MICHEL L. LAPIDUS
http://math.ucr.edu/~lapidus

July 2010

List of Publications

Published Research Journal Articles:

[JA1] "Formules de Moyenne et de Produit pour les Résolvantes Imaginaires d'Opérateurs Auto-Adjoints", [Mean and Product Formulas for Imaginary Resolvents of Self-Adjoint Operators], *Comptes Rendus de l'Académie des Sciences* Paris Sér. A **291** (1980), pp. 451-454; **MR** 81j:47016; **Zbl** 446:47010.

[JA2] "Généralisation de la Formule de Trotter-Lie", [Generalization of the Trotter-Lie Formula], *Comptes Rendus de l'Académie des Sciences* Paris Sér. A **291** (1980), pp. 479-500; **MR** 81k:47092; **Zbl** 447:47023.

[JA3] "Perturbation d'un Semi-groupe par un Groupe Unitaire", [Perturbation of a Semigroup by a Unitary Group], *Comptes Rendus de l'Académie des Sciences* Paris Sér. A **291** (1980), pp. 535-538; **MR** 82d:47046; **Zbl** 447:47022.

[JA4] "Generalization of the Trotter-Lie Formula", *Integral Equations and Operator Theory* **4** (1981), pp. 366-415; **MR** 83e:47057; **Zbl** 463:47824.

[JA5] "Modification de l'Intégrale de Feynman pour un Potentiel Positif Singulier: Approche Séquentielle", [Modification of the Feynman Integral for a Nonnegative Singular Potential: Sequential Approach], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **295** (1982), pp. 1-3; **MR** 83b:81028; **Zbl** 493:35038.

[JA6] "Intégrale de Feynman Modifiée et Formule du Produit pour un Potentiel Singulier Négatif", [Modified Feynman Integral and Product Formula for a Negative Singular Potential], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **295** (1982), pp. 719-722; **MR** 85h:35065; **Zbl** 508:35027.

[JA7] "Valeurs Propres du Laplacien avec un Poids qui Change de Signe", [Eigenvalues of the Laplacian with an Indefinite Weight Function], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **298** (1984), pp. 265-268; **MR** 85j:35139.

[JA8] "Eigenvalues of Elliptic Boundary Value Problems with an Indefinite Weight Function", *Transactions of the American Mathematical Society* **295** (1986), pp. 305-324; **MR** 87j:35282, (with J. Fleckinger).

[JA9] "Product Formula for Imaginary Resolvents with Application to a Modified Feynman

Integral", *Journal of Functional Analysis* **63** (1985), pp. 261-275; **MR** 87c:47059.

[JA10] "Perturbation Theory and a Dominated Convergence Theorem for Feynman Integrals", *Integral Equations and Operator Theory* **8** (1985), pp. 36-62; **MR** 86g:81036.

[JA11] "The Differential Equation for the Feynman-Kac Formula with a Lebesgue-Stieltjes Measure", *Letters in Mathematical Physics* **11** (1986), pp. 1-31; **MR** 87d:58033.

[JA12] "Generalized Dyson Series, Generalized Feynman Diagrams, the Feynman Integral and Feynman's Operational Calculus", *Memoirs of the American Mathematical Society* No. 351, **62** (1986), pp. 1-78, (with G.W. Johnson); **MR** 88f:81034.

[JA13] "The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure and Feynman's Operational Calculus", *Studies in Applied Mathematics* **76** (1987), pp. 93-132.

[JA14] "Remainder Estimates for the Asymptotics of Elliptic Eigenvalue Problems with Indefinite Weights", *Archives for Rational Mechanics and Analysis* **98** (1987), pp. 329-356, (with J. Fleckinger); **MR** 88b:35149.

[JA15] "The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure: An Integral Equation in the General Case", *Integral Equations and Operator Theory* **12** (1989), pp. 163-210.

[JA16] "Une Multiplication Non Commutative des Fonctionnelles de Wiener et le Calcul Opérationnel de Feynman", [A Noncommutative Multiplication of Wiener Functionals and Feynman's Operational Calculus], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **304** (1987), pp. 523-526, (with G.W. Johnson).

[JA17] "Strong Product Integration of Measures and the Feynman-Kac Formula with a Lebesgue-Stieltjes Measure", *Supplemento ai Rendiconti del Circolo Matematico di Palermo,* Ser. II, **17** (1987), pp. 271-312.

[JA18] "Tambour Fractal: Vers une Résolution de la Conjecture de Weyl-Berry pour les Valeurs Propres du Laplacien", [Fractal Drum: Towards a Resolution of the Weyl-Berry Conjecture for the Eigenvalues of the Laplacian], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **306** (1988), pp. 171-175, (with the collaboration of J. Fleckinger).

[JA19] "Noncommutative Operations on Wiener Functionals and Feynman's Operational Calculus", *Journal of Functional Analysis* **81** (1988), pp. 74-99, (with G.W. Johnson).

[JA20] "Schrödinger Operators with Indefinite Weights: Asymptotics of Eigenvalues with Remainder Estimates", *Differential and Integral Equations* **7** (1994), pp. 1389-1418, (with J. Fleckinger).

[JA21] "Fractal Drum, Inverse Spectral Problems for Elliptic Operators and a Partial

Resolution of the Weyl-Berry Conjecture", *Transactions of the American Mathematical Society* **325** (1991), pp. 465-529.

[JA22] "Quantification, Calcul de Feynman Axiomatique et Intégrale Fonctionnelle Généralisée", [Quantization, Axiomatic Feynman's Operational Calculus and Generalized Functional Integral], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **308** (1989), pp. 133-138.

[JA23] "Product Formula for Normal Operators and the Modified Feynman Integral", *Proceedings of the American Mathematical Society* **110** (1990), pp. 449-460, (with A. Bivar Weinholtz).

[JA24] "La Fonction Zêta de Riemann et la Conjecture de Weyl-Berry pour les Tambours Fractals", [The Riemann Zeta-Function and the Weyl-Berry Conjecture for Fractal Drums], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **310** (1990), pp. 343-348, (with C. Pomerance).

[JA25] "Hypothèse de Riemann, Cordes Fractales Vibrantes et Conjecture de Weyl-Berry Modifiée", [The Riemann Hypothesis, Vibrating Fractal Strings and the Modified Weyl-Berry Conjecture], *Comptes Rendus de l'Académie des Sciences* Paris Sér. I Math. **313** (1991), pp. 19-24, (with H. Maier).

[JA26] "The Riemann Zeta-Function and the One-Dimensional Weyl-Berry Conjecture for Fractal Drums", *Proceedings of the London Mathematical Society* (3) **66**, No. 1 (1993), pp. 41-69, (with C. Pomerance).

[JA27] "The Riemann Hypothesis and Inverse Spectral Problem for Fractal Strings", *Journal of the London Mathematical Society* (2) **52**, No. 1 (1995), pp. 15-35, (with H. Maier).

[JA28] "Weyl's Problem for the Spectral Distribution of Laplacians on P.C.F. Self-Similar Fractals", *Communications in Mathematical Physics* **158** (1993), pp. 93-125, (with J. Kigami).

[JA29] "Indefinite Elliptic Boundary Value Problems on Irregular Domains", *Proceedings of the American Mathematical Society* **125** (1995), pp. 513-526, (with J. Fleckinger).

[JA30] "Analysis on Fractals, Laplacians on Self-Similar Sets, Noncommutative Geometry and Spectral Dimensions", *Topological Methods in Nonlinear Analysis,* No. 1, **4** (1994), pp. 137-195.

[JA31] "Eigenfunctions of the Koch Snowflake Drum", *Communications in Mathematical Physics* **172** (1995), pp. 359-376, (with M. Pang).

[JA32] "Fractals and Vibrations: Can You Hear the Shape of a Fractal Drum?", *Fractals* No. 3, **3** (1995), pp. 725-736.

[JA33] "Counterexamples to the Modified Weyl-Berry Conjecture", *Mathematical Proceedings of the Cambridge Philosophical Society*, **119** (1996), pp. 167-178, (with C. Pomerance).

[JA34] "Generalized Minkowski Content and the Vibrations of Fractal Drums and Strings", *Mathematical Research Letters* **3** (1996), pp. 31-40, (with C. Q. He).

[JA35] "Generalized Minkowski Content, Spectrum of Fractal Drums, Fractal Strings and the Riemann Zeta-Function", *Memoirs of the American Mathematical Society* No. 608, **127** (1997), pp. 1-97, (with C. Q. He).

[JA36] "Snowflake Harmonics and Computer Graphics: Numerical Computation of Spectra on Fractal Domains", *International Journal of Bifurcation and Chaos* **6** (1996), pp. 1185-1210, (with J. W. Neuberger, R. J. Renka, and C. A. Griffith). (Includes 23 computer graphics color plates.)

[JA37] "Feynman's Operational Calculus and Evolution Equations", *Acta Applicandae Mathematicae* **47** (1997), pp. 155-211, (with B. DeFacio and G. W. Johnson).

[JA38] "Feynman's Operational Calculus: A Heuristic and Mathematical Introduction", *Annales Mathématiques Blaise Pascal* **3** (1996), pp. 89-102. (Special issue dedicated to the memory of Prof. Albert Badrikian.)

[JA39] "Self-Similarity of Volume Measures for Laplacians on P.C.F. Self-Similar Fractals", *Communications in Mathematical Physics* **217** (2001), pp. 165-180, (with J. Kigami).

[JA40] "Complex Dimensions of Self-Similar Fractal Strings and Diophantine Approximation", *Journal of Experimental Mathematics* No.1, **12** (2003), pp. 41-69, (with M. van Frankenhuysen).

[JA41] "Fractality, Self-Similarity and Complex Dimensions", *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, **72** (2004), Part 1, pp. 349-372, (with M. van Frankenhuijsen). [E-print: arXiv:math.NT/0401156, 2004.]

[JA42] "Random Fractal Strings: Their Zeta Functions, Complex Dimensions and Spectral Asymptotics", *Transactions of the American Mathematical Society* No.1, **358** (2006), pp. 285-314, (with B. Hambly).

[JA43] "A Tube Formula for the Koch Snowflake Curve, with Applications to Complex Dimensions", *Journal of the London Mathematical Society* No. 2, **74** (2006), pp. 397-414, (with E. P. J. Pearse). [E-print: arXiv:math-ph/0412029, 2005.]

[JA44] "Beurling Zeta Functions, Generalized Primes, and Fractal Membranes", *Acta Applicandae Mathematicae* No.1, **94** (2006), pp. 21-48, (with T. Hilberdink). [E-print:

arXiv:math.NT/0410270, 2004.]

[JA45] "Feynman's Operational Calculi: Auxiliary Operations and Related Disentangling Formulas", *Integration: Mathematical Theory and Applications* No. 1, **1** (2008), pp. 29-48, (with G. W. Johnson).

[JA46] "Localization on Snowflake Domains", *Fractals* No. 3, **15** (2007), pp. 255-272, (with B. Daudert). [E-print: arXiv:math.NA/0609798, 2006.]

[JA47] "Nonarchimedean Cantor Set and String", *Journal of Fixed Point Theory and Applications* **3** (2008), pp. 181-190, (with H. Lu). (Special issue dedicated to Vladimir Arnold on the occasion of his Jubilee. Vol. I.) [E-print: IHES/M/08/29, 2008. re.pdf.]

[JA48] "A Trace on Fractal Graphs and the Ihara Zeta Function", *Transactions of the American Mathematical Society* No. 6, **361** (2009), pp. 3041-3070, (with D. Guido and T. Isola). [E-print: arXiv:math.OA/0608060v3, 2008. IHES/M/08/36, 2008.]

[JA49] "Ihara's Zeta Function for Periodic Graphs and Its Approximation in the Amenable Case, *Journal of Functional Analysis*, No. 6, **255** (2008), pp. 1339-1361, (with D. Guido and T. Isola). [E-print: math.OA/0608229. IHES/M/08/37, 2008. re.pdf.]

[JA50] "Dirac Operators and Spectral Triples for some Fractal Sets Built on Curves", *Advances in Mathematics* No. 1, **217** (2008), pp. 42-78, (with E. Christensen and C. Ivan). [E-print: arXiv:math.MG/0610222v2, 2008.]

[JA51] "Fractal Strings and Multifractal Zeta Functions", *Letters in Mathematical Physics* No. 1, **88** (2009), pp. 101-129, (with J. Levy Vehel and J. A. Rock). (Special issue dedicated to the memory of Moshe Flato.) (Springer Open Access: DOI 10.1007/s11005-009-0302-y.) [E-print: arXiv:math-ph/0610015v3, 2009.]

[JA52] "Towards Zeta Functions and Complex Dimensions of Multifractals", *Journal of Complex Variables and Elliptic Equations* No. 6, **54** (2009), 545-559, (with J. A. Rock). (Special issue dedicated to Fractal Analysis.) [E-print: arXiv.math.ph/0810.0789, 2008. IHES/M/08/34, 2008.]

[JA53] "Self-Similar p-Adic Fractal Strings and Their Complex Dimensions", *p-Adic Numbers, Ultrametric Analysis and Applications.* (Russian Academy of Sciences, Moscow, and Springer-Verlag), No. 2, **1** (2009), pp. 167-180, (with H. Lu). [E-print: IHES/M/08/42, 2008.]

[JA54] "Tube Formulas and Complex Dimensions of Self-Similar Tilings", *Acta Applicandae Mathematicae* No. 1, **112** (2010), 91-137 (with E. P. J. Pearse). (Springer Open Access: DOI 10.1007/S10440-010-9562-x) [E-print: arXiv:math.DS/0605527v6, 2010. IHES/M/08/27, 2008.]

In Press Research Journal Articles:

Submitted Research Journal Articles:

[JA55] "Pointwise Tube Formulas for Fractal Sprays and Self-Similar Tilings with Arbitrary Generators", 45 typed pages, 2010, (with Erin P. J. Pearse and Steffen Winter). [E-print: arXiv:math.DG:1006.3807v1]

[JA56] "Partition Zeta Functions, Multifractal Spectra, and Tapestries of Complex Dimensions", 48 types pages, 2010, (with Kate E. Ellis, Michael Mackenzie and John Rock). [E-print: arXiv:math.PH/1007.1467v1, 2010.]

Published Conference Proceedings:

[CP1] "Spectral Theory of Elliptic Problems with Indefinite Weights", in Proc. May-June 1984 Workshop "*Spectral Theory of Sturm-Liouville Differential Operators*", Hans G. Kaper and A. Zettl (Eds.), **ANL-84-73**, Argonne National Laboratory, Argonne, 1984, pp. 159-168.

[CP2] "Product Formula for Imaginary Resolvents, Modified Feynman Integral and a General Dominated Convergence Theorem", Semesterbereich Funcktionalanalysis Sommersemester 84, Mathematisches Institut Eberhard-Karls-Universität Tubingen, R. Nagel, H. Shaefer and U. Schlotterbeck (Eds.), 1984, pp. 9-24.

[CP3] "Asymptotic Distribution of the Eigenvalues of Elliptic Boundary Value Problems and Schrödinger Operators with Indefinite Weights", 14 pages, in "*Partial Differential Equations*", Proc. VIIIth Latin American School of Mathematics, held in July 1986 at IMPA, Rio de Janeiro, Brazil.

[CP4] "Product Formula, Imaginary Resolvents and Modified Feynman Integral", *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society **45** (1986), pp. 109-112; **MR** 87j:47059.

[CP5] "Feynman's Operational Calculus, Generalized Dyson Series and the Feynman Integral", *Contemporary Mathematics*, American Mathematical Society **62** (1987), pp. 437-445, (with G.W. Johnson); **MR** 88c:81025.

[CP6] "The Feynman Integral, The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure and Feynman's Operational Calculus", in "*Path Summation: Achievement and Goals*", S.O. Lundquist *et al*. (Eds.), World Scientific, Singapore, 1988, pp. 327-335.

[CP7] "Can One Hear the Shape of a Fractal Drum? Partial Resolution of the Weyl-Berry Conjecture", in *"Geometric Analysis and Computer Graphics"*, Proc. Workshop on "Differential Geometry, Calculus of Variations and Computer Graphics", held at the MSRI, Berkeley, in May 1988, P. Concus, *et al.* (Eds.), Mathematical Sciences Research

Institute Publications, Vol. 17, Springer-Verlag, New York, 1991, pp. 119-126.

[CP8] "Inverse Spectral Problems for Elliptic Operators on Fractal Drums and the Weyl-Berry Conjecture", in "*Differential Equations and Applications*", Vol. II (Columbus, OH, 1988), Ohio Univ. Press, Athens, 1990, pp. 101-102.

[CP9] "Feynman's Operational Calculus as a Generalized Path Integral", in *"Stochastic Processes. A Festrischift in Honour of Gopinath Kallianpur"*, S. Cambanis, *et al.* (Eds.), 1992, Springer-Verlag, New York, pp. 51-60, (with B. DeFacio and G.W. Johnson).

[CP10] "The Vibrations of Fractal Drums and Waves in Fractal Media", in "*Fractals in the Natural and Applied Sciences*" (A-41), Proc. 2nd IFIP International Conference "Fractals 93", held in London (England, UK) in September 1993, M.M. Novak (Ed.), Elsevier Science B.V., North Holland, 1994, pp. 255-260.

[CP11] "Tube Formulas for Self-Similar Fractals", in: "*Analysis on Graphs and its Applications*", P. Exner, *et al*. (Eds.), *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society **77** (2008), pp. 211-230, (with E. P. J. Pearse). [E-print: arXiv:math.DS/0711.0173v1, 2007. IHES/M/08/28, 2008.]

Published Book Chapters:

[BC1] "The Problem of the Trotter-Lie Formula for Unitary Groups of Operators", Séminaire Choquet: Initiation à l'Analyse, *Publications Mathématiques de l'Université Pierre et Marie Curie* (Paris VI), 20ème année, 1980/81, **46** (1982), pp. 1701-1745; **ZBL** 519:47025.

[BC2] "Spectral and Fractal Geometry: From the Weyl-Berry Conjecture for Fractal Drums to the Riemann Zeta-Function", in "*Differential Equations and Mathematical Physics*", Proc. International Conference on Mathematical Physics and Differential Equations, held in Birmingham in March 1990, C. Bennewitz (Ed.), Academic Press, 1992, pp. 151-182.

[BC3] "Vibrations of Fractal Drums, the Riemann Hypothesis, Waves in Fractal Media, and the Weyl-Berry Conjecture", in *"Ordinary and Partial Differential Equations"*, Proc. Twelth International Conference on the Theory of Partial Differential Equations, held in Dundee (Scotland, UK) in June 1992, Vol. IV, B.D. Sleeman *et al.* (Eds.), Pitman Research Notes in Mathematics Series, **289**, Longman, UK, 1993, pp. 126-209.

[BC4] "Towards a Noncommutative Fractal Geometry? Laplacians and Volume Measures on Fractals", *Contemporary Mathematics*, American Mathematical Society **208** (1997), pp. 211-252.

[BC5] "Computer Graphics and the Eigenfunctions for the Koch Snowflake Drum", in "*Progress* *in Inverse Spectral Geometry"*, Trends in Mathematics, Vol. 1, Birkhäuser-Verlag, Basel and Boston, 1997, pp. 95-109, (with C.A. Griffith). (Includes 10 computer graphics plates).

[BC6] "Complex Dimensions of Fractal Strings and Oscillatory Phenomena in Fractal Geometry and Arithmetic", *Contemporary Mathematics*, American Mathematical Society **237** (1999), pp. 87-105, (with M. van Frankenhuysen).

[BC7] "Spectral Geometry: An Introduction and Background Material for this Volume", in *"Progress in Inverse Spectral Geometry"*, Trends in Mathematics, Vol. 1, Birkhäuser-Verlag, Basel and Boston, 1997, pp. 1-15, (with S.I. Andersson).

[BC8] "A Prime Orbit Theorem for Self-Similar Flows and Diophantine Approximation", *Contemporary Mathematics*, American Mathematical Society **290** (2001), pp. 113-138, (with M. van Frankenhuysen). [E-print: arXiv:math.SP/0111067, 2001.]

[BC9] "T-Duality, Functional Equation, and Noncommutative Stringy Spacetime, in *"Geometries of Nature, Living Systems and Human Cognition: New Interactions of Mathematics with the Natural Sciences and the Humanities"*, L. Boi (Ed.), World Scientific Publ., Singapore, 2005, pp. 3-91.

[The volume in which this research book chapter has appeared is aimed at presenting the research perspectives of several internationally known mathematicians, physicists, biologists and philosophers of science, at the beginning of the 21st century.]

[BC10] "Fractal Geometry and Applications—An Introduction to this Volume", in *Proceedings of Symposia in Pure Mathematics,* **72**, Part 1, American Mathematical Society, Providence, R.I., 2004, pp. 1-25.

[Front article for the two-part volume [EB4]-[EB5]; invited by the publishers of the American Mathematical Society. It provides an introduction to the research area of fractal geometry, describes several of its historical (mathematical) roots, gives an overview of the volume and discusses some of the contributions of the founder of the subject, Benoît Mandelbrot.]

[BC11] "Ihara Zeta Functions for Periodic Simple Graphs", in: "C*-Algebras and Elliptic Theory II", Proceedings of a Conference held at the Banach Center in Warsaw, Poland, D. Burghelea, R. Melrose, *et al*. (Eds.), Trends in Mathematics, Birkhäuser-Verlag, Basel, 2008, pp. 103-121, (with D. Guido and T. Isola). [E-print: arXiv:math.OA/0605753, 2006. IHES/M/08/39, 2008.]

[BC12] "Bartholdi Zeta Functions for Periodic Simple Graphs", in: "*Analysis on Graphs and its Applications*", P. Exner, *et al*. (Eds.), *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society **77** (2008), pp. 109-122, (with D. Guido and T. Isola). [E-print: IHES/M/08/38, 2008.]

[BC13] "Towards the Koch Snowflake Fractal Billiard: Computer Experiments and Mathematical Conjectures", in: "*Gems in Experimental Mathematics*", T. Amdeberhan, L. A. Medina and V. H. Moll (Eds.), *Contemporary Mathematics*, American Mathematical Society, (with R. G. Niemeyer). [E-print: arXiv:math.DS:0912.3948v1, 2009.]

[BC14] "The Geometry of the p-Adic Fractal Strings: A Comparative Survey", in: "*Advances in Non-Archimedian Analysis*", J. Araujo, B. Diarra and A. Escassot (Eds.), *Contemporary Mathematics*, American Mathematical Society, Providence, RI, **551** (2011), pp. 163-206, (with Hung Lu). [E-print: arXiv:1105.2966v1 [math.MG], 2011.]

[BC15] "Partition Zeta Functions, Multifractal Spectra, and Tapestries of Complex Dimensions", in: "*Benoit Mandelbrot*: *A Life in Many Dimensions*", M. Frame (Ed.) World Scientific, Singapore, 2012, 54 typed pages, (with K. E. Ellis, M. C. MacKenzie and J. A. Rock). (Special volume in Memory of Benoit Mandelbrot.) [E-print: arXiv:1007.1467v1 [math-ph], 2010.]

In Press Book Chapters:

Research Memoirs:

[RM1] "Domaine de Dépendance", [Domain of Dependence], Mémoire de l'Université Pierre et Marie Curie (Paris VI), 1978, 45 pages.

Reprinted Articles:

[R1] "Generalized Dyson Series, Generalized Feynman Diagrams, the Feynman Integral and Feynman's Operational Calculus", *Memoirs of the American Mathematical Society* No. 351, **62** (1991), pp. 1-78, (with G.W. Johnson); reprint of [JA12]. [Reprinted by the American Mathematical Society in 1991.]

[R2] "Fractals and Vibrations: Can You Hear the Shape of a Fractal Drum?", in *Fractal Geometry and Analysis: The Mandelbrot Festricht*, Proc. Symposium on "Fractal Geometry and Self-Similar Phenomena" in Honor of Prof. Benoit B. Mandelbrot's 70th Birthday, held in Curaçao (Netherland Antilles, Feb. 1995). C.J.G. Evertsz, H.-O. Peitgen and R.F. Voss (Eds.), World Scientific, Singapore, 1996, pp. 321-332; reprint of [JA35].

Pedagogical Articles:

[PA1] "Creating and Teaching Undergraduate Courses and Seminars in Fractal Geometry: A

Personal Experience", in: *"Fractals, Graphics, and Mathematics Education"*, B. B. Mandelbrot and M. L. Frame (Eds.), Mathematical Association of America, Washington, D. C. (and Cambridge University Press, Cambridge, UK), 2002, pp. 111-116. (Written upon the invitation of Professor Benoit Mandelbrot.)

Encyclopedia Entry:

[EE1] "The Sierpinski Gasket and Carpet", *Kluwer Encyclopedia of Mathematics,* Suppl. Vol. III, Kluwer Academic Publisher, 2002, pp. 364-368.

Undergraduate Research Article:

[URA1] "Fractal Strings and Number Theory: The Harmonic String and the Prime String", *Undergraduate Research Journal* (UCR), **II** (2008), pp. 35-46, (with J. C. Payne). (Survey article.) [re.pdf.]

Research Articles in Preparation:

[Pr1] "Curvature Measures and Tube Formulas for the Generators of Self-Similar Tilings", (with Erin P. J. Pearse).

[Pr2] "Fractal Curvatures and Local Tube Formulas", (with Erin P. J. Pearse and Steffen Winter).

[Pr3] "Fractal Membranes as the Second Quantization of Fractal Strings", (with Ryszard Nest).

[Pr4] "Functional Equations for Zeta Functions Associated with Quasicrystals and Fractal Membranes", (with Ryszard Nest).

[Pr5] "Quasicrystals, Zeta Functions, and Noncommutative Geometry", (with Ryszard Nest).

[Pr6] "Density of Solutions of Dirichlet Polynomial Equations, with Applications to Fractality", (with Machiel van Frankenhuijsen).

[Pr7] "*p*-Adic and Adelic Fractal Strings", (with Hung Lu).

[Pr8] "Spectral Triples for Adelic Fractal Strings and Membranes", (with Hung Lu).

[Pr9] "Partition Zeta Functions of Multifractal Mass Distributions", (with John Rock).

[Pr10] "Complex Fractal Dimensions", (with Erin Pearse and Machiel van Frankenhuijsen).

[Pr11] "Minkowski Measurability of Fractal Sprays and Self-Similar Tilings", (with Steffen Winter and Erin P. J. Pearse).

[Pr12] "Geometry of *p*-Adic Fractal Strings: Zeta Functions, Complex Dimensions and Tube Formulas", (with Hung Lu).

[Pr13] "Families of Periodic Orbits of the Koch Snowflake Fractal Billiard", (with Robert G. Niemeyer).

[Pr14] "Zeta Functions Associated with Fractal Sets in Euclidean Spaces", (with Darko Zubrinic).

[Pr15] "Analytic Continuation of a Class of Multifractal Zeta Functions", (with Driss Essouabri and John A. Rock).

[Pr16] "Invertibility of the Spectral Operator and a Reformulation of the Riemann Hypothesis", (with Hafedh Herichi).

[Pr17] "Spectral Operator and Convergence of Its Euler Product in the Critical Strip", (with Hafedh Herichi).

<u>Research Books</u>:

[RB1] *"The Feynman Integral and Feynman's Operational Calculus"*, Oxford Mathematical Monographs, Oxford Science Publications, *Oxford University Press*, Oxford, London and New York, approx. 800 pages (precisely, 771 + (xviii) pages & 21 illustrations), March 2000. ISBN 0 19 853574 0 (Hbk). (With Gerald W. Johnson.) [Corrected Reprinting, Jan. 2001. ***First Paperback Edition***, *Jan. 2002*. ISBN 0 19 851572 3 (Pbk). Second Reprinting: Jan. 2003. Electronic Edition: forthcoming.] US Library of Congress Classification: QA312.J54 2000.

[This research treatise develops a mathematical theory of the beautiful but challenging subject of the Feynman path integral approach to quantum physics, and of the closely related topic of Feynman's operational calculus for noncommuting operators. It was written over a period of about ten years (Dec. 1989--Dec. 1999) and provides the most complete mathematical treatment of these subjects to date.

Some advantages of the approaches to the Feynman integral which are treated in detail in this book are the following: the existence of the Feynman integral is established for very general potentials in all four cases; under more restrictive but still broad conditions, three of these Feynman integrals agree with one another and with the unitary group from the usual approach to quantum dynamics; these same three Feynman integrals possess pleasant stability properties. The background material in mathematics and physics that motivates the study of the Feynman integral and Feynman's operational calculus is discussed, and detailed proofs are provided for the central results. The last chapter discusses topics in contemporary physics and mathematics (including knot theory and low-dimensional topology) where heuristic Feynman integrals have played a significant role.]

Table of Contents of the Book [RB1]:

Chp. 6: Quantum Dynamics and the Shrödinger Equation. 6.1: Hamiltonian Approach to Quantum Dynamics. 6.2: Transitions Amplitudes and Measurements. 6.3: The Heisenberg Uncertainty Principle. 6.4: Hamiltonian for a System of Particles.

Chp. 7: The Feynman Integral: Heuristic Ideas and Mathematical Difficulties. 7.1: Introduction. 7.2: Feynman's Formula. *Connections with classical mechanics: the method of stationary phase.* 7.3: Heuristic Derivation of the Schrödinger Equation. 7.4: Feynman's Approximation Formula. 7.5: Nelson's Approach via the Trotter Product Formula. *The Trotter product formula.* 7.6: The Approach via Analytic Continuation.

Chp. 8: Semigroups of Operators: An Informal Introduction.

Chp. 9: Linear Semigroups of Operators. 9.1: Infinitesimal Generator. *Integral equation. Evolution equation. Closed unbounded operators.* 9.2: Examples of Semigroups and Their Generators. *The translation semigroup. The heat semigroup. The Poisson semigroup.* 9.3: The Resolvent. 9.4: Generation Theorems. *The Hille-Yosida theorem. Dissipative operators and the Lumer-Phillips theorem.* 9.5: Uniformly Continuous and Weakly Continuous Semigroups. 9.6: Self-Adjoint Operators, Unitary Groups and Stone's Theorem. 9.7: Perturbation Theorems.

Chp. 10: Unbounded Self-Adjoint Operators and Quadratic Forms. 10.1: Spectral Theorem for Unbounded Self-Adjoint Operators. *Multiplication operators. Three useful forms of the spectral theorem.* 10.2: Applications of the Spectral Theorem. *The free Hamiltonian* H_0. *The heat semigroup and unitary group. Standard cores for the free Hamiltonian. Imaginary resolvents.* 10.3: Representation Theorems for Unbounded Quadratic Forms. *Basic definitions and properties. Representation theorems for quadratic forms. The form sum of operators.* 10.4: Conditions on the Potential V for H_0-Form Boundedness.

Chp. 11: Product Formulas with Applications to the Feynman Integral. 11.1: Trotter and Chernoff Product Formulas. *Product formulas for unitary groups.* 11.2: Feynman Integrals via the Trotter Product Formula. *Criteria for essential self-adjointness of positive operators. A brief outline of distribution theory. Kato's distributional inequality. Essential self-adjointness of the Hamiltonian* $H = H_0 + V$. *Conditions on the potential V for* H_0*-operator -boundedness. Feynman integral via the Trotter product formula for unitary groups.* 11.3: Product Formula for Imaginary Resolvents. *Hypotheses and statement of the main result. Proof of the product formula. Consequences, extensions and open problems.* 11.4: Application to the Modified Feynman Integral. *Modified Feynman integral and Schrödinger equation with singular potential. Extensions: Riemannian manifolds and magnetic vector potentials.* 11.5: Dominated Convergence Theorem for the Modified Feynman Integral. *Preliminaries. Perturbation of form sums of self-adjoint operators. Application to a general dominated convergence theorem for Feynman integrals.* 11.6: The Modified Feynman Integral for Complex Potentials. *Product formula for imaginary resolvents of normal operators.*

Application to dissipative quantum systems. 11.7: Appendix: Extended Vitali's Theorem with Application to Unitary Groups. *Extension of Vitali's theorem for sequences of analytic functions. Analytic continuation and product formula for unitary groups.*

Chp. 12: The Feynman-Kac Formula. 12.1: The Feynman-Kac Formula, the Heat Equation and the Wiener Integral. 12.2: Proof of the Feynman-Kac Formula. *Bounded potentials. Monotone convergence theorems for forms and integrals. Unbounded potentials.* 12.3: Consequences.

Chp. 13: Analytic-in-Time or -Mass Operator-Valued Feynman Integrals. 13.1: Introduction. 13.2: The Analytic-in-Time Operator-Valued Feynman Integral. 13.3: Proof of Existence. 13.4: The Feynman Integrals Compared with One Another and with the Unitary Group. Application to Stability Theorems. 13.5: The Analytic-in-Mass Operator-Valued Feynman Integral. *Definition of the analytic-in-mass operator-valued Feynman integral. Nelson's results. Haugsby's results for time-dependent, complex-valued potentials. Further extensions via a product formula for semigroups.* 13.6: The Analytic-in-Mass Modified Feynman Integral. *Existence of the analytic-in-mass modified Feynman integral. Product formula for resolvents: the case of imaginary mass. Comparison with other analytic-in-mass Feynman integrals. Highly singular central potentials—the attractive inverse-square potential.* 13.7: The Analytic-in-Time Operator-Valued Feynman Integral via Additive Functionals of Brownian Motion. *Introductory remarks. The parallel with Section 13.3. Generalized signed measures. The generalized Kato class. Capacity on* P^d. *Smooth measures. Positive continuous additive functionals of Brownian motion. The relationship between smooth measures and PCAFs. The analytic-in-time operator-valued Feynman integral exists for elements of* $S - GK_d$. *Examples.*

Chp. 14: Feynman's Operational Calculus for Noncommuting Operators: An Introduction. 14.1: Functions of Operators. 14.2: The Rules for Feynman's Operational Calculus. *Feynman's time-ordering convention. Feynman's heuristic rules. Two elementary examples.* 14.3: Time-Ordered Peturbation Series. *Perturbation series via Feynman's operational calculus. Perturbation series via a path integral. The origins of Feynman's operational calculus.* 14.4: Making Feynman's Operational Calculus Rigorous. *I: Rigor via path integrals. II: Well-defined and useful formulas arrived at via Feynman's heuristic rules. III: A general theory of Feynman's operational calculus with computations which are rigorous at every stage.* 14.5: Feynman's Operational Calculus via Wiener and Feynman Integrals: Comments on Chapters 15-18.

Chp. 15: Generalized Dyson Series, the Feynman Integral and Feynman's Operational Calculus. 15.1: Introduction. 15.2: The Analytic Operator-Valued Feynman Integral. *Notation and definitions. The analytic (in mass) operator-valued Feynman integral K. Preliminary results.* 15.3: A Simple Generalized Dyson Series ($\eta = \mu + \omega\, \delta_\tau$}). *The Classical Dyson series.* 15.4: Generalized Dyson Series: The General Case. 15.5:

Disentangling via Perturbation Expansions: Examples. *A single measure and potential. Several measures and potentials.* 15.6: Generalized Feynman Diagrams. 15.7: Commutative Banach Algebras of Functionals. *The disentangling algebras* A_t. *The time-reversal map on* A_t *and the natural physical ordering. Connections with Feynman's operational calculus.*

Chp. 16: Stability Results. 16.1: Stability in the Potentials. 16.2: Stability in the Measures.

Chp. 17: The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure and Feynman's Operational Calculus. 17.1: Introduction. *Notation and hypotheses.* 17.2: The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure: Finitely Supported Discrete Part ν. *Integral equation (integrated form of the evolution equation). Differential equation (differential form of the evolution equation). Discontinuities (in time) of the solution. Propagator and explicit solution.* 17.3: Derivation of the Integral Equation in a Simple Case ($\eta = \mu + \omega\, \delta_\tau$). *Sketch of the proof when* ν *is finitely supported.* 17.4: Discontinuities of the Solution to the Evolution Equation. *The time discontinuities. Differential equation and change of initial condition.* 17.5: Explicit Solution and Physical Interpretations. *Continuous measure: uniqueness of the solution. Measure with finitely supported discrete part: propagator and explicit solution. Physical interpretations in the quantum-mechanical case. Physical interpretations in the diffusion case. Further connections with Feynman's operational calculus.* 17.6: The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure: The General Case (Arbitrary Measure η). *Integral equation (integrated form of the evolution equation). Basic properties of the solution to the integral equation. Quantum-mechanical case: reformulation in the interaction (or Dirac) picture. Product integral representation of the solution. Distributional differential equation (true differential form of the evolution equation). Unitary propagators. Scattering matrix and improper product integral. Sketch of the proof of the integral equation.*

Chp. 18: Noncommutative Operations on Wiener Functionals, Disentangling Algebras and Feynman's Operational Calculus. 18.1: Introduction. 18.2: Preliminaries: Maps, Measures and Measurability. 18.3: The Noncommutative Operations * and +°. 18.4: The Functional Integrals K_λ^t and the Operations * and +°. 18.5: The Disentangling Algebras A_t, the Operations * and +°, and the Disentangling Process. *Examples: trigonometric, binomial and exponential formulas.* 18.6: Appendix: Quantization, Axiomatic Feynman's Operational Calculus, and Generalized Functional Integral. *Algebraic and analytic axioms. Consequences of the axioms. Examples: the disentangling algebras and analytic Feynman integrals.*

Chp. 19: Feynman's Operational Calculus and Evolution Equations. 19.1: Introduction and Hypotheses. *Feynman's operational calculus as a generalized path integral.*

Exponentials of sums of noncommuting operators. Disentangling exponentials of sums via perturbation series. Local and nonlocal potentials. Hypotheses. 19.2: Disentangling $\exp[-t\alpha + \int_{0}^{t} \beta(s)\, \mu(ds)]$. 19.3: Disentangling $\exp[-t\alpha + \int_{0}^{t} \beta_1(s)\, \mu_1(ds) + \propto + \int_{0}^{t} \beta_n(s)\, \mu_n(ds)]$. 19.4: Convergence of the Disentangled Series. 19.5: The Evolution Equation. 19.6: Uniqueness of the Solution to the Evolution Equation. 19.7: Further Examples of the Disentangling Process. *Nonlocal potentials relevant to phenomenological nuclear theory.*

Chp. 20: Further Work on or Related to the Feynman Integral. 20.1: Transform Approaches to the Feynman Integral. References to Further Approaches.

A. The Fresnel Integral and Other Transform Approaches to the Feynman Integral. *The Fresnel integral. Properties of the Fresnel integral. An approach to the Feynman integral via the Fresnel integral. Advantages and disadvantages of Fresnel integral approaches to the Feynman integral. The Feynman map. The Poisson process and transforms. A "Fresnel integral" on classical Wiener space. The Banach algebras* Σ *and* $\Phi(H_1)$ *are the same. Consequences of the close relationship between* Σ *and* $\Phi(H_1)$*. More Functions in* $\Phi(H_1)$*. A unified theory of Fresnel integrals: introductory remarks. Background material. A unified theory of Fresnel integrals (continued). The Fresnel classes along with quadratic forms. The classes* $\Gamma^q(H)$ *and* $\Gamma^q(B)$*. Quadratic forms extended. Functions in the Fresnel class of an abstract Wiener space: examples of abstract Wiener spaces. Fourier-Feynman transforms, convolution, and the first variation for functions in* Σ*.*

B. References to Further Approaches to the Feynman Integral. 20.2: The Influence of Heuristic Feynman Integrals on Contemporary Mathematics and Physics: Some Examples.

A. Knot Invariants and Low-Dimensional Topology. *The Jones polynomial invariant for knots and links. Witten's topological invariants via Feynman path integrals. Further developments: Vassiliev invariants and the Kontsevich integral.*

B. Further Comments and References on Subjects Related to the Feynman Integral. *Supersymmetric Feynman path integrals and the Atiyah-Singer index theorem. Deformation quantization: star products and perturbation series. Gauge field theory and Feynman path integrals. String theory: Feynman-Polyakov integrals, and dualities. What lies ahead? Towards a geometrization of Feynman path integrals.*

References.

Index of Symbols. Author Index. Subject Index.

[RB2] *"Fractal Geometry and Number Theory"*. (Subtitle: *"Complex Dimensions of Fractal Strings and Zeros of Zeta Functions"*.) Research Monograph, Birkhäuser-Verlag, Boston, approx. 300 pages (precisely, 268 + (xii) pages & 26 illustrations), January 2000. (With

Machiel van Frankenhuysen.) ISBN 0-8176-4098-3. (Boston). ISBN 3-7643-4098-3 (Basel). US Library of Congress Classification: QA614.86.L36 1999.

[This (refereed) research monograph consists entirely of new research carried out by the authors over a period of about five years (March 1995--Nov. 1999). It develops a mathematical theory of 'complex dimensions' of fractal strings, and of the oscillations intrinsic to the geometry and the spectrum of the associated fractals. In particular, new results about the critical zeros of zeta functions are established and a geometric reformulation of the (Extended) Riemann Hypothesis is obtained in terms of the notion of complex dimension and of the frequency spectrum of fractal strings. On the fractal side, for example, precise explicit formulas are obtained for the volume of tubular neighborhoods of self-similar (and other fractal) strings.

In the long-term, this work is aimed at putting fractal geometry in an arithmetic context and, conversely, at putting various aspects of number theory (such as the theory of Dirichlet series, for example) in a geometric framework.]

[RB3] "*Fractal Geometry, Complex Dimensions and Zeta Functions*", (Subtitle: "*Geometry and Spectra of Fractal Strings*".) Refereed Research Monograph, Springer Monographs in Mathematics, Springer-Verlag, New York, approx. 490 pages (precisely, 460 + (xxiv) pages & 54 illustrations), August 2006. (With Machiel van Frankenhuijsen.) ISBN-10: 0-387-33285-5. e-ISBN: 0-387-35208-2. ISBN-13: 978-0-387-33285-7. US Library of Congress Control Number: 2006929212.

This is a sequel to and a greatly expanded version of the theory of complex fractal dimensions first developed in [RB2]. It contains a large amount of new research material (including several new chapters, sections, and appendices, along with many new examples and applications), almost all of which is directly connected to the work of the authors (and their collaborators) since the publication of [RB2], and a portion of which appeared in print for the first time in the present book.

Overview (of the approx. 200 pages of new material): New chapters on 'self-similar flows' (Chp. 7) and on 'quasiperiodic patterns of self-similar strings' (Chp. 3), providing a much more precise understanding (than in [RB2]) of the complex dimensions of (nonlattice) self-similar fractals (in $\mathbb{R}$) and dynamical systems, as well as of the error terms in the associated 'explicit formulas'. A new geometric description of self-similar fractal strings (in §2.1.1) and a discussion of self-similar strings with multiple generators (in Chp. 2 and throughout the rest of the book). New (and previously entirely unpublished) section (§6.3.3) on an (operator-valued) Euler product attached to the spectrum of a fractal string (generalizing that for the Riemann zeta function $\zeta(s)$ but also converging in the critical strip $0 < \mathrm{Re}\, s < 1$); study of the eigenvalue spectrum of the corresponding 'spectral operator'. New pointwise tube formulas (§8.1.1), with a number of new applications to tube formulas for self-similar strings (in §8.4); new results on average Minkowski content (§8.4.3) and on error terms for nonlattice strings (§8.4.4). New sections (§11.1.1 and §11.4.1) on recent results on zeros of zeta functions in finite arithmetic progressions (extending those of the authors in §9.2 and §9.3 of Chapter 9 of [RB2]). Discussion in the last chapter (Chp. 12) of a number of new topics, including recent results on random fractal strings, quantized strings (fractal membranes), and especially on the complex dimensions of the Koch snowflake curve and its generalizations (via an associated 'tube formula'), as well as of an outline of a higher-dimensional theory of complex dimensions of self-similar systems and fractals. Also, further discussion of a possible cohomological interpretation of the complex dimensions. New appendix (App. A) on Nevanlinna theory and its applications in this context, and a new section (§B.4 of App. B) on 'two-variable zeta functions'. New examples, illustrations, theorems, and proofs, scattered throughout the book.]

Table of Contents of the Book [RB3]:

Generalized Fractal String. *4.2.1: Completion of the harmonic string: Euler product.* 4.3: Generalized Fractal Sprays. 4.4: The Measure of a Self-Similar String. *4.4.1: Measures with a self-similarity property.* 4.5: Notes.

Chp. 5: Explicit Formulas for Generalized Fractal Strings. 5.1: Introduction. *5.1.1: Outline of the proof. 5.1.2: Examples.* 5.2: Preliminaries: The Heaviside Function. 5.3: Pointwise Explicit Formulas. *5.3.1: The order of the sum over the complex dimensions.* 5.4: Distributional Explicit Formulas. *5.4.1: Extension to more general test functions. 5.4.2: The order of the distributional error term.* 5.5: Example: The Prime Number Theorem. *5.5.1: The Riemann-von Mangoldt formula.* 5.6: Notes.

Chp. 6: The Geometry and the Spectrum of Fractal Strings. 6.1: The Local Terms in the Explicit Formulas. *6.1.1: The geometric local terms. 6.1.2: The spectral local terms. 6.1.3: The Weyl term. 6.1.4: The distribution* x^{ω} $(\log x)^{m}$. 6.2: Explicit Formulas for Lengths and Frequencies. *6.2.1: The geometric counting function of a fractal string. 6.2.2: The spectral counting function of a fractal string. 6.2.3: The geometric and spectral partition functions.* 6.3: The Direct Spectral Problem for Fractal Strings. *6.3.1: The density of geometric and spectral states. 6.3.2: The spectral operator and its Euler product.* 6.4: Self-Similar Strings. *6.4.1: Lattice strings. 6.4.2: Nonlattice strings. 6.4.3: The spectrum of a self-similar string.* 6.5: Examples of Non-Self-Similar Strings. *6.5.1: The a-string. 6.5.2: The spectrum of the harmonic string.* 6.6: Fractal Sprays. *6.6.1: The Sierpinski drum. 6.6.2: The spectrum of a self-similar spray.*

Chp. 7: Periodic Orbits of Self-Similar Flows. 7.1: Suspended Flows. *7.1.1: The zeta function of a dynamical system.* 7.2: Periodic Orbits, Euler Product. 7.3: Self-Similar Flows. *7.3.1: Examples of self-similar flows. 7.3.2: The lattice and nonlattice case.* 7.4: The Prime Orbit Theorem for Suspended Flows. *7.4.1: The prime orbit theorem for self-similar flows. 7.4.2: Lattice flows 7.4.3: Nonlattice flows.* 7.5: The Error Term in the Nonlattice Case. *7.5.1: Two generators. 7.5.2: More than two generators.* 7.6: Notes.

Chp. 8: Tubular Neighborhoods and Minkowski Measurability. 8.1: Explicit Formulas for the Volume of Tubular Neighborhoods. *8.1.1: The pointwise tube formula. 8.1.2: Example: the a-string.* 8.2: Analogy with Riemannian Geometry. 8.3: Minkowski Measurability and Complex Dimensions. 8.4: Tube Formulas for Self-Similar Strings. *8.4.1: Generalized Cantor strings. 8.4.2: Lattice self-similar strings. 8.4.3: The average Minkowski content. 8.4.4: Nonlattice self-similar strings.*

Chp. 9: The Riemann Hypothesis and Inverse Spectral Problems. 9.1: The Inverse Spectral Problem. 9.2: Complex Dimensions of Fractal Strings and the Riemann Hypothesis. 9.3: Fractal Sprays and the Generalized Riemann Hypothesis. 9.4: Notes.

Chp. 10: Generalized Cantor Strings and Their Oscillations. 10.1: The Geometry of a Generalized Cantor String. 10.2: The Spectrum of a Generalized Cantor String. *10.2.1: Integral Cantor strings: a-adic analysis of the geometric and spectral oscillations. 10.2.2: Nonintegral Cantor strings: analysis of the jumps in the spectral counting function.* 10.3:

[RB4] "*In Search of the Riemann Zeros*. (Subtitle: *Strings, Fractal Membranes and Noncommutative Spacetimes*".) American Mathematical Society, Providence, R I, 2008, 600 pages (precisely, 558+(xxix) pp.), January, 2008. ISBN-10: 0-8218-422-5. ISBN-13:978-0-8218-4222-5. US Library of Congress Classification: QA333.L37 2007.

[The (physically motivated) theory proposed and developed in this research monograph represents approximately ten years of the author's research on this subject (between about 1996 and 2006). It realizes a synthesis of aspects of string theory, noncommutative geometry, the author (and his collaborators)' theory of fractal strings (now 'quantized' and referred to in this new form as 'fractal membranes') and their complex dimensions, as well as of number theory and arithmetic geometry. In particular, it builds upon and expands—but is also in many ways quite different from—the author's earlier (joint) work in [JA24-27] or in [RB2, RB3]. Much of the material presented in the main part of this book (with the exception of a portion of Chapter 2 and the first section of Chapter 3) is original and published for the first time.]

Table of Contents of the Book [RB4]:

Chp. 4: Noncommutative Models of Fractal Strings: Fractal Membranes and Beyond. 4.1: Connes' Spectral Triple for Fractal Strings. 4.2: Fractal Membranes and the Second Quantization of Fractal Strings. *4.2.1: An alternative construction of fractal membranes.* 4.3: Fractal Membranes and Noncommutative Stringy Spacetimes. 4.4: Cyclic Cohomology and a Possible Interpretation of (Dynamical) Complex Dimensions. *4.4.1: Fractal membranes and quantum deformations: a possible connection with Haran's real and finite primes.* 4.5: Notes.

Chp. 5: Towards an 'Arithmetic Site': Moduli Spaces of Fractal Strings and Membranes. 5.1: The Set of Penrose Tilings: Quantum Space as a Quotient Space. 5.2: The Moduli Space of Fractal Strings: A Natural Receptacle for Zeta Functions. 5.3: The Moduli Space of Fractal Membranes: A Quantized Moduli Space of Fractal Strings. 5.4: Arithmetic Site, Frobenius Flow and the Riemann Hypothesis. *5.4.1: The moduli space of fractal strings and Deninger's arithmetic site. 5.4.2: The moduli space of fractal membranes and (noncommutative) modular flow vs. arithmetic site and Frobenius flow.* (5.4.2a: Factors and their classification. 5.4.2b: Modular theory of von Neumann algebras. 5.4.2c: Reduction of type III to type II factors: automorphisms and flows. 5.4.2d: Noncommutative flows on moduli spaces and the Riemann hypothesis. 5.4.2e: Towards an extended moduli space and flow.) 5.5: Flows of Zeros and Zeta Functions: A Dynamical Interpretation of the Riemann Hypothesis. 5.5.1. *Introduction.* 5.5.2. *Expected Properties of the Flows of Zeros and Zeta Functions.* 5.5.3. *Analogies with Other Geometric, Analytical or Physical Flows.* (5.5.3a. Singular Potentials, Schrödinger Equation and Renormalization Flow. 5.5.3b. KMS Flow and Deformations of Pólya-Hilbert Operators. 5.5.3c. Ricci Flow and Geometric Homogenization. 5.5.3d. Noncommutative KP and Geodesic Flow.) 5.6: Notes.

Appendix A: Vertex Algebras. A.1: Definition of Vertex Algebras: Translation and Scaling Operators. A.2: Basic Properties of Vertex Algebras. A.3: Notes.

Appendix B: The Weil Conjectures and the Riemann Hypothesis. B.1: Varieties Over Finite Fields and Their Zeta Functions. B.2: Zeta Functions of Curves Over Finite Fields and the Riemann Hypothesis. B.3: The Weil Conjectures for Varieties Over Finite Fields. B.4: Notes.

Appendix C: The Poisson Summation Formula, with Applications. C.1: General PSF for Dual Lattices: Scalar Identity and Distributional Form. C.2: Application: Modularity of Theta Functions. C.3: Key Special Case: Self-Dual PSF. C.4: Proof of the General Poisson Summation Formula. C.5: Modular Forms and Their Hecke L-Series. *C.5.1: Modular forms and cusp forms. C.5.2: Hecke operators and Hecke forms. C.5.3: Hecke L-series of a modular form. C.5.4: Modular forms of higher level and their L-functions.* C.6: Notes.

Appendix D: Generalized Primes and Beurling Zeta Functions. D.1: Generalized Primes Π and Integers N. D.2: Beurling Zeta Functions ς_Π. D.3: Analogues of the Prime Number Theorem. D.4: Analytic Continuation and a Generalized Functional Equation for ς_Π. D.5:

Partial Orderings on Generalized Integers. D.6: Notes.

Appendix E: The Selberg Class of Zeta Functions. E.1: Definition of the Selberg Class. E.2: The Selberg Conjectures. E.3: Selected Consequences. E.4: The Selberg Class, Artin L-Series and Automorphic L-Functions: Langlands' Reciprocity Laws. *E.4.1: Selberg's orthonormality conjecture and Artin L-series: Artin's holomorphy conjecture. E.4.2: Selberg's orthonormality conjecture and automorphic representations: Langlands' reciprocity conjecture.* (E.4.2a: Adeles k_A and the linear group $GL(n;k_A)$. E.4.2b: Automorphic representations and automorphic L-series.) E.5: Notes.

Appendix F: The Noncommutative Space of Penrose Tilings and Quasicrystals.
F.1: Combinatorial Coding of Penrose Tilings and Consequences. F.2: Groupoid C*-Algebra and the Noncommutative Space of Penrose Tilings. *F.2.1: Groupoids: definition and examples. F.2.2: The Groupoid convolution algebra. F.2.3: Generalization: groupoids, quasicrystals and noncommutative spaces.* F.3: Quasicrystals: Dynamical Hull and Noncommutative Brillouin Zone. *F.3.1: Mathematical quasicrystals and their generalizations. F.3.2: Translation dynamical system: the hull of a quasicrystal. F.3.3: Typical properties of atomic configurations. F.3.4: The noncommutative Brillouin zone (and groupoid C*-algebra) of a quasicrystal.* F.4: Notes.

Acknowledgements. Bibliography.

Conventions. Index of Symbols. Author Index. Subject Index.

Refereed Edited Research Books:

[EB1] ***<u>Editor:</u>*** *"Progress in Inverse Spectral Geometry"*, Proc. Summer School on "Progress in Inverse Spectral Geometry", held in Stockholm (Sweden) in June-July 1994, *Trends in Mathematics Series*, Vol. 1, Birkhäuser-Verlag, Basel, November 1997, 204 pages, (Co-Editor: Stig I. Andersson). ISBN 3-7643-5755-X (Basel). ISBN 0-8176-5755-X (Boston). US Library of Congress Classification: QA614.95.P78 1997.

[This volume gathers original research contributions or survey expository articles by some of the best experts in spectral geometry, along with a few papers by promising junior investigators selected by the editors. With two exceptions, the editors have reviewed and edited each paper individually.]

[EB2] ***<u>Coordinating Editor:</u>*** *"Harmonic Analysis and Nonlinear Differential Equations"*, (Volume in honor of Prof. Victor L. Shapiro), *Contemporary Mathematics*, Vol. 208, American Mathematical Society, Providence, RI, August 1997, 350 + (xii) pages. (Includes a Dedication and a Preface.) (Co-Editors: Lawrence H. Harper and Adolfo J. Rumbos.) ISBN 0-8218-0565-7 (alk. paper). US Library of Congress Classification: QA403.H223 1997.

[This volume gathers high quality original research contributions or survey expository articles by some of the leading experts in classical and modern analysis, working in the areas of harmonic analysis, nonlinear partial differential and mathematical physics. Each contribution in this volume has been individually refereed according to strict standards set by the American Mathematical Society.]

[EB3] ***Editor:*** *"Dynamical, Spectral and Arithmetic Zeta Functions"*, Proc. Special Session (Amer. Math. Soc. Annual Meeting, San Antonio, Texas, Jan. 1999), *Contemporary Mathematics*, Vol. 290, American Mathematical Society, Providence, RI, December 2001, 195 + (x) pages. (Co-Editor: Machiel van Frankenhuysen.) ISBN 0-8218-2079-6 (alk. paper). US Library of Congress Classification: QA351.A73 1999.

[This volume gathers high quality original research contributions or survey expository articles by some of the leading experts working at the interface of number theory, dynamical systems and/or spectral as well as arithmetic geometry. Each contribution in the volume has been individually refereed according to strict standards set by the American Mathematical Society.]

[EB4] & [EB5]:

Managing Editor: "*Fractal Geometry and Applications*". (Subtitle: "*A Jubilee of Benoît Mandelbrot*".) [Two volumes (Parts 1 & 2), totaling approximately 1,100 pages (precisely, 1,080 + (xxvi) pages).] *Proceedings of Symposia in Pure Mathematics* (PSPUM), American Mathematical Society, Vol. 72, Parts 1 & 2, Providence, RI, Dec. 2004. (Co-Editor: Machiel van Frankenhuijsen.) ISBN 0-8218-3292-1 (set: acid free paper). ISBN 0-8218-3637-4 (part 1: acid free paper). ISBN 0-8218-3638-2 (part 2: acid free paper). US Library of Congress Classification: QA325.F73 2004.

Subtitle of Part 1: *Analysis, Number Theory, and Dynamical Systems*. (Approx. 520 pages; precisely, 508 + (xiii) pages.)

Subtitle of Part 2: *Multifractals, Probability and Statistical Mechanics, Applications*. (Approx. 560 pages; precisely, 546 + (xiii) pages.)

[The PSPUM Series is the most prestigious proceedings series published by the American Mathematical Society. Very few volumes are published every year (or decade). (The preparation of the present two-part volume has taken about three years.) In part for this reason, as the Managing Editor of the volume (i.e., of both Parts 1 and 2), I have devoted a great deal of attention and care to the selection of the invited contributors, the refereeing process (drawing upon more than forty expert referees), and the editing of the volume.. This has been an extremely time-consuming task for me, spanning over hundreds of hours, but one that I think will be ultimately worthwhile and useful to the mathematical as well as the broader scientific community.

The goal of these two books is to give an overview of the field of fractal geometry and of its applications [within mathematics (e.g., harmonic analysis, dynamical systems, number theory, probability, and mathematical physics) as well as to the other sciences (e.g., physics, chemistry, engineering, and computer graphics)], via a careful selection of research expository articles, tutorial articles, and original research papers. It should be accessible and useful to experts and non-experts alike.]

Books in Preparation:

[RB5] **Research Monograph:** *"Feynman's Operational Calculus and Beyond"*. (Subtitle: *"Noncommutativity and Time-Ordering"*.) (With Gerald W. Johnson and Lance Nielsen.)

[RB6] **Research Monograph**: *"Fractal Geometry, Complex Dimensions and Zeta Functions*". (Subtitle: "Geometry and Spectra of Fractal Strings".) Second Edition (Revised and Enlarged Second Edition), Springer Monographs in Mathematics, Springer-Verlag, New York; expected publication date: 2010. (With Machiel van Frankenhuijsen.) [Second edition written upon the request of the publisher, Springer-Verlag. Contract signed with Springer-Verlag on 03/21/2009.)

[TB] **Textbook:** "*An Invitation to Fractal Geometry*". (Subtitle: "*Dimension Theory, Zeta Functions, and Applications*".) (With Robert G. Niemeyer and John A. Rock.)

Theses:

[T1] "*Généralisation de la Formule de Trotter-Lie. Étude de Quelques Problèmes Liés à des Groupes Unitaires"*. (Generalisation of the Trotter-Lie Formula. Study of Several Problems Connected with Unitary Groups.) Thèse de Doctorat de Troisième Cycle (Ph.D. Dissertation). Mathématiques. Université Pierre et Marie Curie (Paris VI), Paris, France, 93 + (iii) pages, 1980.

[T2] *"Formule de Trotter et Calcul Opérationnel de Feynman"*. (Trotter's Formula and Feynman's Operational Calculus.) Thèse de Doctorat D'Etat ès Sciences. Mathématiques. Université Pierre et Marie Curie (Paris VI), Paris, France, 360 + (iv) pages, 1986. [Doctorate ès Sciences Dissertation. Counterpart of a German Habilitation, well beyond a regular Ph.D. It used to be needed to become the analog of a Full Professor at the University in France. It was refereed by anonymous experts and by a University of Paris central commission of specialists composed of well-known mathematicians. The 'Thèse d'Etat' has now been replaced by "l'Habilitation to Direct Research", which the author has also defended at the Université Paris VI (in 1987).]

[Part I: Formules de Trotter et Intégrales de Feynman. Part II: Problèmes aux

Valeurs Propres Elliptiques avec un Poids Indéfini. Part III: Calcul Opérationnel de Feynman. (Part I: Trotter Formulas and Feynman Integrals. Part II: Elliptic Eigenvalue Problems with an Indefinite Weight. Part III: Feynman's Operational Calculus.)]