CONTOUR DYNAMICS AND GLOBAL REGULARITY FOR PERIODIC VORTEX PATCHES AND LAYERS

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Abstract. We study vortex patches for the 2D incompressible Euler equations. Prior works on this problem take the support of the vorticity (i.e., the vortex patch) to be a bounded region. We instead consider the horizontally periodic setting. This includes both the case of a periodic array of bounded vortex patches, and the case of vertically bounded vortex layers. We develop the contour dynamics equation for the boundary of the patch in this horizontally periodic setting, and demonstrate global $C^{1,\varepsilon}$ regularity of this patch boundary. In the process of formulating the problem, we consider different notions of periodic solutions of the 2D incompressible Euler equations, and demonstrate equivalence of these.

Compiled on Tuesday 12 September 2023 at 13:06

1. Introduction

A 2D vortex patch is a solution to the 2D Euler equations for which the vorticity is a constant multiplied by the characteristic function of a domain. We investigate the behavior of vortex patches in an infinite strip periodic in one direction, topologically $S^1 \times \mathbb{R}$, and the corresponding behavior of the vortex patch or layer in the full plane. Our main results are the extension of the $C^{1,\varepsilon}$ global regularity theory for the boundary of the vortex patch to this case, developing and using the appropriate contour dynamics equation for this purpose. Here, and throughout, we fix $\varepsilon \in (0, 1)$.

1.1. **The Euler equations.** We can write the 2D incompressible Euler equations (without forcing) on a domain U in vorticity form as

$$\begin{cases} \partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0 & \text{in } \mathbb{R} \times U, \\ \mathbf{u} = K[\omega] & \text{in } \mathbb{R} \times U, \\ \omega(0) = \omega^0 & \text{in } U. \end{cases} \tag{1.1}$$

Here, ω is the vorticity—the scalar curl of the velocity field $\mathbf{u}$. The vorticity is transported by the velocity field as in $(1.1)_1$, and the velocity field is recovered from the vorticity field by the constitutive law in $(1.1)_2$ so as to be divergence-free and to satisfy any boundary conditions, decay at infinity, or periodicity that might be demanded based, in part, upon the nature of the domain U.

Classically, if $U = \mathbb{R}^2$ and the solution has sufficient decay, one uses the Biot-Savart law as the constitutive law:

$$K[\omega] := K * \omega, \quad K(\mathbf{x}) := \nabla^\perp \left[\frac{1}{2\pi} \log|\mathbf{x}| \right] = \frac{1}{2\pi} \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}. \tag{1.2}$$

Here, K is the Biot-Savart kernel, which we note lies in $L^1_{loc}(\mathbb{R}^2)$, though $K \notin L^p(\mathbb{R}^2)$ for any $p \in [1, \infty]$. To handle solutions having insufficient spatial decay of the vorticity, we must either find an appropriate substitute for the Biot-Savart law or avoid it entirely by using a velocity, pressure formulation.

1.2. **The plane and the cylinder.** In this paper, we will consider two domains: $U = \mathbb{R}^2$ and $U = \Pi$, the infinite flat periodic strip, $S^1 \times \mathbb{R} \cong \mathbb{R}^2/\mathbb{Z} \cong \mathbb{C}/\mathbb{Z}$, which we will most often treat in the form

$$\Pi := \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times \mathbb{R} \text{ with } \left\{-\tfrac{1}{2}\right\} \times \mathbb{R} \text{ identified with } \left\{\tfrac{1}{2}\right\} \times \mathbb{R}. \tag{1.3}$$

We will also find use for the same set as a subset of $\mathbb{R}^2$ or $\mathbb{C}$ without identifying its sides:

$$\Pi_p := \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \mathbb{R} \subseteq \mathbb{R}^2. \tag{1.4}$$

Suppose we have an initial vorticity $\omega^0 = \mathbb{1}_\Omega$ for Ω a bounded domain in Π. We can periodize it to obtain an initial vorticity in $\mathbb{R}^2$ that is periodic in x_1. What results may consist of an infinite number of disconnected domains repeated periodically, one connected, x_1-periodic domain, or a combination of each. Figure 1 displays an example of a simply connected bounded domain in Π yielding an infinite number of copies of the domain in $\mathbb{R}^2$. Figure 2 displays two examples of a non-simply connected domain in Π producing one domain in $\mathbb{R}^2$ periodically repeating in x_1, a so-called *vortex layer*.

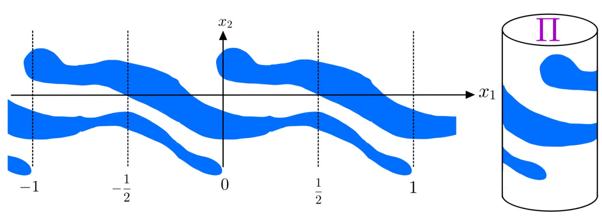


FIGURE 1. *Example of a periodic vortex patch in* $\mathbb{R}^2$ *and in* Π

On the other hand, we could instead formulate the problem by starting with an initial vortex patch in $\mathbb{R}^2$ and periodize it in x_1. If we can translate the evolution of the patch in $\mathbb{R}^2$ to the evolution in Π and back, we can use an understanding of patch behavior in Π to gain

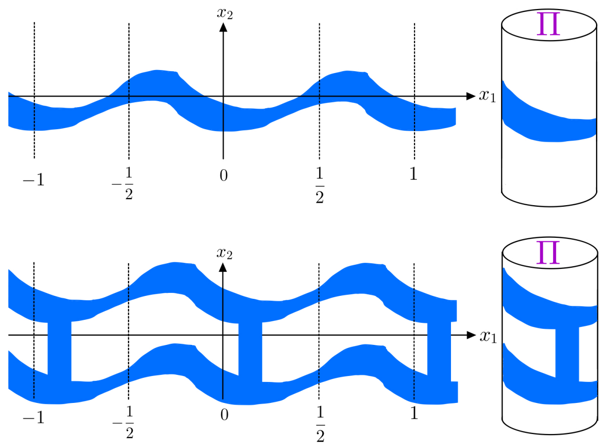


FIGURE 2. *Two examples of a periodic vortex layer in* $\mathbb{R}^2$ *and in* Π

an understanding of the periodic behavior in $\mathbb{R}^2$. The translation back and forth between Π and $\mathbb{R}^2$ is best understood in the more general setting of weak solutions to the 2D Euler equations for bounded vorticity, which includes vortex patch data as a special case.

1.3. **Three types of solutions.** Toward this end, we consider three types of solution to the 2D Euler equations. We summarize the three types of solution briefly now, giving more complete descriptions in later sections.

Type 1 Assume that $\mathbf{u}^0 \in L^\infty(\mathbb{R}^2)$ is divergence-free with $\omega^0 := \operatorname{curl} \mathbf{u}^0 \in L^\infty(\mathbb{R}^2)$ as well. Obtain a bounded vorticity, bounded velocity solution to the the 2D Euler equations on all of $\mathbb{R}^2$ having initial velocity $\mathbf{u}^0$ as done by Serfati in [37].

Type 2 Assume $\mathbf{u}^0 \in L^\infty(\Pi)$ is divergence-free with $\omega^0 := \operatorname{curl} \mathbf{u}^0 \in L^\infty(\Pi)$ as well. Solve the 2D Euler equations in Π, as done in [2, 20, 21].

Type 3 Let $\omega^0 \in L^\infty(\mathbb{R}^2)$ be compactly supported. Solve the 2D Euler equations in vorticity form in all of $\mathbb{R}^2$ with initial vorticity ω^0, but recovering the velocity by applying the Biot-Savart law symmetrically to pairs of the periodically extended copies of ω. This leads to a replacement Biot-Savart kernel, K_∞.

Type 1 and Type 2 solutions are for (potentially) non-decaying velocity and vorticity, but for Type 3 we restrict our attention to vertically decaying solutions, since our primary application is to vortex patch data. Moreover, the convolution $K_\infty * \omega$ cannot be easily defined without some decay assumption.

We will find that all three types of solution are equivalent for a large class of initial data. Since our primary interest is in vortex patches and layers, we will keep things simple by assuming compact support in Π. Assuming, then, that $g \in L^\infty_c(\mathbb{R}^2)$—the space of essentially bounded functions with compact support—we define $\mathcal{P}er(g)$ on Π by

$$\mathcal{P}er(g)(\mathbf{x}) = \sum_{n \in \mathbb{Z}} g(\mathbf{x} - (n, 0)),$$

noting that for each $\mathbf{x}$ the sum has only finitely many nonzero terms. For any measurable function f on Π we define $\mathcal{R}ep(f)$ on $\mathbb{R}^2$ by

$$\mathcal{R}ep(f)(\mathbf{x}) := f(x_1 - \lfloor x_1 + \tfrac{1}{2} \rfloor, x_2).$$

Definition 1.1. *Two functions $g_1, g_2 \in L^\infty_c(\mathbb{R}^2)$ are equivalent, $g_1 \sim g_2$, if $\mathcal{P}er(g_1) = \mathcal{P}er(g_2)$. Figure 3 depicts the support of two functions in the same equivalence class.*

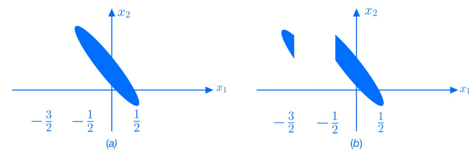


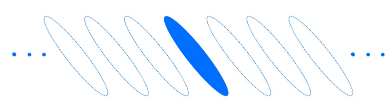
FIGURE 3. *Support of two $L^\infty_c(\mathbb{R}^2)$ functions in the same equivalence class*

Suppose that $g \in L^\infty_c(\mathbb{R}^2)$, and for purposes of illustration, let us treat it as the characteristic function of a bounded domain (our primary application), whose support is depicted as in either (a) or (b) of Figure 3. Below, we construct an initial vorticity from g and depict the support of ω^0 for each type of solution (the time-evolved vorticity being of a similar nature).

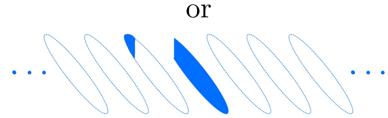
Type 1 Let $\omega^0 = \mathcal{R}ep(\mathcal{P}er(g))$.

Type 2 Let $\omega^0 = \mathcal{P}er(g)$.

Type 3 Let $\omega^0 = g$. The vorticity ω is transported by the flow from the single copy of g, and so is no longer the curl of $\mathbf{u}$. There are, in effect, multiple phantom copies of g matching those of Type 1.



or



The vorticity ω^0 for Type 1 and 2 do not depend upon the representative for the equivalence class, though Type 3 does. We will find, nonetheless, that the velocity field for solutions of Type 3 is independent of the representative.

It is mentioned in [19] that a Type 2 solution is equivalent to a Type 1 solution with periodic velocity and pressure. Following up on this comment, we will show that all three types of solution are equivalent. The equivalence of Type 1 and Type 2 solutions, which applies to a larger class of initial data than we have so far discussed, will rely upon the properties of the pressure required for uniqueness for those two types of solution. The equivalence of Type 3 and Type 2 (and so of Type 1) will rest primarily on showing that solutions of Type 2 reduce to those of Type 3 when the vorticity has sufficient vertical decay. A side benefit of this approach is that it will give the well-posedness of Type 3 solutions. Such a well-posedness result could be obtained by adapting in a fairly straightforward way the approach Marchioro and Pulvirenti take in [31, 32] for the 2D Euler equations, except for subtle points regarding the periodicity of the pressure. It is thus more efficient to leverage the technology developed in [2, 20, 21], though it is more than is strictly needed to develop Type 3 solutions alone.

Specializing to vortex patch data, we will then show how the contour dynamics equation (CDE) is adapted from the classical form, which allows the propagation of regularity of the boundary of a vortex patch to be proved, adapting the argument of Bertozzi and Constantin in [9].

1.4. **Prior work.** Bounded vortex patches evolving under the two-dimensional Euler equations have been well-studied, with global regularity of the boundary being established by Chemin [15] and by Bertozzi and Constantin [9]. Regularity of the vortex patch boundary can also be seen to follow from a more general approach studying level sets of the vorticity, establishing striated regularity, as in the work of Chemin [16] and Serfati [36]. Regularity of bounded vortex patches and/or striated regularity have been established for solutions of related evolution equations as well, such as aggregation equations [8], active transport equations [6], and the surface quasi-geostrophic equation and related systems [14], [22], [29]. None of these problems consider unbounded vortex patches as in the present work.

There are seemingly fewer papers on the evolution of vortex layers. An equation similar to our version of the contour dynamics equation for the motion of the patch/layer boundary was developed in [33], and was subsequently used in [24] for the study of complex singularities in vortex layers. (We mention that the version of the contour dynamics equation developed in the present work lends itself to the study of global regularity.) Atassi, Bernoff, and Lichter study the interaction of a point vortex with a vortex layer [5]. Crowdy gives some exact solutions of vortex layers interacting with solid boundaries [18]. Benedetto and Pulvirenti have shown that vortex layers rigorously approximate vortex sheets in analytic function spaces [7]. Caflisch, Sammartino, and collaborators have considered vortex layers which are not sharp fronts in a series of papers [11], [12], [13], considering how such flows behave in the zero viscosity limit and how such flows may approximate vortex sheets, which represent a more singular vorticity configuration. In these works, they take the vorticity to be exponentially decaying (in the vertical direction) away from a core region, rather than being an indicator function as in the present work. Despite the difference there are similarities to the present work, such as the development of velocity integrals similar to the spatially periodic contour dynamics equation we develop for the periodic patch/layer problem. Further background on vortex layers may be found in [23].

While we are unaware of other works on the global regularity of unbounded vortex patches for the two-dimensional Euler equations, the situation is different for the quasi-geostrophic equation. Rodrigo developed existence theory for a patch which is spatially periodic and vertically unbounded in one direction (similarly to a half-space) [34], [35]. More recently Hunter, Shu, and Zhang have studied the related front solutions of the surface quasi-geostrophic equation [25], [26], [27].

1.5. **Organization of this paper.** We will find many of our calculations much more convenient to perform in the complex plane, yet our results are all real-valued. We describe how to translate back and forth between these settings, largely a matter of notation, in Section 2. In Section 3 we describe the process of symmetrizing in pairs that is behind the Type 1 solutions, which we explore in Section 4. In Section 5 we describe the results of [2, 20, 21] that yield Type 2 solutions, and we use those results in Section 6 to obtain Type 3 solutions. We show the equivalence of the three types of solution in Section 7. In Section 8 we give expressions for the velocity gradient in terms of the vorticity, deferring the proofs to Appendix A. We then specialize to vortex patch solutions for Type 1, 2, and 3 solutions, obtaining their contour dynamics equation in Section 9, and establishing the global-in-time propogation of the regularity of a vortex patch boundary in Section 10.

2. Preliminaries: $\mathbb{R}^2$ and $\mathbb{C}$

2.1. Real to complex translation. Some of our calculations will be more easily performed using complex analysis, though the end results are all real-valued functions. For this we need a means, and a corresponding notation, to switch back and forth between viewing points in the plane as vectors or points in $\mathbb{C}^2$. For this purpose, we will use bold-face letters, such as $\mathbf{x}$ or $\mathbf{u}$, for quantities that are intrinsically elements of $\mathbb{R}^2$ or vector-valued. We define maps,

$$\left\{ \begin{array}{l} \rightarrow : \mathbb{C} \to \mathbb{R}^2, \\ \overrightarrow{x+iy} = (x,y) \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \leftarrow : \mathbb{R}^2 \to \mathbb{C}, \\ \overleftarrow{(x,y)} = x+iy \end{array} \right\}.$$

For a vector $\mathbf{x} = (x, y)$, we define

$$\mathbf{x}^\perp := (-y, x).$$

Hence, $\mathbf{x}^\perp$ is $\mathbf{x}$ rotated 90 degrees counterclockwise.

Lemma 2.1. *Let $z, w \in \mathbb{C}$ and $\cdot$ be the usual dot (inner) product of Euclidean vectors. Then*

$$\begin{aligned} \operatorname{Re}(zw) &= \vec{\overline{z}} \cdot \vec{w}, \\ \operatorname{Im}(zw) &= -\vec{\overline{z}} \cdot \vec{w}^\perp. \end{aligned} \tag{2.1}$$

If $a \in \mathbb{R}$, $z \in \mathbb{C}$,

$$\overrightarrow{az} = a\vec{z}, \quad \overrightarrow{iz} = \vec{z}^\perp, \quad \overleftarrow{\mathbf{v}^\perp} = i\overleftarrow{\mathbf{v}}. \tag{2.2}$$

Also, f is analytic in some domain U if and only if $\operatorname{div} \vec{\overline{f}} = \operatorname{curl} \vec{\overline{f}} = 0$ in U, where for any vector field $\mathbf{v}$,

$$\operatorname{div} \mathbf{v} := \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2}, \quad \operatorname{curl} \mathbf{v} := \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$$

are the divergence and (scalar) curl of $\mathbf{v}$.

The boundary integrals we encounter will be real path integrals, but we will sometimes find it useful to transform them to complex contour integrals as in the following lemma:

Lemma 2.2. *Let $\boldsymbol{\gamma} \colon [a, b] \to \mathbb{C}$ be a Lipschitz-continuous path on which the complex-valued function f is continuous. Let $\boldsymbol{\tau}$ be the unit tangent vector in the direction of $\boldsymbol{\gamma}$ and $\boldsymbol{n}$ the associated unit normal, with $(\boldsymbol{n}, \boldsymbol{\tau})$ in the standard orientation of $(\mathbf{e}_1, \mathbf{e}_2)$. Let $C = \operatorname{image} \boldsymbol{\gamma}$. Then*

$$\text{⨏}_{\boldsymbol{\gamma}} f = \int_C \vec{\overline{f}} \cdot \boldsymbol{\tau} + i \int_C \vec{\overline{f}} \cdot \boldsymbol{n}.$$

Here, ⨏ is a complex contour integral.

Using Lemma 2.1, it is not hard to rewrite the classical Biot-Savart law in the following hybrid real-complex form:

Theorem 2.3. *Assume that $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$. With K as in (1.2),*

$$\mathbf{u}(\mathbf{x}) := K * \omega(\mathbf{x}) = -\overrightarrow{\frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{\omega(\mathbf{y})}{\overleftarrow{\overline{\mathbf{y} - \mathbf{x}}}} \, d\mathbf{y}} \tag{2.3}$$

is divergence-free with $\operatorname{curl} \mathbf{u} = \omega$, and $\mathbf{u}$ is the unique such velocity field in $L^\infty \cap H^1(\mathbb{R}^2)$.

2.2. **The cotangent.**

Lemma 2.4. *For any $z \in \mathbb{C}$ that is not an integer,*

$$\pi \cot \pi z = \frac{1}{z} + 2 \sum_{n=1}^{\infty} \frac{z}{z^2 - n^2} = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n}.$$

Proof. For the first equality see, for instance, Equation (11) in Section 5.2.1 of [3]. The second equality then follows from

$$\frac{z}{z^2 - n^2} = \frac{z}{(z-n)(z+n)} = \frac{1}{2}\left[\frac{1}{z-n} + \frac{1}{z+n}\right]$$

and summing in pairs, n with $-n$. □

Lemma 2.5. *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,*

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{\mathbf{x} + (n, 0)}{|\mathbf{x} + (n, 0)|^2} \cdot \mathbf{y} = \overrightarrow{\pi \cot(\pi \overleftarrow{\overleftarrow{\mathbf{x}}})} \cdot \mathbf{y}.$$

Proof. Letting $z = \overleftarrow{\mathbf{x}}$, $w = \overleftarrow{\mathbf{y}}$, and using $(2.1)_1$, we have

$$\frac{\mathbf{x} + (n, 0)}{|\mathbf{x} + (n, 0)|^2} \cdot \mathbf{y} = \frac{\operatorname{Re}((\overline{z} + n)w)}{|z + n|^2} = \operatorname{Re} \frac{(\overline{z} + n)w}{|z + n|^2} = \operatorname{Re} \frac{w}{z + n}$$

so

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{\mathbf{x} + (n, 0)}{|\mathbf{x} + (n, 0)|^2} \cdot \mathbf{y} = \operatorname{Re}\left[w \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n}\right] = \pi \operatorname{Re}(w \cot \pi z)$$
$$= \overrightarrow{\overline{\pi \cot(\pi \overleftarrow{\mathbf{x}})}} \cdot \mathbf{y} = \overrightarrow{\pi \cot(\pi \overleftarrow{\overleftarrow{\mathbf{x}}})} \cdot \mathbf{y},$$

where we again used $(2.1)_1$. □

2.3. **Useful identities.** The identities in (2.4) and (2.5) are easily verifiable; (2.6) is 4.3.58 of [1].

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{2.4}$$
$$\cosh 2x = 2 \sinh^2 x + 1, \quad \cos 2x = 1 - 2 \sin^2 x, \tag{2.5}$$
$$\cot z = \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}. \tag{2.6}$$

2.4. **Lifting paths and domains.** We will find the need, in the proof of Theorem 9.6, to apply Lemma 2.2 while integrating in Π and apply Cauchy's residue theorem. This could be done directly by introducing a version of the residue theorem for Π, which is a (flat) analytic manifold. Alternately, we can transform integrals in Π to integrals of x_1-periodic functions in $\mathbb{C}$ by *lifting* the domain Ω in Π to a suitable domain $\widetilde{\Omega}$ in $\mathbb{C}$. Our main tool for doing this is the lifting of paths from a topological space to a covering space.

Defining

$$p \colon \mathbb{C} \to \Pi, \quad p(x_1 + ix_2) = x_1 - \lfloor x_1 + \tfrac{1}{2} \rfloor + ix_2,$$

we see that $(\mathbb{C}, p)$ is a covering space of Π (see Section IX.7 of [17], for instance). This will allow us to lift a path in Π to a path in $\mathbb{R}^2$ or $\mathbb{C}$.

Remark 2.6. *$\mathcal{R}ep(f)(\mathbf{x}) = f(p(\mathbf{x}))$, though we do not make direct use of this.*

Definition 2.7. *A path in the topological space X is a continuous map from an interval I to X. The path $\widetilde{\boldsymbol{\gamma}}$ in $\mathbb{C}$ is a lift or lifting of the path $\boldsymbol{\gamma}$ in Π if $p \circ \widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}$.*

Lemma 2.8. *Let $\boldsymbol{\gamma}$ be a finite length continuous path in Π with initial point $\mathbf{x}_0$. For any $\widetilde{\mathbf{x}}_0 \in p^{-1}(\mathbf{x}_0)$, there exists a unique lifting $\widetilde{\boldsymbol{\gamma}}$ with initial point $\widetilde{\mathbf{x}}_0$.*

Proof. This is a classical result; see, for instance, Corollary IX.7.5 of [17]. □

This lifting allows us to relate path integrals in Π to lifted path integrals in $\mathbb{R}^2$ or $\mathbb{C}$:

Lemma 2.9. *Let $\boldsymbol{\gamma}$ be a Lipschitz-continuous path in Π with a lift $\widetilde{\boldsymbol{\gamma}}$ as given by Lemma 2.8. For any any continuous function f on Π,*

$$\int_{\boldsymbol{\gamma}} f = \int_{\widetilde{\boldsymbol{\gamma}}} f \circ p.$$

Moreover, the normal vector field $\boldsymbol{n}$ on $\boldsymbol{\gamma}$ lifts to itself as does $\boldsymbol{\tau}$; that is, $\boldsymbol{n}(\boldsymbol{\gamma}(\alpha)) = \boldsymbol{n}(\widetilde{\boldsymbol{\gamma}}(\alpha))$ for all α in the domain of $\boldsymbol{\gamma}$ (which is the same as the domain of $\widetilde{\boldsymbol{\gamma}}$).

Proof. Suppose that $\boldsymbol{\gamma}\colon [a,b] \to \Pi$, in which case also $\widetilde{\boldsymbol{\gamma}}\colon [a,b] \to \mathbb{C}$ with $p \circ \widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}$. Then

$$\int_{\widetilde{\boldsymbol{\gamma}}} f \circ p = \int_a^b f \circ p(\widetilde{\boldsymbol{\gamma}}(\alpha))\widetilde{\boldsymbol{\gamma}}'(\alpha)\, d\alpha = \int_a^b f(\boldsymbol{\gamma}(\alpha))\boldsymbol{\gamma}'(\alpha)\, d\alpha = \int_{\boldsymbol{\gamma}} f.$$

We used that $\widetilde{\boldsymbol{\gamma}}'(\alpha) = \boldsymbol{\gamma}'(\alpha)$, since locally $\widetilde{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ differ by a constant (if we view $\boldsymbol{\gamma}$ as giving values in Π_p). This also gives that $\boldsymbol{n}$ and $\boldsymbol{\tau}$ lift to themselves. □

Lemma 2.9 is not, however, the entire story when we lift the entire boundary of a domain in Π. An immediate difficulty stems from the ambient space Π, which is topologically a cylinder, having nontrivial fundamental (and first homology) group $\mathbb{Z}$. Let us say that a closed curve on Π wraps around the cylinder n times if it crosses $\{x_1 = 0\}$ (any vertical slice would do) n times counted with sign, positive in one direction, negative in the other (arbitrarily fixing which direction is positive).

A closed path that wraps zero times around the cylinder is homotopic to a point and lifts to a closed path in $\mathbb{C}$. A closed path that wraps around the cylinder n times, however, will lift by Lemma 2.8 to a non-closed path in $\mathbb{C}$ that contains $|n| + 1$ points of $\mathbf{x}_0 + \mathcal{L}$, where we define here and for future use,

$$\mathcal{L} := \{\mathbb{Z}\} \times \{0\}, \quad \mathcal{L}^* := \mathcal{L} \setminus (0,0), \tag{2.7}$$

treated as subsets of $\mathbb{R}^2$ or of $\mathbb{C}$. Since we are lifting paths that are boundary components, they will always be closed in Π, but can wrap only 0 or ± 1 times around the cylinder else they would of necessity self-intersect.

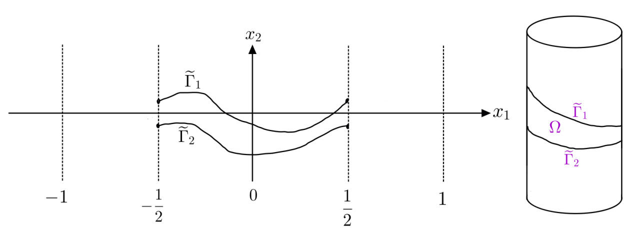


FIGURE 4. *Lifting of $\partial\Omega$ with base points at $x_1 = -\frac{1}{2}$*

Figure 4 shows an example of a domain Ω in Π having two boundary components Γ_1, Γ_2 which lift to non-closed paths $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$. To make a domain from these paths, we could connect $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ with vertical paths at $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$, oppositely oriented, so that the four paths together form the boundary of a lifted domain $\widetilde{\Omega}$.

Equivalently, and in a manner more easily generalizable, we cut the cylinder Π vertically[1] along the line $\ell = \left\{x_1 = \pm\frac{1}{2}\right\}$, which in effect means we view Π in the form suggested in (1.3). For any line segment formed by $\ell \cap \partial\Omega$ we introduce oppositely oriented paths; together, the lifted components of $\partial\Omega$ and these paths, properly oriented, form the boundary components of the lifted domain $\widetilde{\Omega}$.

In lifting these components and paths, however, we need to insure compatible initial points for the paths. To do this, fix any $\mathbf{x}_0$ in Ω. Let $\mathbf{y}$ be any point in Ω and let $\boldsymbol{\gamma}_\mathbf{y}$ be a path connecting $\mathbf{x}_0$ to $\mathbf{y}$. Being a domain, Ω is path-connected so this is always possible. By Lemma 2.8, there is a unique lifting $\widetilde{\boldsymbol{\gamma}}_\mathbf{y}$ of $\boldsymbol{\gamma}_\mathbf{y}$ with initial point $\widetilde{\mathbf{x}}_0$. Then $\widetilde{\Omega} := \cup_{\mathbf{y}\in\Omega} \widetilde{\boldsymbol{\gamma}}_\mathbf{y}$ is the desired lifting of Ω.

Lifted in this way, we have the following lemma:

Lemma 2.10. *Let Ω be a bounded domain in Π and let $\widetilde{\Omega}$ be the lifted domain as described above. Let $\boldsymbol{\gamma}$ be a parameterization of $\partial\Omega$ and $\widetilde{\boldsymbol{\gamma}}$ a parameterization of $\partial\widetilde{\Omega}$. Let f be any continuous complex-valued function. Then*

$$\oint_{\boldsymbol{\gamma}} f = \oint_{\widetilde{\boldsymbol{\gamma}}} f \circ p, \qquad \int_{\partial\Omega} \vec{\vec{f}} \cdot \boldsymbol{\tau} = \int_{\partial\widetilde{\Omega}} (\vec{\vec{f}} \circ p) \cdot \boldsymbol{\tau}, \qquad \int_{\partial\Omega} \vec{\vec{f}} \cdot \boldsymbol{n} = \int_{\partial\widetilde{\Omega}} (\vec{\vec{f}} \circ p) \cdot \boldsymbol{n}.$$

Proof. Follows from Lemma 2.9, since the cuts introduce integrals that cancel in pairs. □

3. Periodized functions and Biot-Savart kernels

Definition 3.1. *Let $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$. We say that the velocity field $\mathbf{u}$ is obtained by symmetrizing in pairs (about 0) if, letting $\omega^{(n)}(\mathbf{x}) = \omega(\mathbf{x} + (n, 0))$, we have*

$$\mathbf{u} = K_{sym}[\omega] := K * \omega + \sum_{n=1}^{\infty} K * \left(\omega^{(-n)} + \omega^{(n)}\right).$$

Definition 3.2. *Let $S = S(\mathbb{R}^2)$ be the Serfati space of bounded, divergence-free vector fields on $\mathbb{R}^2$ having bounded vorticity with norm,*

$$\|\mathbf{u}\|_S := \|\mathbf{u}\|_{L^\infty(\mathbb{R}^2)} + \|\operatorname{curl} \mathbf{u}\|_{L^\infty(\mathbb{R}^2)}.$$

We define $S(\Pi)$ similarly.

Remark 3.3. *As shown in (2.11) of [21], for any $\omega \in L^\infty(\Pi)$ there is a divergence-free vector field $\mathbf{u}$ in $L^\infty(\Pi)$ and so in $S(\Pi)$ for which $\operatorname{curl} \mathbf{u} = \omega$. $S(\mathbb{R}^2)$ is very different, for there is no known general condition on $\omega \in L^\infty(\mathbb{R}^2)$ alone that guarantees a $\mathbf{u}$ in $L^\infty(\mathbb{R}^2)$.*

Proposition 3.4. *For $\omega \in L^\infty_c(\mathbb{R}^2)$, let $\mathbf{u} = K_{sym}[\omega]$ as in Definition 3.1. Then $\mathbf{u} \in S(\mathbb{R}^2)$ with $\operatorname{curl} \mathbf{u} = \operatorname{curl} K_{sym}[\omega] = \mathcal{R}ep(\omega)$. Further,*

$$\mathbf{u} = K_{sym}[\omega] = K_\infty * \omega, \quad K_\infty(\mathbf{x}) := \overrightarrow{-\frac{i}{2}\cot \pi \overleftleftarrows{\mathbf{x}}} = \frac{1}{2}\left[\overrightarrow{\cot \pi \overleftleftarrows{\mathbf{x}}}\right]^\perp, \tag{3.1}$$

[1]In pathological cases, we would have to perturb this cut to avoid producing an infinite number of boundary components, but we will not explore this issue further.

where we note that K_∞ is periodic in x_1 with period 1 as is $\mathbf{u}$. We also have,

$$K_\infty(\mathbf{x}) = K(\mathbf{x}) + H(\mathbf{x}), \tag{3.2}$$

where H is harmonic on $\mathbb{R}^2 \setminus \mathcal{L}^*$, where $\mathcal{L}^*$ is defined in (2.7).

Proof. Applying Theorem 2.3, we have

$$I_n := K * \left(\omega^{(-n)} + \omega^{(n)}\right)(\mathbf{x}) = \overrightarrow{-\frac{i}{2\pi}\int_{\mathbb{R}^2} \left[\frac{\omega(\mathbf{y})}{\overleftarrow{\mathbf{y}-\mathbf{x}} - n}\right] + \left[\frac{\omega(\mathbf{y})}{\overleftarrow{\mathbf{y}-\mathbf{x}} + n}\right] d\mathbf{y}},$$

so

$$\overleftarrow{I_n} = -\frac{i}{2\pi}\int_{\mathbb{R}^2} 2\frac{\overleftarrow{\mathbf{y}-\mathbf{x}}}{(\overleftarrow{\mathbf{y}-\mathbf{x}})^2 - n^2}\omega(\mathbf{y})\, d\mathbf{y}.$$

From Definition 3.1 with Lemma 2.4, then (the compact support of ω allows us to interchange integration and summation),

$$\overleftarrow{\mathbf{u}(\mathbf{x})} = -\frac{i}{2}\int_{\mathbb{R}^2} \cot(\pi(\overleftarrow{\mathbf{y}-\mathbf{x}}))\omega(\mathbf{y})\, d\mathbf{y},$$

and (3.1) follows from (2.2). Since the singularity of $\cot(\pi z)$ at $z = 0$ is like $1/(\pi z)$ and ω is compactly supported, we see that the above integral lies in $L^\infty(\mathbb{R}^2)$. Since the curl of each I_n is $\omega^{(-n)} + \omega^{(n)}$ while its divergence is zero and the sum converges absolutely and uniformly, we know that $\operatorname{div} \mathbf{u} = 0$ and $\operatorname{curl} \mathbf{u} = \mathcal{R}ep(\omega)$.

But $\cot z = \frac{1}{z} + h(z)$ on $\mathbb{C} \setminus \mathcal{L}^*$, where h is analytic. From this (3.2) follows. □

Proposition 3.5. *If $\omega_1 \sim \omega_2$ in $L^\infty_c(\mathbb{R}^2)$ as in Definition 1.1 then $K_\infty * \omega_1 = K_\infty * \omega_2$.*

Proof. For any $\omega \in L^\infty_c(\mathbb{R}^2)$,

$$\begin{aligned}
K_\infty * \mathcal{P}er(\omega)(\mathbf{x}) &= \int_\Pi K_\infty(\mathbf{x}-\mathbf{y})\mathcal{P}er(\omega)(\mathbf{y})\, d\mathbf{y} = \int_{\Pi_p} K_\infty(\mathbf{x}-\mathbf{y}) \sum_{n\in\mathbb{Z}} \omega(\mathbf{y} - (n,0))\, d\mathbf{y} \\
&= \int_{\Pi_p} \sum_{n\in\mathbb{Z}} K_\infty(\mathbf{x} - (\mathbf{y} - (n,0)))\omega(\mathbf{y} - (n,0))\, d\mathbf{y} \\
&= \sum_{n\in\mathbb{Z}} \int_{\Pi_p} K_\infty(\mathbf{x} - (\mathbf{y} - (n,0)))\omega(\mathbf{y} - (n,0))\, d\mathbf{y} = \sum_{n\in\mathbb{Z}} \int_{\Pi_p - (n,0)} K_\infty(\mathbf{x}-\mathbf{y})\omega(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{R}^2} K_\infty(\mathbf{x}-\mathbf{y})\omega(\mathbf{y})\, d\mathbf{y} = K_\infty * \omega(\mathbf{x}).
\end{aligned}$$

We were able to interchange the integral and sum here because for any fixed $\mathbf{x}$, the compact support of ω makes all but a finite number of terms in the sum zero. Hence, if $\omega_1 \sim \omega_2$ then $K_\infty * \omega_1 = K_\infty * \mathcal{P}er(\omega) = K_\infty * \omega_2$. □

We will see in Section 5.1 that K_∞ also serves as the Biot-Savart kernel on Π.

4. Type 1: Periodized solutions

We review here results, obtained variously in [4, 28, 37, 38], on bounded vorticity, bounded velocity solutions to the 2D Euler equations in $\mathbb{R}^2$.

Definition 4.1. *Fix $T > 0$ and let $u \in L^\infty(0,T;S) \cap C([0,T] \times \mathbb{R}^2)$ with vorticity $\omega := \operatorname{curl} u$. We say that u is a bounded weak Eulerian solution to the Euler equations without forcing if, on the interval $[0,T]$, $\partial_t \omega + u \cdot \nabla \omega = 0$ as a distribution on $(0,T) \times \mathbb{R}^2$. We say that u is a Lagrangian solution if $\omega(t, X(t,x)) = \omega(0,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^2$, where X is the flow map for u (noting that u has sufficient regularity to insure the existence of a unique classical flow map).*

Let a_R be a radial cutoff function: $a_R(\cdot) = a(\cdot/R)$ for any $R > 0$, where $a \in C^\infty(\mathbb{R}^2)$ is radially symmetric and equal to 1 in a neighborhood of the origin. For definitiveness, we will assume that $a \equiv 1$ on $B_1(0)$, $a \equiv 0$ on $B_2(0)^C$, and $|a| \leqslant 1$ on $\mathbb{R}^2$.

Theorem 4.2 ([28]). *Any weak solution to the Euler equations (Eulerian or Lagrangian) with $\mathbf{u} \in L^\infty(0,T;S) \cap C([0,T] \times \mathbb{R}^2)$ having vorticity ω with $\mathbf{u}(0) = \mathbf{u}^0$, $\omega(0) = \omega^0$, must satisfy, for some $\mathbf{U}_\infty \in C([0,T])^2$, the Serfati identity,*

$$\begin{aligned} u^j(t) - (u^0)^j &= U^j_\infty(t) + (aK^j) * (\omega(t) - \omega^0) \\ &\quad - \int_0^t \left(\nabla\nabla^\perp \left[(1-a)K^j \right] \right) *\cdot (u \otimes u)(s)\, ds, \end{aligned} \tag{4.1}$$

$j = 1, 2$, and the renormalized Biot-Savart law,

$$\mathbf{u}(t) - \mathbf{u}^0 = \mathbf{U}_\infty(t) + \lim_{R \to \infty} (a_R K) * (\omega(t) - \omega^0) \tag{4.2}$$

on $[0,T] \times \mathbb{R}^2$. Furthermore, the corresponding pressure is of the form,

$$p(t, \mathbf{x}) = -\mathbf{U}'_\infty(t) \cdot \mathbf{x} + q(t, \mathbf{x}), \tag{4.3}$$

where q grows sublinearly at infinity.

Theorem 4.2 characterizes solutions to the 2D Euler equations that have bounded vorticity and bounded velocity: their existence and uniqueness under the condition that (4.1) holds is shown, for $\mathbf{U}_\infty \equiv 0$, in [37] and elaborated on in [4], their extension to a general $\mathbf{U}_\infty$ being a simple matter. Uniqueness under the assumption of sublinear growth of the pressure is established in [38].

Combining these results leads to the following:

Theorem 4.3. *Let $\mathbf{u}^0 \in S(\mathbb{R}^2)$ and set $\omega^0 = \operatorname{curl} \mathbf{u}^0$. There exists a solution $(\mathbf{u}, p)$ to the 2D Euler equations with $\mathbf{u} \in L^\infty(0,T;S) \cap C([0,T] \times \mathbb{R}^2)$ having initial velocity $\mathbf{u}^0$. Existence and uniqueness hold if we require that the solution satisfy any one (and hence all) of (4.1) through (4.3) with $\mathbf{U}_\infty \equiv 0$.*

5. Type 2: Solutions in an infinite periodic strip

Let BUC(Π) be the space of bounded, uniformly continuous functions, noting that any vector field in $S(\Pi)$ lies in BUC(Π). Well-posedness of solutions to the Navier-Stokes equations for initial velocity in BUC(Π) was established by Afendikov and Mielke in [2]. Building on this, Gallay and Slijepčević in [21] (and see the comments in [19]) obtained improved bounds for the case where the initial velocity lies in $S(\Pi)$, having established properties of the pressure in [20]. These works are for the Navier-Stokes equations, but as the authors point out, the pertinent estimates are uniform in small viscosity and hold for solutions to the Euler equations as well (by repeating the argument with the viscous terms missing or by using known vanishing viscosity results).

In Theorem 5.3 we give the well-posedness result as derived from [2, 20, 21], but for this we need to first explore some aspects of the analysis in these references.

5.1. **Biot-Savart kernels.** The authors of [2,20,21] orient their periodic strip (infinite cylinder) horizontally and S^1 is, in effect, parametrized from 0 to 1. Let (x_1', x_2') be the coordinates for the horizontal strip of [2,20,21], while we will keep (x_1, x_2) for our vertical strip. Rotating the horizontal strip 90 degrees counterclockwise induces the change of variables,

$$x_1' \mapsto x_2, \quad x_2' \mapsto -x_1.$$

The Biot-Savart kernel on Π used in [2] and (2.7) of [20] is $\nabla^\perp G$, where

$$G(x_1', x_2') := \frac{1}{4\pi} \log\left(2\cosh(2\pi x_1') - 2\cos(2\pi x_2')\right)$$

is the Green's function for the Dirichlet Laplacian on Π. In (x_1, x_2) variables,

$$G(x_1, x_2) := \frac{1}{4\pi} \log\left(2\cosh(2\pi x_2) - 2\cos(2\pi x_1)\right). \tag{5.1}$$

Lemma 5.1. *We have $K_\infty = \nabla^\perp G$. Moreover, $G(\mathbf{x}) = (2\pi)^{-1} \log \rho(\mathbf{x})$, where*

$$\rho(\mathbf{x}) := \left(\sin^2(\pi x_1) + \sinh^2(\pi x_2)\right)^{\frac{1}{2}}. \tag{5.2}$$

Proof. From (2.5), $2\cosh(2\pi x_2) - 2\cos(2\pi x_1) = 4\rho(\mathbf{x})^2$, gives our alternate expression for G (noting that the Green's function on Π is unique up to an additive constant). From (2.6) and (5.1), we have

$$\nabla^\perp G(x_1, x_2) = \frac{1}{2\pi} \frac{(-\pi\sinh(2\pi x_2), \pi\sin(2\pi x_1))}{\cosh(2\pi x_2) - \cos(2\pi x_1)} = \frac{1}{2}\overrightarrow{\cot(\pi\overline{z})}^\perp, \tag{5.3}$$

matching the expression for K_∞ in (3.1). Here, we used (2.6). □

Lemma 5.2. *The function $\log \rho(\mathbf{x}) - \log|\mathbf{x}|$ is harmonic on $\mathbb{R}^2 \setminus \mathcal{L}^*$, where ρ is defined in (5.2).*

Proof. Letting $z = \overleftarrow{\mathbf{x}}$, we have, using (2.4),

$$\log \rho(\mathbf{x}) - \log|\mathbf{x}| = \frac{1}{2}\log\left|\frac{\rho(\mathbf{x})^2}{|\mathbf{x}|^2}\right| = \frac{1}{2}\log\left|\frac{\sin z}{z}\right|^2 = \log\left|\frac{\sin z}{z}\right| = \operatorname{Re}\log\frac{\sin z}{z},$$

which is the real part of a function that is complex analytic on $\mathbb{C} \setminus \mathcal{L}^*$. □

5.2. **Mean horizontal values.** As observed below Lemma 2.2 of [2], although $K_\infty \in L^1_{loc}(\Pi)$, $K^2_\infty \in L^1(\Pi)$ (accounting for the different orientation of the strip). Moreover, convolution with K^1_∞ can be handled by subtracting from u^2 its mean horizontal value to give it mean value zero. We summarize here this process as described on page 1748 of [20].

If $\mathbf{v}(t) \in S(\Pi)$, the mean value of $v^2(t)$ along the horizontal line segment $x_2 = a$ is independent of $a \in \mathbb{R}$, and if $(\mathbf{v}, p)$ solves the Euler equations on Π then it is independent of time as well. Hence, we can define

$$m_2(t) = m_2[\mathbf{v}(t)] := \langle v^2(t) \rangle, \tag{5.4}$$

the mean value of $v^2(t)$ along any such horizontal line segment and we will have $\langle v^2(t)\rangle = \langle v^2_0\rangle$.

The mean value of $v^1(t)$, however, will depend upon x_2, so we write

$$m_1(t, x_2) = m_1[\mathbf{v}(t)](x_2) := \int_{-\frac{1}{2}}^{\frac{1}{2}} v^1(t, x_1, x_2)\, dx_1.$$

Similarly, we define

$$\langle\omega\rangle(t,x_2) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \omega(t,x_1,x_2)\,dx_1$$

and $\widehat{\omega}(t,x_1,x_2) := \omega(t,x) - \langle\omega\rangle(t,x_2)$. Also,

$$\langle\omega\rangle(t,x_2) = \langle\partial_1 u^2 - \partial_2 u^1\rangle(t,x_2) = -\langle\partial_2 u^1\rangle(t,x_2) = -\partial_2 m_1(t,x_2). \tag{5.5}$$

A form of the Biot-Savart law given in (2.5, 2.6) of [20] (suppressing the time variable) is

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} -m_1(x_2) \\ m_2 \end{pmatrix} + \int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} K_\infty(\mathbf{x}-\mathbf{y})\widehat{\omega}(\mathbf{y})\,dy_1\,dy_2. \tag{5.6}$$

We note here that in transforming from the expression as written in [20], a velocity (v^1, v^2) in (x_1', x_2') becomes $(v^2, -v^1)$ in (x_1, x_2), which accounts for the minus sign in $-m_1(x_2)$.

5.3. **Type 2 solutions.** We can now summarize the known result we need for Type 2 solutions:

Theorem 5.3 ([2,20,21]). *For $\mathbf{v}^0 \in S(\Pi)$ with $\langle v_0^2\rangle = 0$ there exists a unique solution $(\mathbf{v}, q)$ to the Euler equations,*

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla q = 0 & \text{in } [0,\infty)\times\Pi, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } [0,\infty)\times\Pi, \\ \mathbf{v}(0) = \mathbf{v}^0 & \text{in } \Pi \end{cases} \tag{5.7}$$

for which $m_2(t) \equiv 0$ with $\mathbf{v} \in C([0,\infty); BUC(\Pi)) \cap L^\infty([0,\infty); S(\Pi))$ and pressure $q \in W^{1,\infty}([0,\infty)\times\Pi)$. The pressure is given by[2]

$$q = -(u^2)^2 + 2K_\infty^2 * (\omega u^1).$$

The solutions are Eulerian in velocity and satisfy the vorticity equation. Moreover, $\mathbf{u}$ can be recovered from ω by the Biot-Savart law as in (5.6).

5.4. **Compactly supported vorticity.** As a prelude to obtaining Type 3 solutions, let us consider the special case of Type 2 solutions that we can obtain when the vorticity is compactly supported in Π. First, we specialize the Biot-Savart law in (5.6) to compactly supported vorticity.

Lemma 5.4. *Let $\mathbf{v} \in S(\Pi)$ with $\omega := \operatorname{curl}\mathbf{v}$ compactly supported in Π. Then $m_1(-\infty) + m_1(\infty) \equiv m_2 \equiv 0$ if and only if $\mathbf{v} = K_\infty * \omega$.*

Proof. Since $\langle\omega\rangle = -\partial_2 m_1$, we have

$$\begin{aligned} I^j &:= \left[\int_{-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} K_\infty(\mathbf{x}-\mathbf{y})\langle\omega\rangle(\mathbf{y})\,dy_1\,dy_2\right]^j \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\infty}^{\infty} K_\infty^j((x-x', y-y'))\partial_2 m_1(y')\,dy'\,dx'. \end{aligned}$$

Lemma 5.7, below, gives that $I^2 = 0$.

We now consider I^1. Because ω is compactly supported within some $[-1/2,1/2]\times[-R_0,R_0]$, so, too, are $\langle\omega\rangle$ and then, by (5.5), $\partial_2 m_1$. Choose $\varphi \in C_C^\infty(\mathbb{R})$ equal to 1 on $[-R,R]$ and

[2] $+2K_\infty^2$ is $-\partial_2 G$ in (2.8) of [20]: we have made the transformation from a horizontal to a vertical strip.

equal to zero outside $[-R+1, R+1]$ where we will choose $R \geqslant R_0$ more precisely later. Let $m_1^\varepsilon = \eta_\varepsilon * m_1$, where η_ε is a (compactly supported) Friedrich's mollifier. As in [2], we treat K_∞^1 as a distribution on Π with φm_1^ε a test function. Since also $K_\infty^1 \in L^1_{loc}(\Pi)$, we have, for fixed $\mathbf{x}$,

$$\begin{aligned} I^1 &= \lim_{\varepsilon \to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} K_\infty^1((x - x', y - y'))\varphi(y)\partial_2 m_1^\varepsilon(y')\, dy'\, dx' \\ &= \lim_{\varepsilon \to 0} K_\infty^1 * (\varphi \partial_2 m_1^\varepsilon) = \lim_{\varepsilon \to 0} K_\infty^1 * \partial_2(\varphi m_1^\varepsilon) - \lim_{\varepsilon \to 0} K_\infty^1 * (\partial_2 \varphi\, m_1^\varepsilon). \end{aligned}$$

Now,

$$\partial_2 K_\infty^1 = -\partial_2^2 G = -\Delta G + \partial_1^2 G = -\delta + \partial_1^2 G,$$

where G is the Green's function for the Dirichlet Laplacian on Π as in (5.1) and δ is the Dirac delta function on Π. Hence,

$$\partial_2(\varphi m_1^\varepsilon) = m_1^\varepsilon(x_2) - \int_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_1^2 G((x - x', y - y'))\, dx' m_1^\varepsilon(y')\, dy' = m_1^\varepsilon(x_2),$$

where the integral vanishes after integrating by parts, since G is periodic in x_1. Hence,

$$I^1 = m_1(x_2) - \lim_{\varepsilon \to 0} K_\infty^1 * (\partial_2 \varphi m_1^\varepsilon),$$

and this equality holds regardless of our choice of $R \geqslant R_0$. Therefore, if we can evaluate $K_\infty^1 * (\partial_2 \varphi m_1^\varepsilon)$ in the limit as $R \to \infty$, it will be its common value for all $R \geqslant R_0$.

We see from (5.3) that $K_\infty^1(x-y) \to \pm 1/2$ as $y_2 \to \pm\infty$ and $\partial_2 K_\infty^1(x-y) \to 0$ as $y_2 \to \pm\infty$, so

$$\begin{aligned} \lim_{R \to \infty} K_\infty^1 * (\partial_2 \varphi m_1^\varepsilon) &= \lim_{R \to \infty} \left(\int_{-R-1}^{-R} + \int_R^{R+1} \right) \partial_2 \varphi K_\infty^1(x - y) m_1^\varepsilon \\ &= \lim_{R \to \infty} \left[(K_1^\infty m_1^\varepsilon)(-R) - (K_1^\infty m_1^\varepsilon)(R) \right] - \lim_{R \to \infty} \left(\int_{-R-1}^{-R} + \int_R^{R+1} \right) \varphi \partial_2 K_\infty^1(x - y) m_1^\varepsilon \\ &= -\frac{1}{2} \lim_{R \to \infty} \left[m_1^\varepsilon(-R) + m_1^\varepsilon(R) \right]. \end{aligned}$$

We also used here that $\partial_2 m_1^\varepsilon = -\eta_\varepsilon * \partial_2 \langle \omega \rangle = 0$ for $R \geqslant R_0$. Since this limit gives the value for all $R \geqslant R_0$, we can take $\varepsilon \to 0$ to conclude that

$$I_1 = m_1(x_2) + \frac{1}{2} \left[m_1(-\infty) + m_1(\infty) \right].$$

Returning to (5.6), then, we see that

$$\mathbf{v}(t, \mathbf{x}) = \frac{1}{2} \begin{pmatrix} m_1(-\infty) + m_1(\infty) \\ 2m_2 \end{pmatrix} + (K_\infty * \omega(t)(\mathbf{x}). \tag{5.8}$$

This shows that $m_1(-\infty) + m_1(\infty) \equiv 0$ and $m_2 \equiv 0$ if and only if $\mathbf{v} = K_\infty * \omega$. □

Corollary 5.5. *Let $\omega \in L_c^\infty(\Pi)$. Then $\mathbf{v} = K_\infty * \omega$ is the unique element in $S(\Pi)$ for which* $\operatorname{curl} \mathbf{v} = \omega$, *$m_2[\mathbf{v}] = 0$, and $m_1[\mathbf{v}](-\infty) + m_1[\mathbf{v}](\infty) = 0$.*

Proposition 5.6. *Assume that $\omega^0 \in L_c^\infty(\Pi)$, $\mathbf{v}^0 = K_\infty * \omega^0$, and $\mathbf{v}$ is a Type 2 solution as in Theorem 5.3 with $\mathbf{v}$ given by (5.6). Then $\mathbf{v}(t) = K_\infty * \omega(t)$ for all t.*

Proof. It follows from Lemma 5.4 that $m_1(0, -\infty) + m_1(0, \infty) = 0$. But as observed following (2.11) of [20], $\partial_t m_1 = -\langle u^2 \omega \rangle$, which we note vanishes for all sufficiently large x_2 because of the compact support of ω. Hence, $m_1(t, -\infty) + m_1(t, \infty) = 0$ for all t. We conclude from (5.8) that $\mathbf{v}(t) = K_\infty * \omega(t)$ for all t. □

We used Lemma 5.7 in the proof of Lemma 5.4, above.

Lemma 5.7. *For all $y \in \mathbb{R}$, $K^1_\infty(x_1, x_2)$ is even in x_1 and odd in x_2, while $K^2_\infty(x_1, x_2)$ is odd in x_1 and even in x_2.*

Proof. This follows directly from (5.3), since $\nabla^\perp G = K_\infty$. □

6. Type 3: Solutions with a periodized kernel

Theorem 6.1. *Let $\omega^0 \in L^\infty_c(\mathbb{R}^2)$. There exists a solution μ to*

$$\begin{cases} \partial_t \mu + \mathbf{w} \cdot \nabla \mu = 0 & \text{in } [0, \infty) \times \mathbb{R}^2, \\ \mathbf{w} = K_\infty * \mu & \text{in } [0, \infty) \times \mathbb{R}^2, \\ \mu(0) = \omega^0 & \text{in } \mathbb{R}^2. \end{cases}$$

Moreover, $\operatorname{curl} \mathbf{w} = \mathcal{R}ep(\mathcal{P}er(\mu))$*, and* $\mathbf{w} \in L^\infty(0, T; S) \cap C([0, T] \times \mathbb{R}^2)$ *is the unique solution to*

$$\begin{cases} \partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla r = 0 & \text{in } [0, \infty) \times \mathbb{R}^2, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } [0, \infty) \times \mathbb{R}^2, \\ \mathbf{w}(0) = K_\infty * \mu(0) & \text{in } \mathbb{R}^2, \end{cases} \tag{6.1}$$

with the uniqueness criteria being that r is periodic. Finally, $r \in L^\infty([0, T] \times \mathbb{R}^2)$.

Proof. From Proposition 3.4 we know that $K_\infty * \omega^0 \in L^\infty(\mathbb{R}^2)$ and is periodic in x_1 with period 1; hence, abusing notation, we can set $\mathbf{v}^0 = K_\infty * \omega^0|_\Pi$ and obtain by Theorem 5.3 a unique solution $(\mathbf{v}, q)$ to (5.7) for which q is periodic in x_1 and $m_2(t) \equiv 0$. Since $\operatorname{curl} \mathbf{v}^0 = \omega^0|_\Pi$ is compactly supported and so $\operatorname{curl} \mathbf{v}$ remains compactly supported for all time, we know from Proposition 5.6 that $\mathbf{v} = K_\infty * \operatorname{curl} \mathbf{v}$. So letting $\zeta = \operatorname{curl} \mathbf{v}$, we see that

$$\begin{cases} \partial_t \zeta + \mathbf{v} \cdot \nabla \zeta = 0 & \text{in } [0, \infty) \times \Pi, \\ \mathbf{v} = K_\infty * \zeta & \text{in } [0, \infty) \times \Pi, \\ \zeta(0) = \omega^0 & \text{in } \Pi. \end{cases}$$

Setting $\mathbf{w} = \mathbf{v}$, $\mu = \zeta$ gives the desired solution of Type 3. Moreover, since $q(t)$ is periodic, we can let $r = \mathcal{R}ep(q)$, and we obtain a unique solution to (6.1). □

7. Three types of solution are equivalent

For certain classes of initial data, our three types of solution are equivalent. The equivalence of Type 1 and Type 2 holds for a broader class, so we first prove it in Theorem 7.1. The equivalence of the third type holds for initial data in $L^\infty_c(\mathbb{R}^2)$, as we show in Theorem 7.2. This includes vortex patch data, our application in Section 9.

Theorem 7.1. *Let $\mathbf{v}^0 \in S(\Pi)$ and periodize it to give $\mathbf{u}^0 = \mathcal{R}ep(\mathbf{v}^0) \in S(\mathbb{R}^2)$. Let $(\mathbf{u}, p)$ be the solution of Type 1 with initial velocity $\mathbf{u}^0$ given by Theorem 4.3 and let $(\mathbf{v}, q)$ the solution of Type 2 with initial velocity $\mathbf{v}^0$ given by Theorem 5.3. Then $\mathcal{R}ep(\mathbf{v}) = \mathbf{u}$.*

Proof. We have $\operatorname{curl} \mathbf{v}(0) = \operatorname{curl} \mathbf{u}^0|_\Pi$, where we abuse notation somewhat. From Theorem 5.3, we have a pressure q with $q(t) \in L^\infty(\Pi)$ for which

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla q = 0 & \text{in } [0,\infty) \times \Pi, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } [0,\infty) \times \Pi, \\ \mathbf{v}(0) = \mathbf{v}^0 & \text{in } \Pi. \end{cases} \tag{7.1}$$

Since $\mathcal{R}ep(\mathbf{v})$ and $\mathcal{R}ep(q)$ are x_1-periodic with period 1, we can set $\widetilde{\mathbf{v}} = \mathcal{R}ep(\mathbf{v})$ and $\widetilde{q} = \mathcal{R}ep(q)$, and both will lie in $L^\infty([0,T] \times \mathbb{R}^2)$ with $\operatorname{curl} \widetilde{\mathbf{v}}(t) = \mathcal{R}ep(\operatorname{curl} \mathbf{v}(t))$. Thus, $\widetilde{\mathbf{v}}$ is $\mathbf{v}$ periodized and $\operatorname{curl} \widetilde{\mathbf{v}}$ is $\operatorname{curl} \mathbf{v}$ periodized, meaning that (7.1) in effect holds on Π_p translated by $(n, 0)$ for any integer n, so we see that

$$\begin{cases} \partial_t \widetilde{\mathbf{v}} + \widetilde{\mathbf{v}} \cdot \nabla \widetilde{\mathbf{v}} + \nabla \widetilde{q} = 0 & \text{in } [0,\infty) \times \mathbb{R}^2, \\ \operatorname{div} \widetilde{\mathbf{v}} = 0 & \text{in } [0,\infty) \times \mathbb{R}^2, \\ \widetilde{\mathbf{v}}(0) = \mathbf{u}^0 & \text{in } \mathbb{R}^2. \end{cases} \tag{7.2}$$

We see that $(\widetilde{\mathbf{v}}, \widetilde{q})$ is a solution to the Euler equations on $[0,\infty) \times \mathbb{R}^2$. Manifestly, $\widetilde{\mathbf{v}}$, $\operatorname{curl} \widetilde{\mathbf{v}}$, and $\widetilde{q}$ each lie in $L^\infty([0,\infty) \times \mathbb{R}^2)$, being periodic in x_1. Hence, $\widetilde{\mathbf{v}}$ is a bounded velocity, bounded vorticity solution to the Euler equations on $[0,\infty) \times \mathbb{R}^2$. Because the pressure $\widetilde{q}$ grows sublinearly it is, in fact, the (unique) Serfati solution (it satisfies the Serfati identity), as follows from Theorem 4.3. Therefore, $\mathbf{u} = \mathbf{v}$. □

Theorem 7.2. *For $\omega^0 \in L^\infty_c(\mathbb{R}^2)$, let $\mathbf{u}^0 = K_{sym}[\omega^0]$ be obtained by symmetrizing in pairs as in Definition 3.1, and let $\mathbf{v}^0 = K_\infty * \mathcal{P}er(\omega^0)$. Let $(\mathbf{u}, p)$, $(\mathbf{v}, q)$ be the Type 1, 2 solutions with initial velocity $\mathbf{u}^0$, $\mathbf{v}^0$ and let $\mathbf{w}^0$ be the velocity field for the Type 3 solution given by Theorem 6.1. Then $\mathcal{R}ep(\mathbf{v}) = \mathbf{u} = \mathbf{w}$.*

Proof. Theorem 7.1 gives $\mathcal{R}ep(\mathbf{v}) = \mathbf{u}$, while $\mathcal{R}ep(\mathbf{v}) = \mathbf{w}$ is inherent in the proof of Theorem 6.1. □

8. The velocity gradient

The following expression for $\nabla(K * \omega)$ is classical (see, for instance, Proposition 2.20 of [30]):

Lemma 8.1. *Assume that $\omega \in L^\infty(\mathbb{R}^2)$ is compactly supported and let $\mathbf{u} = K * \omega$. Then*

$$\nabla \mathbf{u}(\mathbf{x}) = \omega(\mathbf{x}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_{\mathbb{R}^2} \nabla K(\mathbf{x} - \mathbf{y}) \omega(\mathbf{y}) \, d\mathbf{y},$$

where we can write,

$$\nabla K(\mathbf{x}) = \frac{1}{2\pi} \frac{\sigma(\mathbf{x})}{|\mathbf{x}|^2}, \quad \sigma(\mathbf{x}) := \frac{1}{|\mathbf{x}|^2} \begin{pmatrix} 2x_1 x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1 x_2 \end{pmatrix}.$$

The analog for the K_∞ kernel is Lemma 8.2.

Lemma 8.2. *Assume that $\omega \in L^\infty(\mathbb{R}^2)$ is compactly supported and let $\mathbf{u} = K_\infty * \omega$. Then*

$$\nabla \mathbf{u}(\mathbf{x}) = \sum_{n \in \mathbb{Z}} \frac{\omega(\mathbf{x} + (n,0))}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_{\mathbb{R}^2} \nabla K_\infty(\mathbf{x} - \mathbf{y}) \omega(\mathbf{y}) \, d\mathbf{y},$$

where ρ is as in (5.2) and where we can write,

$$\nabla K_\infty(\mathbf{x}) = \frac{\pi}{2} \frac{\beta(\mathbf{x})}{\rho(\mathbf{x})^2},$$

where

$$\beta(\mathbf{x}) = \frac{1}{2\rho(\mathbf{x})^2}\begin{pmatrix} \sin(2\pi x_1)\sinh(2\pi x_2) & \cos(2\pi x_1)\cosh(2\pi x_2) - 1 \\ \cos(2\pi x_1)\cosh(2\pi x_2) - 1 & -\sin(2\pi x_1)\sinh(2\pi x_2) \end{pmatrix}.$$

Proof. The proof is given in Appendix A. □

Remark 8.3. *Like σ, the matrix β is symmetric with trace zero. Near the origin, $\rho(\mathbf{x})^2 \approx \pi^2|\mathbf{x}|^2$, and we can see that $\beta(\mathbf{x}) \approx 4\pi^2|\mathbf{x}|^2/(2\pi^2|\mathbf{x}|^2) \approx 2 \approx \sigma(\mathbf{x})$, and so $\nabla K_\infty(\mathbf{x}) \approx 2\pi/(2\pi^2|\mathbf{x}|^2) \approx 1/(\pi|\mathbf{x}|^2) \approx \nabla K(\mathbf{x})$. Also like σ, β_{11} and β_{22} integrate to zero over circles centered at the origin, but unlike σ, neither β_{12} nor β_{21} integrate to zero.*

We have the following immediate corollary of Lemma 8.2:

Corollary 8.4. *Let $\mathbf{v} \in S(\Pi)$ with $\omega = \operatorname{curl}\mathbf{v}$ compactly supported and let $\mathbf{u} = K_\infty * \omega$. Then*

$$\nabla\mathbf{u}(\mathbf{x}) = \frac{\omega(\mathbf{x})}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_\Pi \nabla K_\infty(\mathbf{x}-\mathbf{y})\omega(\mathbf{y})\, d\mathbf{y}$$

and ∇K_∞ can be written as in Lemma 8.2.

9. Contour Dynamics Equations

First we review the Contour Dynamics Equation (CDE) for a classical vortex patch—the characteristic function of a bounded, simply connected domain evolving under the vorticity equation for the Euler equations on all of $\mathbb{R}^2$—then turn to the CDE for Type 2 solutions.

In what follows we use the Lipschitz space *Lip* and homogeneous Lipschitz space *lip*. On $U \subseteq \mathbb{R}^d$ for $d \geqslant 1$, we define their semi-norm and norm,

$$\|f\|_{lip(U)} := \sup_{x \neq y \in U} \frac{|f(x)-f(y)|}{|x-y|}, \quad \|f\|_{Lip(U)} := \|f\|_{L^\infty(U)} + \|f\|_{lip(U)}.$$

9.1. Classical vortex patches.

In the classical setting of a vortex patch in $\mathbb{R}^2$, we have Theorems 9.1 and 9.2, as in Proposition 8.6 of [30] and the derivation of the classical CDE that appears before it.

In what follows, ω_0 is a fixed, nonzero real constant.

Theorem 9.1. *Let $\boldsymbol{\gamma}\colon [0,2\pi] \to \mathbb{R}^2$ be a C^1 counterclockwise[3] parameterization of the boundary of a bounded, simply connected domain Ω. Then*

$$\mathbf{u}(\mathbf{x}) = -\frac{\omega_0}{2\pi}\int_0^{2\pi} \log|\mathbf{x} - \boldsymbol{\gamma}(\alpha)|\partial_\alpha\boldsymbol{\gamma}(\alpha)\, d\alpha \tag{9.1}$$

is the unique divergence-free vector field decaying at infinity for which $\operatorname{curl}\mathbf{u} = \omega_0\mathbb{1}_\Omega$.

Now let us suppose that Ω is a simply connected bounded domain in $\mathbb{R}^2$ with a $C^{1,\varepsilon}$ boundary. Let $\mathbf{u}$ be the unique weak solution to the Euler equations with initial vorticity $\omega^0 := \omega_0\mathbb{1}_\Omega$ and let X be the flow map for $\mathbf{u}$. Then we know that the vorticity $\omega(t) = \omega_0\mathbb{1}_{\Omega_t}$, where $\Omega_t = X(t,\Omega)$.

Let $\boldsymbol{\gamma}(0,\cdot)$ be a C^1-regular counterclockwise parameterization of $\Gamma = \partial\Omega$. Define a parameterization of $\partial\Omega_t = X(t,\Gamma)$ by $\boldsymbol{\gamma}(t,\cdot) := X(t,\boldsymbol{\gamma}(0,\cdot))$. The log-Lipschitz regularity of $\mathbf{u}(t)$ induces $C^{c(t)}$-regularity of the flow map $X(t,\cdot)$ with $c(t) \in (0,1)$ and decreasing with time, as in Lemma 8.2 of [30]. This is insufficient regularity to obtain a C^1-parameterization of

[3]In [30], the patch boundary is parameterized clockwise, but $(\boldsymbol{\tau}, \boldsymbol{n})$ is in the standard $(\mathbf{e}_1, \mathbf{e}_2)$ orientation; the two resulting sign changes between [30] and us cancel, so there is no sign change in our expressions.

$\partial\Omega_t$, so let us *suppose* that our (classical) solution has $\mathbf{u} \in C(0,T;lip)$. Then $\boldsymbol{\gamma}(t,\cdot)$ is a C^1-parameterization of $\partial\Omega_t$.

Since we assumed $\partial\Omega$ is $C^{1,\varepsilon}$, we could give $\boldsymbol{\gamma}(0,\cdot)$ $C^{1,\varepsilon}$-regularity, but this does not itself ensure that $\boldsymbol{\gamma}(t,\cdot)$ is $C^{1,\varepsilon}$: proving that is tantamount to establishing the propagation of regularity of the vortex patch boundary.

Theorem 9.2. *Let $\mathbf{u}(t,\mathbf{x})$ be given by (9.1) applied with $\boldsymbol{\gamma}(t,\cdot)$; that is,*

$$\mathbf{u}(t,\mathbf{x}) := -\frac{\omega_0}{2\pi}\int_0^{2\pi} \log|\mathbf{x}-\boldsymbol{\gamma}(t,\alpha)|\partial_\alpha\boldsymbol{\gamma}(t,\alpha)\,d\alpha.$$

Then $\mathbf{u}$ is a weak solution to the 2D Euler equations on $[0,T]\times\mathbb{R}^2$ with $\mathbf{u}\in C(0,T;Lip)$ if and only if $\boldsymbol{\gamma}$ is a $C^1([-T,T];C([0,2\pi]))\cap C([-T,T];C^1([0,2\pi]))$ solution to the contour dynamics equations (CDE),

$$\frac{d}{dt}\boldsymbol{\gamma}(t,\alpha) = -\frac{\omega_0}{2\pi}\int_0^{2\pi} \log|\boldsymbol{\gamma}(t,\alpha)-\boldsymbol{\gamma}(t,\alpha')|\partial_{\alpha'}\boldsymbol{\gamma}(t,\alpha')\,d\alpha'. \tag{9.2}$$

Theorems 9.1 and 9.2 were expressed for simply connected domains. As pointed out on page 330 of [30], the only difference for multiply connected domains is that the integrals in (9.1) and (9.2) are summed over each component of the boundary.

Theorem 9.3. *Theorems 9.1 and 9.2 hold for bounded, multiply connected domains if we evaluate and sum each of the boundary integrals over each boundary component.*

We view (9.2) as a form of the Euler equations applying specifically to a vortex patch: it comes directly from (9.1), which we view as a form of the Biot-Savart law that recovers the velocity from the vorticity, as it is encoded by $\boldsymbol{\gamma}$. We work, now, to obtain replacements for these expressions that apply to periodized vortex patches. This is a matter of deriving the CDE for a solution to the Euler equations and showing, conversely, that any solution to the CDE satisfies the Euler equations.

9.2. **Type 2 solutions.** Turning to Type 2 solutions, we make the following assumptions on Ω:

Assumption 9.4. *Assume that $\Omega\subseteq\Pi$ is bounded with a finite number of boundary components, $\Gamma_1,\dots,\Gamma_J$, each $C^{1,\varepsilon}$ regular.*

With Ω as in Assumption 9.4, we let $\mathbf{u}$ be the unique Type 2 solution having initial vorticity $\omega^0 := \omega_0 \mathbb{1}_\Omega$ with $m_2 \equiv m_1(t,-\infty)+m_2(t,\infty)\equiv 0$ given by Theorem 5.3 and Proposition 5.6 (m_1, m_2 are defined in Section 5.2). Set

$$\Omega_t := X(t,\Omega), \quad \Gamma_{t,j} := X(t,\Gamma_j),$$

noting that because $X(t,\cdot)$ is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$, $\Gamma_{t,j}$ is the j^{th} of the J components of $\partial\Omega_t$. We then define a parameterization $\boldsymbol{\gamma}_j$ of $\Gamma_{t,j}$ as we parameterized $\partial\Omega_t$ in Section 9.1, setting $\boldsymbol{\gamma}_j(t,\cdot) := X(t,\boldsymbol{\gamma}_j(0,\cdot))$. As in that section, a priori, we do not even know that $\boldsymbol{\gamma}_j(t)$ has C^1 regularity for $t>0$; proving that it has $C^{1,\varepsilon}$ regularity is the ultimate goal (of Section 10).

We show in Theorems 9.5 and 9.6 that the analog of Theorem 9.3 holds for Type 2 solutions.

Theorem 9.5. *Let Ω be as in Assumption 9.4, and for each j, let $\boldsymbol{\gamma}_j\colon[0,2\pi]\to\mathbb{R}^2$ be a C^1 counterclockwise parameterization of the boundary component Γ_j. With ρ as in (5.2),*

$$\mathbf{u}(\mathbf{x}) = -\frac{\omega_0}{2\pi}\sum_{j=1}^J\int_0^{2\pi} \log\rho(\mathbf{x}-\boldsymbol{\gamma}_j(\alpha))\partial_\alpha\boldsymbol{\gamma}_j(\alpha)\,d\alpha \tag{9.3}$$

is the unique divergence-free vector field in $S(\Pi)$ *having curl equal to* $\omega_0 \mathbb{1}_\Omega$ *for which* $m_2 = 0$ *and* $m_1(-\infty) + m_1(\infty) = 0$*.*

Proof. By Corollary 5.5, we know that $\mathbf{u} = K_\infty * \omega$ is the unique divergence-free vector field in $S(\Pi)$ having curl equal to $\omega_0 \mathbb{1}_\Omega$ for which $m_2 = 0$ and $m_1(-\infty) + m_1(\infty) = 0$. Then we have, using Lemma 5.1 and parameterizing $\Gamma_{t,j}$ by arc length from 0 to ℓ_j, setting $\mathbf{y}(s) = \boldsymbol{\gamma}_j(\alpha(s))$,

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &= K_\infty * \omega(\mathbf{x}) = \nabla^\perp G * \omega(\mathbf{x}) = \frac{\omega_0}{2\pi} \int_\Omega \nabla^\perp \log \rho(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\
&= -\frac{\omega_0}{2\pi} \int_\Omega \nabla_{\mathbf{y}}^\perp \log \rho(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{\ell_j} \log \rho(\mathbf{x} - \mathbf{y}(s))(-n^2, n^1) \, ds \\
&= -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{\ell_j} \log \rho(\mathbf{x} - \mathbf{y}(s)) \boldsymbol{\tau}(s) \, ds = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{2\pi} \log \rho(\mathbf{x} - \boldsymbol{\gamma}_j(\alpha)) \partial_\alpha \boldsymbol{\gamma}_j(\alpha) \, d\alpha.
\end{aligned}$$

Here $(n^1, n^2) = \boldsymbol{n}$ and $(-n^2, n^1) = \boldsymbol{\tau}$ (see Lemma 2.2), and we used that

$$\partial_\alpha \boldsymbol{\gamma}_j(\alpha) \, d\alpha = \frac{\partial_\alpha \boldsymbol{\gamma}_j(\alpha)}{|\partial_\alpha \boldsymbol{\gamma}_j(\alpha)|} |\partial_\alpha \boldsymbol{\gamma}_j(\alpha)| \, d\alpha = \boldsymbol{\tau}(s) \, ds.$$

From this, (9.3) follows. □

Theorem 9.6. *Let* $\mathbf{u}$ *be the Type 2 solution described above and assume that each* $\boldsymbol{\gamma}_j$ *is in* $C^1([-T, T]; C([0, 2\pi])) \cap C([-T, T]; C^1([0, 2\pi]))$*. Then*

$$\mathbf{u}(t, \mathbf{x}) = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{2\pi} \log \rho(\mathbf{x} - \boldsymbol{\gamma}_j(t, \alpha)) \partial_\alpha \boldsymbol{\gamma}_j(t, \alpha) \, d\alpha \tag{9.4}$$

and lies in $C(0, T; Lip)$*. Moreover, each* $\boldsymbol{\gamma}_k$ *satisfies the CDE,*

$$\frac{d}{dt} \boldsymbol{\gamma}_k(t, \alpha) = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{2\pi} \log \rho(\boldsymbol{\gamma}_k(t, \alpha) - \boldsymbol{\gamma}_j(t, \alpha')) \partial_\alpha \boldsymbol{\gamma}_j(t, \alpha) \, d\alpha'. \tag{9.5}$$

Conversely, if each $\boldsymbol{\gamma}_k$ *in* $C^1([-T, T]; C([0, 2\pi])) \cap C([-T, T]; C^1([0, 2\pi]))$ *satisfies (9.5) then* $\mathbf{u}$ *given by (9.4) is a Type 2 solution with* $\mathbf{u} \in C(0, T; Lip)$ *and* $m_2 \equiv m_1(t, -\infty) + m_2(t, \infty) \equiv 0$*.*

Proof. The forward direction follows directly from Theorem 9.5.

For the converse, we parallel the proof of Proposition 8.6 of [30], which consists of two steps: (1) Show that $\mathbf{u}$ given by (9.4) is divergence-free with curl $\mathbf{u} = \mathbb{1}\Omega_{0,t}$. (2) Show that $\mathbf{u}$ solves the 2D Euler equations.

To prove (1), let $\mathbf{u}$ be given by (9.4). Reparameterizing by arc length as in the proof of Theorem 9.5,

$$\mathbf{u}(t, \mathbf{x}) = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_0^{\ell_j} \log \rho(\mathbf{x} - \mathbf{y}(s)) \boldsymbol{\tau}(s) ds = -\frac{\omega_0}{2\pi} \sum_{j=1}^J \int_{\Gamma_{t,j}} \log \rho(\mathbf{x} - \cdot) \boldsymbol{\tau}.$$

To apply div and curl to this expression, we use that for a constant vector field $\mathbf{w}$ and scalar function g, $\operatorname{div}(g\mathbf{w}) = \nabla g \cdot \mathbf{w}$ and $\operatorname{curl}(g\mathbf{w}) = \nabla^\perp g \cdot \mathbf{w}$. Also, letting $\mathbf{v} = (2\pi)^{-1} \nabla^\perp \log \rho(\mathbf{x} - \cdot)$

and $f = \overline{\overleftarrow{\mathbf{v}}}$, we see that

$$\begin{aligned}
\operatorname{curl} \mathbf{u}(t, \mathbf{x}) &= -\frac{\omega_0}{2\pi} \sum_{j=1}^{J} \int_{\Gamma_{t,j}} \nabla^\perp \log \rho(\mathbf{x} - \cdot) \cdot \boldsymbol{\tau} = -\omega_0 \sum_{j=1}^{J} \int_{\Gamma_{t,j}} \vec{\overline{f}} \cdot \boldsymbol{\tau} = -\omega_0 \int_{\partial \Omega_t} \vec{\overline{f}} \cdot \boldsymbol{\tau}, \\
\operatorname{div} \mathbf{u}(t, \mathbf{x}) &= -\frac{\omega_0}{2\pi} \sum_{j=1}^{J} \int_{\Gamma_{t,j}} \nabla \log \rho(\mathbf{x} - \cdot) \cdot \boldsymbol{\tau} = \frac{\omega_0}{2\pi} \sum_{j=1}^{J} \int_{\Gamma_{t,j}} \nabla^\perp \log \rho(\mathbf{x} - \cdot) \cdot \boldsymbol{n} \\
&= \omega_0 \sum_{j=1}^{J} \int_{\Gamma_{t,j}} \vec{\overline{f}} \cdot n = \omega_0 \int_{\partial \Omega_t} \vec{\overline{f}} \cdot n.
\end{aligned}$$

Up to this point, we have been integrating over paths in Π treated as $\mathbb{R}^2/\mathcal{L}$, but we wish to apply Lemma 2.2, which obliges us to work in $\mathbb{C}$. To do this, we lift Ω_t to $\widetilde{\Omega}_t$ as described in Section 2.4. Applying Lemmas 2.2 and 2.10 (writing f in place of $f \circ p$ by viewing f as x_1-periodic with period 1) gives for all $\mathbf{x}$ not lying on $\partial \widetilde{\Omega}_t$ (a set of measure 0),

$$\omega_0 \oint_{\partial \Omega_t} f = \omega_0 \oint_{\partial \widetilde{\Omega}_t} f = \omega_0 \int_{\partial \widetilde{\Omega}_t} \vec{\overline{f}} \cdot \boldsymbol{\tau} + i \omega_0 \int_{\partial \widetilde{\Omega}_t} \vec{\overline{f}} \cdot n = -\operatorname{curl} \mathbf{u}(t, \mathbf{x}) + i \operatorname{div} \mathbf{u}(t, \mathbf{x}).$$

But we see from Lemma 5.1 that $\mathbf{v} = K_\infty(\mathbf{x} - \cdot)$ and that

$$f = \frac{1}{2} \overline{\overleftarrow{\overrightarrow{\cot(\pi \overline{z})}^{\perp}}} = \frac{1}{2} \overline{\overleftarrow{\overrightarrow{\overline{i \cot(\pi z)}}}} = \frac{1}{2} \overline{\overline{i \cot(\pi z)}} = -\frac{i}{2} \cot(\pi z),$$

where we used (2.2) and the identity $\overline{i \overline{z}} = -iz$. The complex meromorphic function f has simple poles at each point in $\mathbf{x} + \mathcal{L}$ with residue $(-2\pi)^{-1} i$. By the residue theorem, then, summing over all points of $\mathcal{L}$ lying inside $\partial \widetilde{\Omega}_t$—that is, lying in $\widetilde{\Omega}_t$,

$$\omega_0 \oint_{\partial \widetilde{\Omega}_t} f = \operatorname{Re} \left(2\pi i \omega_0 \sum_n \operatorname{Res}(f, (n, 0)) \right) = \omega_0 \operatorname{Re} \left(\frac{2\pi i}{-2\pi i} \sum_n 1 \right) = -\omega_0 n.$$

But $\widetilde{\Omega}_t$ can contain at most one point of $\mathbf{x} + \mathcal{L}$ else the lift given in Section 2.4 would map $\mathbf{x}$ to more than one point in $\mathbb{C}$ (which would mean it is not a lift). We see, then, that

$$\operatorname{curl} \mathbf{u}(t, \mathbf{x}) = -\omega_0 \oint_{\partial \Omega_t} f = -\omega_0 \oint_{\partial \widetilde{\Omega}_t} f = \omega_0 \mathbb{1}_{\Omega_t}(t, \mathbf{x}) = \omega(t, \mathbf{x}).$$

We conclude that for all $t \in [0, T]$, $\operatorname{div} \mathbf{u} = 0$ and $\operatorname{curl} \mathbf{u} = \omega = \omega_0 \mathbb{1}_{\Omega_t}$. Directly from (9.4), we know that $\mathbf{u} \in L^\infty(\Pi)$ and hence $\mathbf{u} \in S(\Pi)$. It follows from Theorem 9.5 applied with $\gamma_j(t, \cdot)$ in place of γ_j for any fixed t that $m_2[\mathbf{u}(t)] = 0$ and $m_1[\mathbf{u}(t)](-\infty) + m_1[\mathbf{u}](\infty) = 0$.

Using (1), the proof of (2) that $\mathbf{u}$ solves the 2D Euler equations on the time interval $[-T, T]$ proceeds just as it does in the proof of Proposition 8.6 on page 334 of [30]. □

Remark 9.7. *We can view Type 2 solutions as equivalent to Type 1 or 3 solutions by virtue of Theorem 7.2. For vortex patches it is most natural to start with an $\Omega \in \Pi$ satisfying Assumption 9.4 and lift it to $\mathbb{R}^2$ as in Section 2.4 to give Ω_0. It is also possible to start with a domain in $\mathbb{R}^2$, and use it to obtain via the $\mathcal{P}er$ operator a domain in Π, but there are no simple general conditions to guarantee that the boundary of the domain in Π is regular.*

10. Regularity of a vortex patch boundary

To prove the propagation of regularity of a vortex patch boundary for our Type 1, 2, or 3 solutions, it will be easiest to work with Type 2 solutions, the result then immediately following for the other two types by Theorem 7.2. We will prove, in Theorem 10.1, that for Type 2 solutions, the regularity of the boundary of a periodic vortex patch is maintained for all time, as in the classical case.

Theorem 10.1. *Let Ω be as in Assumption 9.4 and let $\Omega_t = X(t, \Omega)$ for a Type 2 solution. Then $\partial\Omega_t$ is $C^{1,\varepsilon}$ for all time. The analogous result holds for Type 1 and 3 solutions.*

Proof. We describe only how the proof differs from the now classical proof as presented in Chapter 8 of [30]. There are two main steps to the proof given in [30]: First, show local-in-time existence of a $C^{1,\varepsilon}$ solution to the CDE (based on [10]) then show that the solution extends globally in time (based on [9]).

Local-in-time $C^{1,\varepsilon}$ solutions: In brief, the first step is to define the function F on the space $B^{1,\varepsilon}$ of closed $C^{1,\varepsilon}$ paths in Π by (we have translated this to Type 2 solutions) by

$$F(\boldsymbol{\gamma}(\beta)) := \frac{\omega_0}{2\pi} \int_0^{2\pi} \log \rho(\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)) \partial_\alpha \boldsymbol{\gamma}(\alpha) \, d\alpha.$$

Here, F is as defined for each boundary component separately, we suppress the sums over each boundary component for notational simplicity. First show that $F \colon \mathcal{O}^M \to B^{1,\varepsilon}$ is Lipschitz-continuous on the open subset

$$\begin{aligned} \mathcal{O}^M &:= \left\{ \boldsymbol{\gamma} \in B^{1,\varepsilon} \colon |\boldsymbol{\gamma}|_* > M^{-1}, \, \|\boldsymbol{\gamma}'\|_{L^\infty} < M \right\}, \\ |\boldsymbol{\gamma}|_* &:= \inf_{\alpha \neq \alpha'} \frac{\boldsymbol{\gamma}(\alpha) - \boldsymbol{\gamma}(\alpha')}{|\alpha - \alpha'|} \end{aligned}$$

for some $M > 0$. A Picard fixed point theorem (Theorem 8.3 of [30]) then assures a local-in-time solution to the ODE,

$$\frac{d\boldsymbol{\gamma}}{dt} = F(\boldsymbol{\gamma}), \quad \boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0 \in \mathcal{O}^M,$$

with $\boldsymbol{\gamma} \in C^1([-T, T]; \mathcal{O}^M)$ for a T that depends upon M.

To adapt the argument in [30] to Type 2 solutions, we decompose $\log \rho(\mathbf{x})$ as follows. Let $\varphi \in C_0^\infty(\Pi)$ be a radially symmetric cutoff function supported on $B_{1/4}(0)$ with $\varphi \equiv 1$ on $B_{1/8}(0)$. Then

$$\begin{aligned} \log \rho(\mathbf{x}) &= \varphi(\mathbf{x}) \log|\mathbf{x}| + R(\mathbf{x}), \\ R(\mathbf{x}) &:= \varphi(\mathbf{x}) \left[\log \rho(\mathbf{x}) - \log|\mathbf{x}|\right] + (1 - \varphi(\mathbf{x})) \log \rho(\mathbf{x}). \end{aligned}$$

Recall that on Π, we use coordinates in which $\mathbf{x} = (x_1, x_2)$ with $-1/2 \leqslant x_1 < 1/2$. Because $\varphi(\mathbf{x}) = 0$ for $|x_1| > 1/4$, the function $\varphi(\mathbf{x}) \log|\mathbf{x}|$ is in $C^\infty(\Pi \setminus (0,0))$. Also, $\log \rho(\mathbf{x})$ is harmonic away from the origin, so $R(\mathbf{x}) \in C^\infty(\Pi)$, as follows from Lemma 5.2. In particular, $\varphi(\mathbf{x}) \log|\mathbf{x}|$ and $R(\mathbf{x})$ are well-defined as functions on Π.

It follows that for each component of $\partial\Omega_{t,0}$, $F = F_1 + F_2$, where

$$\begin{aligned} F_1(\boldsymbol{\gamma}(\beta)) &:= \frac{\omega_0}{2\pi} \int_0^{2\pi} \varphi(\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)) \log|\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)| \partial_\alpha \boldsymbol{\gamma}(\alpha) \, d\alpha, \\ F_2(\boldsymbol{\gamma}(\beta)) &:= \frac{\omega_0}{2\pi} \int_0^{2\pi} R(\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)) \partial_\alpha \boldsymbol{\gamma}(\alpha) \, d\alpha. \end{aligned}$$

Other than the cutoff function, which introduces no real difficulties, F_1 is the same expression as in the classical setting and is estimated in $B^{1,\varepsilon}$ in the same manner. We note that applying $d/d\beta$ to $F_1(\boldsymbol{\gamma}(\beta))$ leads to a singularity in the integrand at $\alpha = \beta$. The key to estimating F_1 is treating $dF_1/d\beta$, beginning in Lemma 8.7 of [30], as a principal value integral. The situation is no different here than in [30].

Similarly, for F_2, the key is bounding $dF_2/d\beta$ in C^ε. This is much simpler than bounding $dF_1/d\beta$, for we have

$$\frac{d}{d\beta}F_2(\boldsymbol{\gamma}(\beta)) = \frac{\omega_0}{2\pi}\int_0^{2\pi} \nabla R(\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)) \cdot \partial_\beta \boldsymbol{\gamma}(\beta))\, \partial_\alpha \boldsymbol{\gamma}(\alpha)\, d\alpha.$$

Then for any α,

$$\begin{aligned} &\|\nabla R(\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)) \cdot \partial_\beta \boldsymbol{\gamma}(\beta))\, \partial_\alpha \boldsymbol{\gamma}(\alpha)\|_{C^\varepsilon} \\ &\qquad \leqslant |\partial_\alpha \boldsymbol{\gamma}(\alpha)| \|\nabla R\|_{C^\varepsilon(\Pi)} \|\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)\|_{lip}^\varepsilon \|\partial_\beta \boldsymbol{\gamma}(\beta)\|_{C^\varepsilon(0,2\pi)}. \end{aligned}$$

But, $|\partial_\alpha \boldsymbol{\gamma}(\alpha)| \leqslant \|\boldsymbol{\gamma}\|_{Lip} < M$ and $\|\boldsymbol{\gamma}(\beta) - \boldsymbol{\gamma}(\alpha)\|_{lip} = \|\boldsymbol{\gamma}\|_{lip} < M$. Hence,

$$\left\|\frac{d}{d\beta}F_2(\boldsymbol{\gamma}(\beta))\right\|_{C^\varepsilon(0,2\pi)} \leqslant CM^2|\omega_0|\|\boldsymbol{\gamma}\|_{C^\varepsilon}.$$

We see, then, that the bounds in Lemma 8.10 of [30] hold, and the proof of local-in-time existence is completed as in [30].

Global-in-time $C^{1,\varepsilon}$ solutions: The proof of the global existence of a $C^{1,\varepsilon}$ solution to the CDE is the same as in Section 8.3.3 of [30], except that Corollary 8.4 is used to obtain $\nabla \mathbf{u}$. By virtue of Proposition 3.4, the estimates differ little from those for classical vortex patches.

This completes the proof for Type 2 solutions. The result for Types 1 and 3 solutions then follows directly, exploiting the lifting of domains described in Section 2.4. □

Appendix A. Proof of the formula for ∇u

Before giving the proof of the singular integral operator formula for ∇u of Lemma 8.2, let us calculate $\nabla K_\infty(\mathbf{x})$ to obtain the expression for β. Letting

$$\xi(\mathbf{x}) = \rho(\mathbf{x})^2 = \sin^2(\pi x_1) + \sinh^2(\pi x_2),$$

we have $\partial_1 \rho(\mathbf{x}) = \pi \sin(2\pi x_1)$, $\partial_2 \rho(\mathbf{x}) = \pi \sinh(2\pi x_2)$. Then from Lemma 5.1, we have $G(\mathbf{x}) = (2\pi)^{-1} \log \rho(\mathbf{x}) = (4\pi)^{-1} \log \xi(\mathbf{x})$, so

$$K_\infty(\mathbf{x}) = \nabla^\perp G(\mathbf{x}) = \frac{\nabla^\perp \xi(\mathbf{x})}{4\pi \xi(\mathbf{x})} = \frac{(-\partial_2 \xi(\mathbf{x}), \partial_1 \xi(\mathbf{x})}{4\pi \xi(\mathbf{x})} = \frac{(-\sinh(2\pi x_2), \sin(2\pi x_1))}{4\xi(\mathbf{x})}.$$

Remark A.1. *As in Remark 8.3, near the origin,* $\xi(\mathbf{x}) = \rho(\mathbf{x})^2 \approx \pi^2|\mathbf{x}|^2$*. Hence,* $G(\mathbf{x}) \approx (1/4\pi)\log(\pi^2|\mathbf{x}|^2) \approx C + (1/2\pi)\log|x|$*, like the fundamental solution to the Laplacian on* $\mathbb{R}^2$*. Then* $K_\infty(\mathbf{x}) \approx 2\pi|\mathbf{x}|/(4\xi(\mathbf{x})) \approx 2\pi|\mathbf{x}|/(4\pi^2|\mathbf{x}|^2) = 1/(2\pi|x|)$*, as it is for the Biot-Savart kernel on* $\mathbb{R}^2$*.*

Taking another derivative,

$$\nabla K_\infty(\mathbf{x}) = \frac{1}{4}\begin{pmatrix} -\partial_1 \dfrac{\sinh(2\pi x_2)}{\xi(\mathbf{x})} & -\partial_2 \dfrac{\sinh(2\pi x_2)}{\xi(\mathbf{x})} \\[2ex] \partial_1 \dfrac{\sin(2\pi x_1)}{\xi(\mathbf{x})} & \partial_2 \dfrac{\sin(2\pi x_1)}{\xi(\mathbf{x})} \end{pmatrix}$$

$$= -\frac{1}{4\xi(\mathbf{x})^2}\begin{pmatrix} -\sinh(2\pi x_2)\partial_1\xi(\mathbf{x}) & -\sinh(2\pi x_2)\partial_2\xi(\mathbf{x}) \\ \sin(2\pi x_1)\partial_1\xi(\mathbf{x}) & \sin(2\pi x_1)\partial_2\xi(\mathbf{x}) \end{pmatrix}$$

$$+ \frac{1}{4\xi(\mathbf{x})}\begin{pmatrix} 0 & -2\pi\cosh(2\pi x_2) \\ 2\pi\cos(2\pi x_1) & 0 \end{pmatrix}$$

$$= -\frac{1}{4\xi(\mathbf{x})^2}\begin{pmatrix} -\sinh(2\pi x_2)\pi\sin(2\pi x_1) & -\sinh(2\pi x_2)\pi\sinh(2\pi x_2) \\ \sin(2\pi x_1)\pi\sin(2\pi x_1) & \sin(2\pi x_1)\pi\sinh(2\pi x_2) \end{pmatrix}$$

$$+ \frac{1}{4\xi(\mathbf{x})^2}\begin{pmatrix} 0 & -2\pi\cosh(2\pi x_2)\xi(\mathbf{x}) \\ 2\pi\cos(2\pi x_1)\xi(\mathbf{x}) & 0 \end{pmatrix}$$

$$= \frac{\pi}{4\xi(\mathbf{x})^2}\begin{pmatrix} \sinh(2\pi x_2)\sin(2\pi x_1) & \sinh^2(2\pi x_2) \\ -\sin^2(2\pi x_1) & -\sin(2\pi x_1)\sinh(2\pi x_2) \end{pmatrix}$$

$$+ \frac{\pi}{4\xi(\mathbf{x})^2}\begin{pmatrix} 0 & -2\cosh(2\pi x_2)\xi(\mathbf{x}) \\ 2\cos(2\pi x_1)\xi(\mathbf{x}) & 0 \end{pmatrix}$$

$$= \frac{\pi}{2\rho(\mathbf{x})^4}\begin{pmatrix} \alpha_{11}(\mathbf{x}) & \alpha_{12}(\mathbf{x}) \\ \alpha_{21}(\mathbf{x}) & \alpha_{22}(\mathbf{x}) \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{11}(\mathbf{x}) &= -\alpha_{22}(\mathbf{x}) = \tfrac{1}{2}\sinh(2\pi x_2)\sin(2\pi x_1), \\ \alpha_{12}(\mathbf{x}) &= \tfrac{1}{2}\left[\sinh^2(2\pi x_2) - 2\cosh(2\pi x_2)\xi(\mathbf{x})\right], \\ \alpha_{21}(\mathbf{x}) &= \tfrac{1}{2}\left[-\sin^2(2\pi x_1) + 2\cos(2\pi x_1)\xi(\mathbf{x})\right]. \end{aligned}$$

Using (2.5) and $\cosh^2 x - \sinh^2 x = 1$, we see that

$$\begin{aligned} 2\alpha_{12}(\mathbf{x}) &= \sinh^2(2\pi x_2) - 2\cosh(2\pi x_2)(\sin^2(\pi x_1) + \sinh^2(\pi x_2)) \\ &= \sinh^2(2\pi x_2) - 2\cosh(2\pi x_2)\sin^2(\pi x_1) - \cosh(2\pi x_2)(\cosh(2\pi x_2) - 1) \\ &= -1 + \cosh(2\pi x_2)(1 - 2\sin^2(\pi x_1)) = \cosh(2\pi x_2)\cos(2\pi x_1) - 1, \\ 2\alpha_{21}(\mathbf{x}) &= -\sin^2(2\pi x_1) + 2\cos(2\pi x_1)(\sin^2(\pi x_1) + \sinh^2(\pi x_2)) \\ &= -\sin^2(2\pi x_1) + \cos(2\pi x_1)(1 - \cos(2\pi x_1)) + 2\cos(2\pi x_1)\sinh^2(\pi x_2)) \\ &= -1 + \cos(2\pi x_1)(1 + 2\sinh^2(\pi x_2)) = \cos(2\pi x_1)\cosh(2\pi x_2) - 1. \end{aligned}$$

Thus,

$$\nabla K_\infty(\mathbf{x}) = \frac{\pi}{2}\frac{\beta(\mathbf{x})}{\rho(\mathbf{x})^2},$$

where

$$\beta(\mathbf{x}) = \frac{1}{2\rho(\mathbf{x})^2} \begin{pmatrix} \sin(2\pi x_1)\sinh(2\pi x_2) & \cos(2\pi x_1)\cosh(2\pi x_2) - 1 \\ \cos(2\pi x_1)\cosh(2\pi x_2) - 1 & -\sin(2\pi x_1)\sinh(2\pi x_2) \end{pmatrix},$$

as given in Lemma 8.2.

Proof of Lemma 8.2. Let $M \in (H^1(\Omega))^{2\times 2}$ be arbitrary. We will show that

$$(\nabla \mathbf{u}, M) = \left(\sum_{n\in\mathbb{Z}} \frac{\omega(\mathbf{x} + (n,0))}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, M \right) + \frac{1}{2\pi} \text{ p. v.} \int_{\mathbb{R}^2} \nabla K_\infty(\mathbf{x} - \mathbf{y}) M(\mathbf{y})\, d\mathbf{y},$$

giving the action of $\nabla \mathbf{u} \in H^{-1}(\mathbb{R}^2)$ on any test function in $H^1(\mathbb{R}^2)$, and thus establishing our expression for $\nabla \mathbf{u}$.

For any $r \in (0,1)$, we let

$$U_r = \bigcup_{n\in\mathbb{Z}} B_r(\mathbf{x} + (n,0)).$$

Then

$$\begin{aligned}(\nabla \mathbf{u}, M) &= (\mathbf{u}, \operatorname{div} M) = (K_\infty * \omega, \operatorname{div} M) = \lim_{r\to 0} \int_{U_r^C} K_\infty * \omega(\mathbf{x}) \operatorname{div} M(\mathbf{x})\, d\mathbf{x} \\ &= -\lim_{r\to 0} \int_{U_r^C} \nabla(K_\infty * \omega)(\mathbf{x})\, M(\mathbf{x})\, d\mathbf{x} - \lim_{r\to 0} \int_{\partial U_r} (\nabla M \cdot \boldsymbol{n}) K_\infty * \omega\, dS =: I + II.\end{aligned}$$

We used here that $\mathbf{u}$ is integrable and that the orientation of ∂U is opposite that of ∂U^C. The limit in I gives the principal value integral in our expression for ∇u. Noting that the compact support of ω makes the sum below finite,

$$\begin{aligned} II &= \sum_{n\in\mathbb{Z}} \lim_{r\to 0} \int_{\partial B_r(\mathbf{x}+(n,0))} (\nabla M \cdot \boldsymbol{n}) K_\infty * \omega\, dS \\ &= \sum_{n\in\mathbb{Z}} \lim_{r\to 0} \int_{\partial B_r(\mathbf{x})} (\nabla M(\cdot + (n,0)) \cdot \boldsymbol{n}) K_\infty * \omega\, dS \\ &= \sum_{n\in\mathbb{Z}} \lim_{r\to 0} \int_{\partial B_r(\mathbf{x})} (\nabla M(\cdot + (n,0)) \cdot \boldsymbol{n}) K * \omega\, dS \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, M(\cdot + (n,0)) \right) = \sum_{n\in\mathbb{Z}} \left(\frac{\omega(\mathbf{x} - (n,0))}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, M \right). \end{aligned}$$

We used that $K_\infty(\mathbf{y})$ becomes $K(\mathbf{y})$ in the limit of small $\mathbf{y}$, and then evaluated the limit of the boundary integral as in the classical case. □

Acknowledgements

DMA is grateful to the National Science Foundation for support through grant DMS-1907684.

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