GENERATORS AND DEGREES OF TORUS KNOTS IN $\mathbb{C}P^2$

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ABSTRACT
The $\mathbb{C}P^2$-genus of a knot $K$ is the minimal genus over all isotopy classes of smooth, compact, connected and oriented surfaces properly embedded in $\mathbb{C}P^2 - B^4$ with boundary $K$. We compute the $\mathbb{C}P^2$-genus and realizable degrees of $(-2,q)$-torus knots for $3 \leq q \leq 11$ and $(2,q)$-torus knots for $3 \leq q \leq 17$. The proofs use gauge theory and twisting operations on knots.

Keywords: Smooth genus; $\mathbb{C}P^2$-genus; twisting; blow-up.

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1. Introduction
Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. A knot is a smooth embedding of $S^1$ into the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\pm \infty\}$. All knots are oriented. Let $K$ be a knot in $\partial(\mathbb{C}P^2 - B^4) \cong S^3$, where $B^4$ is an embedded open 4-ball in $\mathbb{C}P^2$. The $\mathbb{C}P^2$-genus of a knot $K$, denoted by $g_{\mathbb{C}P^2}(K)$, is the minimal genus over all isotopy classes of smooth, compact, connected and oriented surfaces properly embedded in $\mathbb{C}P^2 - B^4$ with boundary $K$. If $K$ bounds a properly embedded 2-disk in $\mathbb{C}P^2 - B^4$, then $K$ is called a slice knot in $\mathbb{C}P^2$. A similar definition could be made for any 4-manifold and that this is a generalization of the 4-ball genus.

Recall that $\mathbb{C}P^2$ is the closed 4-manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0,0,0)\}$ defined by $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$ where $\lambda \in \mathbb{C}^*$, i.e. $\mathbb{C}P^2 = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^*$. An element of $\mathbb{C}P^2$ is denoted by its homogeneous coordinates $[x : y : z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{C}P^2 | x = 0\}$ ($H \cong \mathbb{C}P^1$) generates the second homology group $H_2(\mathbb{C}P^2; \mathbb{Z})$ (see Gompf and Stipsicz [12]). Since $H \cong \mathbb{C}P^1$, then the standard generator of $H_2(\mathbb{C}P^2; \mathbb{Z})$ is denoted, from
Given a realizable degree, is the underlying surface $\Sigma$ properly embedded in $\mathbb{C}P^2 - B^4$ with the complex orientations.

A class $\xi \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z}) \cong H_2(\mathbb{C}P^2 - B^4; \mathbb{Z}) \to H_2(\mathbb{C}P^2; \mathbb{Z}).$$

Let $d$ be an integer, then the degree-$d$ smooth slice genus of a knot $K$ in $\mathbb{C}P^2$ is the least integer $g$ such that $K$ is the boundary of a smooth, compact and orientable genus $g$ surface $\Sigma_g$ properly embedded in $\mathbb{C}P^2 - B^4$ with boundary $K$ in $\partial(\mathbb{C}P^2 - B^4)$ and degree $d$, i.e.

$$[\Sigma_g, \partial \Sigma_g] = d \gamma \in H_2(\mathbb{C}P^2 - B^4, \partial(\mathbb{C}P^2 - B^4); \mathbb{Z}).$$

By the above identification, we also have: $[\Sigma_g] = d \gamma \in H_2(\mathbb{C}P^2 - B^4, \mathbb{Z})$. If such a surface can be given explicitly, then we say that the degree $d$ is realizable. The $\mathbb{C}P^2$-genus of a knot $K$, denoted by $g_{\mathbb{C}P^2}(K)$, is the minimum over these over all $d$.

**Question 1.1.** Given a realizable degree, is the underlying surface $\Sigma_g$ unique, up to isotopy?

An interesting question is to find the $\mathbb{C}P^2$-genus and the realizable degree(s) of knots in $\mathbb{C}P^2$. In this paper, we compute the $\mathbb{C}P^2$-genus and realizable degrees of a finite collection of torus knots.

**Theorem 1.1.**

1. $g_{\mathbb{C}P^2}(T(-2,3)) = 0$ with realizable degree $d \in \{\pm 2, \pm 3\}$.
2. $g_{\mathbb{C}P^2}(T(-2,q)) = 0$ for $q = 5, 7$ and $9$ with respective realizable degrees $\pm 3, \pm 4$ and $\pm 4$.
3. $g_{\mathbb{C}P^2}(T(-2,11)) = 1$ with possible degree(s) $d \in \{\pm 4, \pm 5\}$.

Note that for any $0 < p < q$, $T(p,q)$ is obtained from $T(2,3)$ by adding $(p-1)(q-1) - 2$ half-twisted bands. Then, there is a genus $\frac{(p-1)(q-1)-2}{2}$ cobordism between $T(2,3)$ and $T(p,q)$. We conjecture that the $\mathbb{C}P^2$-genus of a $(p,q)$-torus knot is equal to the genus of the cobordism between $T(2,3)$ and $T(p,q)$.

**Conjecture 1.1.** $g_{\mathbb{C}P^2}(T(2,q)) = \frac{(p-1)(q-1)-1}{2}$.

We answer this conjecture by the positive for all $(2,q)$-torus knots with $3 \leq q \leq 17$.

**Theorem 1.2.**

1. $g_{\mathbb{C}P^2}(T(2,3)) = 0$ with realizable degree $d = 0$.
2. $g_{\mathbb{C}P^2}(T(2,q)) = \frac{q-7}{2}$ for $5 \leq q \leq 17$ with respective possible degree(s)
   - $d \in \{0, \pm 1\}$ if $q \in \{5, 7, 9, 11\}$, and
   - $d \in \{0, \pm 1, \pm 3\}$ if $q \in \{13, 15, 17\}$. 
2. Twisting Operations and Sliceness in 4-Manifolds

Let \( K \) be a knot in the 3-sphere \( S^3 \), and \( D^2 \) a disk intersecting \( K \) in its interior. Let \( n \) be an integer. A \(-\frac{1}{n}\)-Dehn surgery along \( C = \partial D^2 \) changes \( K \) into a new knot \( K_n \) in \( S^3 \). Let \( \omega = \text{lk}(\partial D^2, L) \). We say that \( K_n \) is obtained from \( K \) by \((n, \omega)\)-twisting (or simply twisting). Then, we write \( K \xrightarrow{(n, \omega)} K_n \), or \( K \xrightarrow{(n, \omega)} (n, \omega) \). We say that \( K_n \) is \( n \)-twisted provided that \( K \) is the unknot (see Fig. 1).

An easy example is depicted in Fig. 2, where we show that the right-handed trefoil \( T(2, 3) \) is obtained from the unknot \( T(2, 1) \) by a \((+1, 2)\)-twisting. (In this case \( n = +1 \) and \( \omega = +2 \).)

There is a connection between twisting of knots in \( S^3 \) and dimension four: Any knot \( K_{n-1} \) obtained from the unknot \( K \) (or more generally, a smooth slice knot in the 4-ball) by a \((-1, \omega)\)-twisting is smoothly slice in \( \mathbb{C}P^2 \) with degree \( \omega \) realizable by the twisting disk \( \Delta \), i.e. there exists a properly embedded smooth disk \( \Delta \subset \mathbb{C}P^2 - B^4 \) such that \( \partial \Delta = K_{n-1} \) and \([\Delta] = \omega \gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})\). For convenience of the reader, we give a sketch of a proof due to Miyazaki and Yasuhara [21]: We assume \( K \cup C \subset \partial h^0 \cong S^3 \), where \( h^0 \) denotes the 4-dimensional 0-handle (\( h^0 \cong B^4 \)). The unknot \( K \) bounds a properly embedded smooth disk \( \Delta \) in \( h^0 \). Then, performing a \((-1)\)-twisting is equivalent to adding a 2-handle \( h^2 \), to \( h^0 \) along \( C \) with framing \(+1\). It is known that the resulting 4-manifold \( h^0 \cup h^2 \) is \( \mathbb{C}P^2 - B^4 \) (see Kirby [18] for example). In addition, it is easy to verify that \([\Delta] = \omega \gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})\).

More generally, we can prove, using Kirby calculus [18] and some twisting manipulations, that an \((n, \omega)\)-twisted knot in \( S^3 \) bounds a properly embedded smooth disk \( \Delta \) in a punctured standard four manifold of the form \( n\mathbb{C}P^2 - B^4 \) if \( n > 0 \) (see Fig. 3), or \( |n|\mathbb{C}P^2 - B^4 \) if \( n < 0 \). The second homology of \([\Delta]\) can be computed from \( n \) and \( \omega \).

\[
\omega = \text{lk}(K, C) \quad (\omega = 0)
\]

\(-\frac{1}{n}\)–Dehn surgery along \( C \)

\begin{center}
\begin{tikzpicture}
  \node at (0,0) (K) {\( K \)};
  \node at (1,-1) (C) {\( C \)};
  \draw[->] (K) -- (C);
  \draw[->] (K) -- (K+2,0);
  \draw[->] (K) -- (K+2,-1);
  \draw[->] (C) -- (C+1,0);
  \draw[->] (C) -- (C+1,-1);
  \draw[->] (K) .. controls +(-1,1) and +(-1,-1) .. (K);
  \draw[->] (C) .. controls +(1,1) and +(1,-1) .. (C);
  \draw (K) node[below] {\( (n, \omega)\)-twisting};
  \draw (C) node[below] {\( n\)-full twists};
  \draw (K+2,0) node[above] {\( K_n \)};
  \draw (C+1,0) node[below] {\( \partial \Delta = K_{n-1} \)};
  \draw (K) node[below] {\( \omega = \text{lk}(K, C) \)};
\end{tikzpicture}
\end{center}

Fig. 1.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) (T21) {\( T(2, 1) \)};
  \node at (1,0) (C) {\( C \)};
  \draw[->] (T21) -- (T21+1,0);
  \draw[->] (T21) -- (T21+1,-1);
  \draw[->] (C) -- (C+1,0);
  \draw[->] (C) -- (C+1,-1);
  \draw (T21) node[below] {\( (+1, 2)\)-twisting along \( C \)};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node at (0,0) (T23) {\( T(2, 3) \)};
  \node at (1,0) (C) {\( C \)};
  \draw[->] (T23) -- (T23+1,0);
  \draw[->] (T23) -- (T23+1,-1);
  \draw[->] (C) -- (C+1,0);
  \draw[->] (C) -- (C+1,-1);
  \draw (T23) node[below] {\( (+1, 2)\)-twisting along \( C \)};
\end{tikzpicture}
\end{center}

Fig. 2.
Examples.

(1) Song and Goda and Hayashi proved in [11] that $T(p, p + 2)$ (for any $p \geq 5$) is obtained by a single $(+1)$-twisting along an unknot. This implies that their corresponding left-handed torus knots are smoothly slice in $\mathbb{C}P^2$ (see [2]). In [5], we proved that the realizable degree of $T(-p, p + 2)$ in $\mathbb{C}P^2$ is $p + 1$ (for any $p \geq 5$).

(2) Any unknotting number one knot is $(−1)$-twisted (see Fig. 4), and then it is smoothly slice in $\mathbb{C}P^2$. In particular, the double of any knot is smoothly slice in $\mathbb{C}P^2$.

**Question 2.1.** Is there a knot which is topologically but not smoothly slice in $\mathbb{C}P^2$?

The proof of Theorem 2.1 can be found in [4]:

**Theorem 2.1.** If a knot $K$ is obtained by a single $(n, \omega)$-twisting from an unknot $K_0$ along $C$, then its inverse $-K$ is obtained by a single $(n, -\omega)$-twisting from the unknot $-K_0$ along $C$.

Note that $T(-p, 4p \pm 1)$ ($p \geq 2$) is obtained from the unknot $T(-1, 4p \pm 1)$ by a $(−1, 2p)$-twisting (see Fig. 5). Therefore, Theorem 2.2 is deduced from Kirby calculus.

**Theorem 2.2.** $T(-p, 4p \pm 1)$ ($p \geq 2$) is smoothly slice in $\mathbb{C}P^2$ with realizable degree $d = 2p$. 
We refer the reader to my Ph.D thesis [2] for more details on twisting operations on knots in $S^3$.

3. Preliminaries

Litherland gave an algorithm to compute the $x$-signatures of torus knots.

**Theorem 3.1 (Litherland [20]).** Let $\xi = e^{2i\pi x}$, $x \in \mathbb{Q}$ (with $0 < x < 1$), then

$$\sigma_\xi(T(p, q)) = \sigma_{\xi^+} - \sigma_{\xi^-},$$

$$\sigma_{\xi^+} = \# \left\{ (i, j) \middle| 1 \leq i \leq p-1 \quad \text{and} \quad 1 \leq j \leq q-1 \quad \text{such that} \quad x - 1 < \frac{i}{p} + \frac{j}{q} < x \quad \text{(mod 2)} \right\},$$

$$\sigma_{\xi^-} = \# \left\{ (i, j) \middle| 1 \leq i \leq p-1 \quad \text{and} \quad 1 \leq j \leq q-1 \quad \text{such that} \quad x < \frac{i}{p} + \frac{j}{q} < x + 1 \quad \text{(mod 2)} \right\}$$

($i$ and $j$ are integers)

If $y_{i,j} = \frac{i}{p} + \frac{j}{q}$, then $x - 1 < y_{i,j} < x$ (mod 2) is equivalent to

$$0 < y_{i,j} < x \quad \text{or} \quad x + 1 < y_{i,j} < 2.$$

The signature of a knot is $\sigma(k) = \sigma_{-1}(k)$ obtained by assigning $x = \frac{1}{2}$ and the Tristram $d$-signature ($d \geq 3$ and prime) corresponds to $x = \frac{d-1}{2d}$ which we denote by $\sigma_d(k) = \sigma_{e^{\frac{i\pi}{d}}}$. (Tristram [24]).

In the following, $b_2^+(X)$ (respectively, $b_2^-(X)$) is the rank of the positive (respectively, negative) part of the intersection form of the oriented, smooth and compact 4-manifold $X$. Let $\sigma(X)$ denote the signature of $M^4$. Then a class $\xi \in H_2(X, \mathbb{Z})$ is said to be characteristic provided that $\xi \cdot x = x \cdot x$ for any $x \in H_2(X, \mathbb{Z})$ where $\xi \cdot x$ stands for the pairing of $\xi$ and $x$, i.e. their Kronecker index and $\xi^2$ for the self-intersection of $\xi$ in $H_2(M^4, \mathbb{Z})$. 

Fig. 5.
Theorem 3.2 (Gilmer and Viro [10, 25]). Let $X$ be an oriented, compact 4-manifold with $\partial X = S^3$, and $K$ a knot in $\partial X$. Suppose $K$ bounds a surface of genus $g$ in $X$ representing an element $\xi$ in $H_2(X, \partial X)$.

1. If $\xi$ is divisible by an odd prime $d$, then:
   \[
   \left| \frac{d^2-1}{2d^2} \xi^2 - \sigma(X) - \sigma(d) \right| \leq \dim H_2(X; \mathbb{Z}_d) + 2g.
   \]

2. If $\xi$ is divisible by 2, then:
   \[
   \left| \frac{\xi^2}{2} - \sigma(X) - \sigma(K) \right| \leq \dim H_2(X; \mathbb{Z}_2) + 2g.
   \]

The following theorem gives a lower bound for the genus of a characteristic class embedded in a 4-manifold:

Theorem 3.3 (Acosta [1], Fintushel [8], Yasuhara [27]). Let $X$ be a smooth closed oriented simply connected 4-manifold with $m = \min(b_2^+(X), b_2^-(X))$ and $M = \max(b_2^+(X), b_2^-(X))$, and assume that $m \geq 2$. If $\Sigma$ is an embedded surface in $X$ of genus $g$ so that $[\Sigma]$ is characteristic, then

\[
g \geq \begin{cases} 
\left\lfloor \frac{\Sigma, \Sigma - \sigma(X)}{8} \right\rfloor + 2 - M, & \text{if } \Sigma, \Sigma \leq \sigma(X) \leq 0 \quad \text{or } 0 \leq \sigma(X) \leq \Sigma, \Sigma, \\
9\left\lfloor \frac{\Sigma, \Sigma - \sigma(X)}{8} \right\rfloor + 2 - M, & \text{if } \sigma(X) \leq \Sigma, \Sigma \leq 0 \quad \text{or } \Sigma, \Sigma \leq \sigma(X), \\
\left\lfloor \frac{\Sigma, \Sigma - \sigma(X)}{8} \right\rfloor + 2 - m, & \text{if } \sigma(X) \leq 0 \quad \text{or } \Sigma, \Sigma \leq \sigma(X).
\end{cases}
\]

Using the knot filtration on the Heegaard Floer complex $\hat{CF}$, Ozsvath and Szabo introduced in [23] an integer invariant $\tau(K)$ for knots. They showed that $|\tau(T(p, q))| = \frac{p+q}{2} - \frac{(p-q)^2}{4}$ (see [23, Corollary 1.7]). In addition, they give a lower bound for the genus of a surface $\Sigma$ bounding a knot in a 4-manifold. To state their result, let $X$ be a smooth, oriented four-manifold with $\partial X = S^3$ and with $b^+(X) = b_1(X) = 0$. According to Donaldson’s celebrated theorem [3], the intersection form of $W$ is diagonalizable. Writing a homology class $[\Sigma] \in H_2(X)$ as $[\Sigma] = s_1e_1 + \cdots + s_be_b$, where $e_i$ are an orthonormal basis for $H_2(X; \mathbb{Z})$, and $s_i \in \mathbb{Z}$, we can define the $L^1$ norm of $[\Sigma]$ by $|[\Sigma]| = |s_1| + \cdots + |s_b|$. Note that this is independent of the diagonalization (since the basis $e_i$ is uniquely characterized, up to permutations and multiplications by $\pm 1$, by the orthonormality condition). We then have the following bounds on the genus of $[\Sigma]$:

Theorem 3.4 (Ozsvath and Szabo [23]). Let $X$ be a smooth, oriented four-manifold with $b_2^+(X) = b_1(X) = 0$, and $\partial X = S^3$. If $\Sigma$ is any smoothly embedded
surface-with-boundary in $X$ whose boundary lies on $S^3$, where it is embedded as the knot $K$, then we have the following inequality:

$$\tau(K) + \frac{|\Sigma| + |\Sigma| |\Sigma|}{2} \leq g(\Sigma).$$

4. Proof of Statements

To prove Theorems 1.1 and 1.2, we need the following lemma.

**Lemma 4.1.** Let $d$ be an odd prime number. Then the $d$-signature of a $(2, q)$-torus knot ($q \geq 3$) is given by the formula:

$$\sigma_d((T(2, q))) = -(q - 1) + 2\left\lfloor \frac{q}{2d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less or equal to $x$.

**Proof.** We use Litherland’s algorithm to compute $\sigma_d((T(2, q)))$. In this case, $y_1, j = \frac{1}{p} + \frac{1}{q}$ and $x = \frac{d - 1}{2d}$. Therefore,

- $1 + \frac{d - 1}{2d} < \frac{1}{p} + \frac{1}{q} < 2$ is equivalent to $1 + \left\lfloor \frac{(2d - 1)q}{2d} \right\rfloor \leq j < q - 1$.
- $\frac{d - 1}{2d} < \frac{1}{p} + \frac{1}{q} < 1 + \frac{d - 1}{2d}$ is equivalent to $1 \leq j \leq \left\lfloor \frac{(2d - 1)q}{2d} \right\rfloor$.

Litherland’s algorithm yields that $\sigma_d((T(2, q))) = (q - 1) - 2\left\lfloor \frac{(2d - 1)q}{2d} \right\rfloor$. It is easy to check that this is equivalent to $\sigma_d((T(2, q))) = -(q - 1) + 2\left\lfloor \frac{q}{2d} \right\rfloor$. □

4.1. Proof of Theorem 1.1

**Proof.**

(1) It is easy to check that $T(-2, 3)$ is obtained by a single $(1, -2)$-twisting and also by a single $(1, 3)$-twisting from the unknot, and therefore $T(-2, 3)$ is smoothly slice in $\mathbb{C}P^2$, or equivalently, $g_{\mathbb{C}P^2}(T(-2, 3)) = 0$. Theorems 3.2 and 2.1 yield that the only possible degrees are $d \in \{\pm 2, \pm 3\}$; realizable by the twisting disks.

(2) Note that $T(-2, 5)$ can be obtained from the unknot by a single $(1, -3)$-twisting (see Fig. 6), which proves that $T(-2, 5)$ is smoothly slice in $\mathbb{C}P^2$ with degree $d = -3$ (see [21]). Theorems 3.2 and 2.1 yield that the only possible degrees are $d = \pm 3$; realizable by the twisting disks.

(3) Theorem 2.2 yields that $T(-2, 7)$ and $T(-2, 9)$ are slice with degree $d = 4$.

We can deduce from Theorems 3.2 and 2.1 that the only realizable degrees are $d = \pm 4$.

(4) To show that $g_{\mathbb{C}P^2}(T(-2, 11)) = 1$ and $d \in \{\pm 4, \pm 5\}$, we first notice that $T(-2, 11)$ is obtained from $T(-2, 9)$ by adding two half-twisted bands. By Theorem 2.2, $T(-2, 9)$ is smoothly slice in $\mathbb{C}P^2$. Thus $g_{\mathbb{C}P^2}(T(-2, 11)) \leq 1$. 

To show that $g_{CP^2}(T(-2,11)) = 1$, let $\Sigma_g$ be a minimal genus smooth, compact, connected and oriented surface in $CP^2 - B^4$ with boundary $T(-2,11)$, and assume that $[\Sigma_g] = d_\gamma \in H_2(CP^2 - B^4, S^3, \mathbb{Z})$.

**Case 1.** If $d$ is even, then by Theorem 3.2.(2), $|\frac{d^2}{2} - \sigma(T(-2,11)) - 1| \leq 1 + 2g$. By A.G. Tristram [24], $\sigma(T(-2,11)) = 10$, then $d$ satisfies $20 - 4g \leq d^2 \leq 24 + 4g$. Therefore, $g = 1$ and $d = \pm 4$ are the only possibilities.

**Case 2.** Assume now that $d$ is odd. We can check that $T(2,11)$ is obtained from the unknot $T(-2,1)$ by a single $(6,2)$-twisting. It was proved in [21] and [7], using Kirby’s calculus on the Hopf link [18], that this yields the existence of a properly embedded disk $D$ using Kirby’s calculus on the Hopf link [18], that this yields the existence of a compact, connected and oriented surface in $CP^2 - B^4$ such that $[D] = -2\alpha + 6\beta$ and $\partial D = T(2,11)$. The genus $g$ surface $\Sigma = \Sigma_g \cup D$ satisfies $[\Sigma_g \cup D] = d_\gamma - 2\alpha + 6\beta \in H_2(CP^2 \# S^2 \times S^2, \mathbb{Z})$. Note that $\Sigma$ is a characteristic class and $[\Sigma] = d_\gamma - 2\alpha + 6\beta \leq 24$. Assume first that $|d| \geq 7$, so blowing up $\Sigma \subset S^2 \times S^2 \# CP^2$ a number of times equal to $d^2 - 24$ gives a genus $g$ surface $\Sigma \subset CP^2 \# S^2 \times S^2 \# (d^2 - 24) CF^2 = X$ (the proper transform) with $[\Sigma]^2 = 0$. If $e_i$ denotes the homology class of the exceptional sphere in the $i$th blow-up ($i = 1, 2, \ldots, d^2 - 24$), then $[\Sigma] = d_\gamma - 2\alpha + 6\beta - \sum_{i=1}^{d^2-24} e_i \in H_2(X, \mathbb{Z})$. The last inequality of Theorem 3.3 yields that $g \geq \frac{|\sum e_i|}{8}$, which is equivalent to $g \geq \frac{d^2-24}{8}$; which contradicts the assumptions $g \leq 1$ and $|d| \geq 7$. Therefore, if $d$ is odd then $d \in \{\pm 1, \pm 3, \pm 5\}$ and $g = 1$.

(a) To exclude $d \in \{\pm 1, \pm 3\}$, let $\Sigma_1$ be a genus-one smooth, compact, connected and oriented surface in $CP^2 - B^4$ with boundary $T(-2,11)$, such that $[\Sigma_1] = d_\gamma \in H_2(CP^2 - B^4, S^3, \mathbb{Z})$. Thus, the surface with the other orientation $(\Sigma_1, \partial \Sigma_1) \subset (CP^2 - B^4, S^3)$ is a genus-one surface bounding $T(2,11)$ such that $[\Sigma_1] = \pm \sigma(T(2,11)) \in H_2(CP^2 - B^4, S^3, \mathbb{Z})$. By Theorem 3.4, we have $\tau(T(2,11)) + \frac{|\Sigma_1|^2 + |\Sigma_1|^2}{2} \leq g(\Sigma_1)$. Since $\tau(T(2,11)) = 5$, $|\Sigma_1| = |d|$, and $[\Sigma_1]^2 = -d^2$, then $5 + \frac{|d| + d^2}{2} \leq 1$, a contradiction.

(a) If $d = \pm 5$, then by Lemma 4.1, we have $\sigma_5(T(-2,11)) = 8$ and then Theorem 3.2.(2) yields that $g = 1$ and $d = \pm 5$ are two possibilities. □
4.2. Proof of Theorem 1.2

To prove Theorem 1.2, we recall the definition of band surgery:

**Band surgery.** Let $L$ be a $\mu$-component oriented link. Let $B_1, \ldots, B_\nu$ be mutually disjoint oriented bands in $S^3$ such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha'_i$, where $\alpha_1, \alpha'_1, \ldots, \alpha_\nu, \alpha'_\nu$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \cdots \cup \partial B_\nu$ is also a link $L'$.

**Definition 4.2.** If $L'$ has the orientation compatible with the orientation of $L - \bigcup_{i=1, \ldots, \nu} \alpha_i \cup \alpha'_i$ and $\bigcup_{i=1, \ldots, \nu} (\partial B_i - \alpha_i \cup \alpha'_i)$, then $L'$ is called the link obtained by the band surgery along the bands $B_1, \ldots, B_\nu$. If $\mu - \nu = 1$, then this operation is called a fusion.

**Example 4.3.** Let $L_{p,q} = K_1^1 \cup \cdots \cup K_\mu^1 \cup K_1^2 \cup \cdots \cup K_\eta^2$ denote the $((p, 0), (q, 0))$-cable on the Hopf link with linking number 1 (see Fig. 7). Then, $T(2, 9)$ can be obtained from $L_{2,4}$ by fusion (see Fig. 8).

**Example 4.4.** Any $(p, 2kp+1)$-torus knot $(k > 0)$ is obtained from $L_{p,kp}$ by adding $(p - 1)(k + 1)$ bands (see Yamamoto’s construction in [26]). This construction can be generalized to any $(p, q)$-torus.

For convenience of the reader, we give a smooth surface that bounds $L_{p,q}$ in $T^4 - J$ ($J$ is a 4-ball); due to Kawamura (see [14, 15]): Consider $T^4 = T^2 \times T^2$.

![Fig. 7. The link $L_{p,q}$](image1)

![Fig. 8.](image2)
where $T^2 = [0, 1] \times [0, 1]/\sim$ such that $(0, t) \sim (1, t)$ and $(s, 0) \sim (s, 1)$, and define $E$ and $J$ by:

$$E = \bigcup_{k=1,\ldots,p} \left( \frac{k}{p+1}, \frac{k}{p+1} \right) \times T^2 \cup \bigcup_{k=1,\ldots,q} T^2 \times \left( \frac{k}{q+1}, \frac{k}{q+1} \right)$$

and $J = \left[ \frac{1}{p+2}, \frac{q+1}{p+2} \right]^2 \times \left[ \frac{1}{q+2}, \frac{q+1}{q+2} \right]^2$. The 4-ball $J$ contains all self-intersections of $E$ and we have:

**Theorem 4.5 (Kawamura [14, 15]).** $\partial(E - J) = E \cap \partial J \subset \partial J$ is the link $L_{p,q}$.

Auckly proved the following in [6].

**Theorem 4.6.** $0$ is a basic class of $T^4$.

To prove Theorem 1.2, we need Proposition 4.7 and Lemma 4.8.

**Proposition 4.7.** If $K_{p,q}$ is a knot obtained from $L_{p,q}$ by fusion and $\Sigma_g$ a smooth, compact, connected and oriented surface properly embedded in $\mathbb{CP}^2 - B^4$ with boundary $K_{p,q}$ in $\partial(\mathbb{CP}^2 - B^4)$, then

$$2pq - d^2 + |d| \leq 2(p + q + g) - 2.$$

**Proof.** By Theorem 4.5, there exists a surface $E$ and a 4-ball $J$, such that: $\partial(E - J) = L_{p,q}$ (see Fig. 9). Since $K_{p,q}$ is obtained from $L_{p,q}$ by fusion, then there exists a $(p + q + 1)$-punctured sphere $\hat{F}$ in $S^3 \times [0, 1] \subset J$ such that we can identify this band surgery with $\hat{F} \cap (S^3 \times \{1/2\})$, and $\partial \hat{F} = L_{p,q} \cup K_{p,q}$ with $L_{p,q}$ lies in $S^3 \times \{0\} \cong \partial J \times \{0\}$ and $K_{p,q}$ lies in $S^3 \times \{1\} \cong \partial J \times \{1\}$. The 3-sphere $S^3 \times \{1\}(\cong \partial J \times \{1\})$ bounds a 4-ball $B^4 \subset J$. The surface $F = (E - J) \cup \hat{F}$ is a

![Fig. 9. The surface $\Sigma = (E - J) \cup \hat{F} \cup \Sigma_g$.](image-url)
smooth surface properly embedded in $T^4 - B^4$, and with boundary $K_{p,q}$. Since $K_{p,q}$ bounds a genus $g$ surface $\Sigma_g \subset \mathbb{C}P^2 - B^4$, then $K_{p,q}$ bounds a properly embedded genus $g$ surface $\Sigma_g \subset \mathbb{C}P^2 - B^4$ such that $[\Sigma_g] = \pm d \gamma \in H_2(\mathbb{C}P^2 - B^4, S^3; \mathbb{Z})$. The smooth surface $\Sigma = F \cup \Sigma_g$ in $T^4 \# \mathbb{C}P^2$ satisfies $[\Sigma]^2 = F^2 + (\Sigma_g)^2$. Since $F$ and $E$ are homologous, then $F^2 = E^2 = 2pq$ which implies that $[\Sigma]^2 = 2pq - d^2$. By Theorem 4.6, $0$ is a basic class for $T^4$, then the basic class of $T^4 \# \mathbb{C}P^2$ (the blowup of $T^4$) is $K = \pm \gamma$ (see [9]), and therefore $|K . \Sigma| = |d|$. Since $g(E - B^4) = p + q$, then $g(\Sigma) = p + q + g$. The adjunction inequality proved by Kronheimer and Mrowka [22] implies that $[\Sigma]^2 + |K . \Sigma| \leq 2g(\Sigma) - 2$. Therefore, $2pq - d^2 + |d| \leq 2(p + q + g) - 2$.

**Lemma 4.8.** Let $(\Sigma_g, \partial \Sigma_g) \subset (\mathbb{C}P^2 - B^4, S^3)$ be a genus-minimizing smooth, compact, connected and oriented surface properly embedded in $\mathbb{C}P^2 - B^4$ with boundary $T(2, q)$ and let

$$\Sigma_g = d \gamma \in H_2(\mathbb{C}P^2 - B^4; \mathbb{Z}).$$

1. If $d$ is even, then $g = \frac{d-3}{2}$ and $d = 0$. Therefore Conjecture 1.1 holds in case $d$ is even.

2. Conjecture 1.1 holds in case $d = \pm 1$.

**Proof.**

1. For any $q > 0$, we can check that $T(2, q)$ is obtained from $T(2, 3)$ by adding $q - 3$ half-twisted bands, then there is a genus $\frac{q-3}{2}$ cobordism between $T(2, 3)$ and $T(2, q)$. Since $T(2, 3)$ is slice in $\mathbb{C}P^2$, then $g \leq \frac{q-3}{2}$. Since $d$ is even, then by Theorem 3.2(1), $\left|\frac{d}{4} - 1 - \sigma(T(2, q))\right| \leq 1 + 2g$. By Tristram [24], $\sigma(T(2, q)) = -(q - 1)$, and then $\frac{d}{4} + \frac{d-3}{2} \leq g$ which implies that $\frac{q-3}{2} \leq g$ and $d = 0$. Therefore, Conjecture 1.1 holds in case $d$ is even.

2. To prove that Conjecture 1.1 holds in case $d = \pm 1$, note that $T(2, q)$ is obtained from $L(2, \frac{q-3}{2})$ by fusion, and then apply Proposition 4.7.

**Proof of Theorem 1.2.** If $d$ is even, then by Lemma 4.8(2), $g_{\mathbb{C}P^2}(T(2, q)) = \frac{q-3}{2}$ for $3 \leq q \leq 17$ and the only possible degree is $d = 0$; realizable by the twisting disk $\Delta$. If $d$ is odd, then by Lemma 4.8, we can assume, from now on, that $d \in \mathbb{Z} - \{\pm 1\}$.

1. If $q = 3$ then it is not hard to check that $T(2, 3)$ can be obtained by a single $(-1, 0)$-twisting from the unknot. This implies that $T(2, 3)$ is smoothly slice in $\mathbb{C}P^2$, or equivalently $g_{\mathbb{C}P^2}(T(2, 3)) = 0$. To prove that $d = 0$ is the only possibility, let $(\Delta, \partial \Delta) \subset (\mathbb{C}P^2 - B^4, S^3)$ be a smooth 2-disk such that $\partial \Delta = T(2, 3)$, and assume that $|\Delta| = d \gamma \in H_2(\mathbb{C}P^2 - B^4, S^3)$. It is easy to check that $T(2, 1) \overset{(-2,2)}{\sim} T(2, 3)$. By [21] and [7], there exists a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that $[D] = 2\alpha + 2\beta \in H_2(S^2 \times S^2 - B^4, S^3, \mathbb{Z})$ and $\partial D = T(-2, 3)$. The genus $g$ surface $\Sigma = \Sigma_g \cup T(2, 3)$, $D$ satisfies
[\Sigma] = d\gamma + 2\alpha + 2\beta \in H_2(\mathbb{CP}^2 \# S^2 \times S^2; \mathbb{Z}) and then \([\Sigma]^2 = d^2 + 8\). Blowing up \Sigma a number of times equal to \(d^2 + 8\) gives a genus \(g\) surface \(\tilde{\Sigma} \subset \mathbb{CP}^2 \# S^2 \times S^2 \# (d^2 + 8)\mathbb{CP}^2 = X\) (the proper transform) with \([\tilde{\Sigma}]^2 = 0\). The last inequality of Theorem 3.3 yields that \(g \geq \frac{d^2 + 8}{2}\). Therefore, \(T(2,3)\) is not slice, a contradiction.

(2) For \(g = 5\), note that \(T(-2,1) \sim \frac{(-2,3)}{T(-2,5)}\). By the same argument as in case \(q = 3\), Theorem 3.3 yields that \(g \geq \frac{d^2 + 7}{8}\). This would contradict the assumptions \(g \leq 2\) and \(|d| \neq 1\).

(3) For \(g = 7\), we can also notice that \(T(2,1) \sim \frac{(-4,2)}{T(-2,7)}\). By a similar argument, we get a genus \(g\) surface \(\Sigma = \Sigma_g \cup_{T(2,7)} D\) such that \([\Sigma] = d\gamma + 2\alpha + 4\beta \in H_2(\mathbb{CP}^2 \# S^2 \times S^2; \mathbb{Z})\). Since \([\Sigma]^2 = d^2 + 16\), then blowing up \(\Sigma\) a number of times equal to \(d^2 + 16\) gives a genus \(g\) surface \(\tilde{\Sigma} \subset \mathbb{CP}^2 \# S^2 \times S^2 \# (d^2 + 16)\mathbb{CP}^2 = X\) with \([\tilde{\Sigma}]^2 = 0\). The last inequality of Theorem 3.3 yields that \(g \geq \frac{d^2 + 15}{8}\). This would contradict the assumptions \(g \leq 2\) and \(|d| \neq 1\).

(4) The case \(q = 9\) is similar to \(q = 7\) since \(T(-2,1) \sim \frac{(-4,2)}{T(-2,9)}\), then we can conclude from Theorem 3.3 that \(g \geq \frac{d^2 + 15}{8}\). Since \(g \leq 3\), then the only possibilities are \(d = \pm 3\) and \(g = 3\); excluded by Theorem 3.2(2) and Lemma 4.1 (\(\sigma_3(T(2,9)) = -6\)).

(5) For \(g = 11\), we can check that \(T(2,1) \sim \frac{(-6,2)}{T(-2,11)}\). By a similar argument, we get a surface \(\Sigma\) such that \([\Sigma] = d\gamma + 2\alpha + 6\beta \in H_2(\mathbb{CP}^2 \# S^2 \times S^2; \mathbb{Z})\) and \([\Sigma]^2 = d^2 + 23\). Blowing up \(\Sigma\) a number of times equal to \(d^2 + 24\) gives a surface \(\tilde{\Sigma} \subset \mathbb{CP}^2 \# S^2 \times S^2 \# (d^2 + 24)\mathbb{CP}^2 = X\) such that \([\tilde{\Sigma}] = d\gamma + 2\alpha + 6\beta - \sum_{i=1}^{d^2 + 24} e_i \in H_2(X, \mathbb{Z})\) and then \([\tilde{\Sigma}]^2 = 0\). Since \(\sigma(X) = -d^2 - 23\), then Theorem 3.3 implies that \(g \geq \frac{d^2 + 22}{8}\). Since \(g \leq 4\), then the only possibilities are \(d = \pm 3\) and \(g = 4\); excluded by Theorem 3.2(2) and Lemma 4.1 (\(\sigma_3(T(2,11)) = -8\)).

(6) For \(q = 13\), we can easily check that \(T(2,1) \sim \frac{(-6,2)}{T(-2,13)}\), and Lemma 4.1 yields that \(\sigma_3(T(2,13)) = -8\). Then, the argument is similar to the case \(q = 11\).

(7) For \(q = 15\), we have \(T(2,1) \sim \frac{(-8,2)}{T(-2,15)}\). Theorem 3.3 implies that \(g \geq \frac{d^2 + 31}{8}\); which excludes the cases where \(|d| \geq 5\). Lemma 4.1 yields that \(\sigma_3(T(2,15)) = -10\); which yields that the case \(d = \pm 3\) and \(g = 5\) are two possibilities.

(8) For \(q = 17\), we have \(T(2,1) \sim \frac{(-8,2)}{T(-2,17)}\). Lemma 4.1 yields that \(\sigma_3(T(2,17)) = -12\). Then the argument is similar to the case \(q = 15\).

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References

