Representations of modular skew group algebras

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Notation

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- The *skew group algebra* $AG = A \otimes_k kG$ as vector spaces, with multiplication determined by:

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- Examples: regular group algebras, algebra of matrices, etc.

Motivation

When $|G|$ is invertible, it has been shown that AG and A share many properties. For instance:

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3. (Dionne, etc) AG is *piecewise hereditary* if so is A.
4. (Martinez) If A is graded and the action of G preserves grading, AG is *Koszul* if and only if so is A.

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- **Question:** For arbitrary G, under what conditions do A and AG still share these properties?

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- We obtain two functors:

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  They are exact and preserve projective modules.
- **Proposition:** If AG has a property specified in (1-3) [1] above, so does AH (in particular, A). The converse statement is also true if $|G : H|$ is invertible (or equivalently, H contains a Sylow p-subgroup of G).

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- **Proposition:** If $\operatorname{gldim} AS < \infty$, then the action of S on E is free.
- **Proposition:** Suppose that $p \geqslant 5$ and A is a finite dimensional algebra which is not local. If AS is of finite representation type, then the action of S on E is free.

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- A must have the same property; and
- the action of S on E must be free.
- **Question:** Are these two conditions sufficient?

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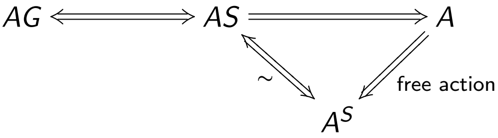
- Consequently, AS and A^S are Morita equivalent.
- **Proposition:** If A has a property specified in (1-3), so does A^S.
- **Answer:** Yes!



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2. If the action of S on E is free, then AG, AS, A^S, and A have the same global dimension, *finitistic dimension*, and *strong global dimension*.
3. If $p \geqslant 5$ and A is a finite dimensional algebra which is not local. Then AG is of finite representation type if and only if so is A, and S acts freely on E.

Transporter categories

- Let $\mathcal{P}$ be a finite connected poset on which every element in G acts as an automorphism. The Grothendick construction $\mathcal{T} = G \propto \mathcal{P}$ is called a *transporter category*.

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- The category algebra $k\mathcal{T}$ is a skew group algebra.
- **Theorem:** For $p \geqslant 5$, $k\mathcal{T}$ is of finite representation type if and only if $\mathcal{P}$ is trivial and kG is of finite representation type, or kG is semisimple and the incidence algebra $k\mathcal{P}$ is of finite representation type.

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- **Theorem (Happel & Zacharia):** A is piecewise hereditary if and only if it has finite strong global dimension.
- **Theorem:** AG is piecewise hereditary if and only if so is A, and the action of S on E is free.

References

1. L. Li, *Representations of modular skew group algebras*, to appear in Tran. Amer. Math. Soc.
2. L. Li, *Piecewise hereditary skew group algebras*, preprint.