Noetherian Property of Infinite EI Categories

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- $\text{End}_{\mathcal{C}}([n])$ is precisely $S_n$. 

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- Used to study representations of all symmetric groups simultaneously, in particular the representation stability, and asymptotic behavior such as polynomial growth.

Theorem (CEF):
FI is locally Noetherian over any field of characteristic 0; that is, sub-representations of finitely generated representations are still finitely generated.

The above result (proved by representations of symmetric groups and branching rule) is shown to be true over arbitrary (left) Noetherian rings by CEFN, using a completely different combinatorial approach.
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- $\text{FI}_W$, where $W$ is a Weyl group.
EI categories

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All above categories are examples of *locally finite EI categories* of type $A_\infty$, which are small categories such that every endomorphism is invertible and satisfy:

- for every pair $x, y \in \text{Ob} \ C$, $|C(x, y)|$ is finite;
- objects are indexed by $\mathbb{N} \cup \{0\}$, and $C(j, s) \circ C(i, j) = C(i, s)$. 

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Noetherian Property of Infinite EI Categories
Let $C$ be a small EI category. A $k$-linear representation of $C$ is a covariant functor from $C$ to the category of vector spaces over $k$. The category algebra $kC$ is the vector space spanned by all morphisms and equipped with a multiplication induced by composition of morphisms. A representation of $C$ is precisely a $kC$-module. The category algebra is never Noetherian when $\text{Ob } C$ is infinite; but $kC$-mod can still be abelian, and this happens if and only if $kC$ is locally Noetherian.
Representations and category algebras

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- **Transitivity:** each $G_j = \mathcal{C}(j,j)$ acts transitively on $\mathcal{C}(j - 1, j)$. 
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- **Transitivity:** each \( G_j = C(j, j) \) acts transitively on \( C(j - 1, j) \).
- It turns out that \( G_j \) acts transitively on \( C(i, j) \) for all \( i \leq j \).
- Take a representative \( \alpha_j \in C(j, j + 1) \) for each \( j \). Then \( C \) can be depicted as

\[
G_0 \xrightarrow{\alpha_0} G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \xrightarrow{\alpha_3} \ldots
\]
For $j > i$, define $\alpha_{i,j} = \alpha_{j-1} \circ \ldots \circ \alpha_i$, which belongs to $C(i,j)$. 

$\text{Motivation}$

$\text{Our Project}$

$\text{Further remarks}$

Stabilizers
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- For $j > i$, define $\alpha_{i,j} = \alpha_{j-1} \circ \ldots \circ \alpha_{i}$, which belongs to $C(i,j)$.
- Define $H_{i,j} = \text{Stab}_{G_j}(\alpha_{i,j})$. 
Stabilizers

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- Define $H_{i,j} = \text{Stab}_{G_j}(\alpha_{i,j})$.
- Since $G_j$ acts transitively on $C(i,j)$, one has $C(i,j) \cong G_j / H_{i,j}$. 
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Define $H_{i,j} = \text{Stab}_{G_j}(\alpha_{i,j})$.

Since $G_j$ acts transitively on $C(i,j)$, one has $C(i,j) \cong G_j/H_{i,j}$.

The composition with $\alpha_{j+1}$ gives a map $C(i,j) \to C(i,j+1)$, which induces a map

$$\phi_{i,j} : H_{i,j} \backslash G_j / H_{i,j} \to H_{i,j+1} \backslash G_{j+1} / H_{i,j+1}.$$
Theorem (G-L): Let $C$ be a locally finite EI category of type $A_\infty$. If $C$ satisfies the transitivity condition and the following bijectivity condition:

**Bijectivity:** For $i \geq 0$ and $j \gg i$, $\phi_{i,j}$ is bijective.

then the category algebra $kC$ is locally Noetherian, where $k$ is a field of characteristic 0.
Main result

- **Theorem (G-L):** Let $\mathcal{C}$ be a locally finite EI category of type $A_\infty$. If $\mathcal{C}$ satisfies the transitivity condition and the following bijectivity condition:
  - **Bijectivity:** For $i \geq 0$ and $j \gg i$, $\phi_{i,j}$ is bijective.
  
  then the category algebra $k\mathcal{C}$ is locally Noetherian, where $k$ is a field of characteristic 0.

- This theorem applies to $\text{FI}$, $\text{FI}_q$, and $\text{FI}_W$. 
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Putman, Sam, Snowden have shown that $\mathcal{FI}_q$ and some other categories are locally Noetherian over arbitrary Noetherian rings.
Noetherian property over Noetherian rings

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- Putman, Sam, Snowden have shown that $\mathcal{FI}_q$ and some other categories are locally Noetherian over arbitrary Noetherian rings.
- Can our results extend to arbitrary Noetherian rings?
Koszul property

- FI and FI_q have a natural grading. Therefore, they are graded categories.
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- FI and FI have a non-trivial self-embedding $\mathcal{D} \rightarrow \mathcal{C}$ with $\mathcal{D} \cong \mathcal{C}$, which gives rise to a degree shift functor.
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- With respect to this self-embedding, $k\mathcal{C} \downarrow_{\mathcal{D}}^\mathcal{C}$ has a special decomposition.
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- FI and FI have a non-trivial self-embedding \( \mathcal{D} \to \mathcal{C} \) with \( \mathcal{D} \cong \mathcal{C} \), which gives rise to a degree shift functor.
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- **Theorem** (G-L): A certain combinatorial condition guarantees the previous properties. In particular, categories satisfying this condition are Koszul over fields with characteristic 0.
Koszul property

- **FI** and **FI**\(_q\) have a natural grading. Therefore, they are graded categories.
- **FI** and **FI** have a non-trivial self-embedding \(D \to C\) with \(D \cong C\), which gives rise to a degree shift functor.
- With respect to this self-embedding, \(kC \downarrow^C_D\) has a special decomposition.

**Theorem** (G-L): A certain combinatorial condition guarantees the previous properties. In particular, categories satisfying this condition are Koszul over fields with characteristic 0.

**Corollary**: Every EI category having the same objects and non-invertible morphisms as **FI** or **FI**\(_q\) is Koszul over a field of characteristic 0.