\,\|v(0)\|_{L^2} \\ &\le C\|u^0\|_{L^2}\,\|v\|_{L^\infty([0,T];L^2)} \le C\sqrt{\nu}.\end{aligned}$$

We conclude from all these inequalities that

$$\nu \int_0^T \|\omega(u)\|^2_{L^2(\Gamma_{c\nu})} \to 0 \text{ as } \nu \to 0$$

is a sufficient condition for the vanishing viscosity limit to hold, as, too, is Kato's condition involving ∇u in place of $\omega(u)$. The necessity follows easily from the energy inequality. The condition on the energy density in the boundary layer given in [6] follows by the same slight modifications to the above argument made in [6].

Remark 7.1. The reason Kato's argument is a bit more involved and less direct is that he carefully avoids any problematical integrations by using v to construct the legitimate test function, $u - \overline{u} + v$, which he can apply to the Navier-Stokes equations. The energy inequality for the Navier-Stokes equations along with pretty much the same estimates given above are then sufficient to complete the argument rigorously.

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