[1]

TENSOR FUNCTORS ON A CERTAIN CATEGORY CONSTRUCTED FROM SPHERICAL CATEGORIES

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Abstract. We construct functors from a certain algebraic category $\overline{\mathcal{A}lg}$, defined by Hopf algebra generators and relations, to the category of vector spaces, based on spherical categories. The category $\overline{\mathcal{A}lg}$ is proposed by Habiro to be isomorphic to the cobordism category of once-punctured surfaces. If the proposal is proved valid, the result of this paper would imply a construction of a TQFT functor based on a spherical category.

1. Introduction

Quantum invariants of 3-manifolds has been developed since Reshetikhin and Turaev [15] constructed their 3-manifold invariants from the representation category of $U_q(\mathfrak{s}l_2)$, which are considered to be equal to Witten's Chern-Simons path integral [20]. Later Turaev and Viro [18] constructed their invariant, also from $U_q(\mathfrak{s}l_2)$. In this construction the braiding structure of the representation category was not used. It was proved that the Turaev-Viro invariant is the square of the absolute value of the Reshetikhin-Turaev invariant for the same closed 3-manifold [19, 17, 16]. Both these two invariants extend to Topological Quantum Field Theory, i.e., a symmetric, monoidal functor from the category of closed oriented surfaces and compact, oriented 3-dimensional cobordisms to the category of finite dimensional vector spaces.

The construction of Turaev-Viro type of invariants from more general categories without braiding has been done by Ocneanu [8], using the category of bimodules arising from subfactors. The construction of Turaev-Viro type 3-manifold invariants from spherical categories is given in [1] and [9].

On the other hand, Crane and Yetter [7, 21] and Kerler [12], independently, introduced the category $\mathcal{C}ob$ of connected surfaces with one boundary component and their cobordisms, and discovered a braided Hopf algebra structure in $\mathcal{C}ob$. In [11], Kerler attempted to give a presentation of $\mathcal{C}ob$ as a braided category. More precisely, he defined a

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category $\mathcal{A}lg$ which is entirely described in algebraic terms, and constructed a surjective functor from $\mathcal{A}lg$ to $\mathcal{C}ob$. He proposed in [11, Theorem 2] that there should be additional relations imposed on $\mathcal{A}lg$, with possible modification, so that the surjection will become an isomorphism. Recently, Bobtcheva and Piergallini [4] gave a partial answer to this problem, namely, the additional relations so that the functor $\mathcal{A}lg \to \mathcal{C}ob$ is surjective on (punctured) closed 3-manifolds.

Habiro proposed a complete set of relations for the category $\mathcal{C}ob$ in a paper in preparation [10]. [1] It would enable us to describe $\mathcal{C}ob$ in a purely algebraic manner, and thus construction of (punctured) TQFT is reduced to assigning each generator of $\mathcal{C}ob$ a new morphism, and checking that the new morphisms satisfy the relations in $\mathcal{C}ob$.

In this paper we construct a tensor functor from Habiro's category using a spherical category, in a perspetive that, with Habiro's upcoming paper, it would extend to (punctured) Turaev-Viro type TQFT. In Section 2, we describe the result of [10] and state the main theorem of this paper. In Section 3, we define a category $\mathcal{S}$, through which we define a TQFT functor. In Section 4 we define the morphisms corresponding to the generators of $\mathcal{M}or(\mathcal{A}lg)$ in $\mathcal{S}$. In Section 5 we give the proof that the morphisms given in the previous section satisfy the relations given in Section 2. Note that this paper has an overlap with [6], where an algorithm for Reschetikhin-Turaev invariants from the categorical center $\mathcal{Z}(\mathcal{C})$ of a spherical category $\mathcal{C}$ is given in terms of $\mathcal{C}$. In this paper we propose an explicit construction of a TQFT in terms of $\mathcal{C}$, granted that Habiro's result will be proved. Namely the category $\overline{\mathcal{A}lg}$ defined by Habiro is expected to be isomorphic to the category $\mathcal{C}ob$ of connected surface with one boundary with cobordisms, and in this paper we construct a tensor functor defined on $\overline{\mathcal{A}lg}$. Notice that Hopf diagrams in [6] are similar to the morphisms of $\overline{\mathcal{A}lg}$. However $\overline{\mathcal{A}lg}$ has more relations to ensure isomorphism to $\mathcal{C}ob$. The category of Hopf diagrams is not expected to be isomorphic to $\mathcal{C}ob$. (It should be surjective.)

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[1] The author was informed that the result for the case of closed is also given in in [5].

2. Algebraic presentation of $\mathcal{C}ob$ and the TQFT functor

Definition 2.1. *([10], c.f. [11]) $\mathcal{A}lg$ is a monoidal category, where the objects are given by*

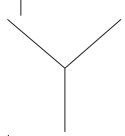
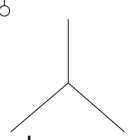
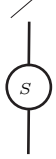
$$\mathcal{O}bj\mathcal{A}lg = \mathbb{Z}_{\geq 0},$$

$\mathcal{M}or\mathcal{A}lg$ is freely generated by the following generators:

- $\eta \in \mathcal{A}lg(0,1)$
- $\mu \in \mathcal{A}lg(2,1)$
- $\varepsilon \in \mathcal{A}lg(1,0)$
- $\Delta \in \mathcal{A}lg(1,2)$
- $S \in \mathcal{A}lg(1,1)$
- $S^{-1} \in \mathcal{A}lg(1,1)$
- $c \in \mathcal{A}lg(0,2)$
- $v = v_+ \in \mathcal{A}lg(0,1)$
- $v_- \in \mathcal{A}lg(0,1)$
- $d \in \mathcal{A}lg(2,0)$
- $\psi \in \mathcal{A}lg(2,2)$
- $\psi^{-1} \in \mathcal{A}lg(2,2)$,

where by $\mathcal{A}lg(k,l)$ we denote the set of morphisms of $\mathcal{A}lg$ from k to l. We denote by $1_n \in \mathcal{A}lg(n,n)$ the identity morphism, in particular we denote $1_1 = 1$.

The generators of $\mathcal{A}lg$ have graphical expression as follows. We do not use them in this paper, but we include the pictures nonetheless, for a comparison with those in [11], and for those who wish to visualize the relations of the generators introduced in Definition 2.2.

- $\eta =$ 
- $\mu =$ 
- $\varepsilon =$ 
- $\Delta =$ 
- $S =$ 

- $S^{-1} =$ 

- $c =$ 
- $v = v_+ =$ 

- $v_- =$ 
- $d =$ 
- $\psi =$ 
- $\psi^{-1} =$ 

Definition 2.2. *Let $\mathcal{A}lg$ as above. We define a new category $\overline{\mathcal{A}lg}$ as follows:*

- $\mathcal{O}bj\overline{\mathcal{A}lg} = \mathcal{O}bj\mathcal{A}lg$
- $\mathcal{M}or\overline{\mathcal{A}lg} = \mathcal{M}or\mathcal{A}lg/relations$,

where the relations are given by the following.

(1) $\psi_{k,1}(f \otimes 1) = (1 \otimes f)\psi_{m,1}$ *for any generator* $f \in \mathrm{Mor}(m,k)$, *where* $\psi_{1,1} = \psi$, $\psi_{i,1} = (\psi_{i-1,1} \otimes 1)(1_{i-1} \otimes \psi)$,
(2) $\psi_{1,k}(1 \otimes f) = (f \otimes 1)\psi_{1,m}$ *for any generator* $f \in \mathrm{Mor}(m,k)$, *where* $\psi_{i,l} = (1_{l-1} \otimes \psi)(\psi_{i,l-1} \otimes 1)$
(3) $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$
(4) $\mu(\eta \otimes 1) = \mu(1 \otimes \eta) = 1$
(5) $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$
(6) $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = 1$
(7) $\varepsilon\eta = 1$,
(8) $\Delta\eta = \eta \otimes \eta$,
(9) $\varepsilon\mu = \varepsilon \otimes \varepsilon$,
(10) $\Delta\mu = (\mu \otimes \mu)(1 \otimes \psi \otimes 1)(\Delta \otimes \Delta)$,
(11) $\mu(1 \otimes S)\Delta = \eta\epsilon = \mu(S \otimes 1)\Delta$,
(12) $S^{-1}S = SS^{-1} = 1$,
(13) $\mu(1 \otimes v) = \mu(v \otimes 1)$,
(14) $\mu(v \otimes v^{-1}) = \eta$,
(15) $v\varepsilon = 1$,
(16) $vS = v$.
(17) $\Delta v = (\mu \otimes \mu)(1 \otimes c \otimes 1)(v \otimes v)$,
(18) $\eta = (1 \otimes \varepsilon)c = (\varepsilon \otimes 1)c$,
(19) $(\mu \otimes 1^{\otimes 2})(1 \otimes c \otimes 1)c = (1 \otimes \Delta)c$,

(20) $(1^{\otimes 2} \otimes \mu)(1 \otimes c \otimes 1)c = (\Delta \otimes 1)c$,
(21) $(1 \otimes d)(c \otimes 1) = 1$,
(22) $(d \otimes 1)(1 \otimes c) = 1$,
(23) $\mu(v \otimes v) = \mu c$,
(24) $(L \otimes 1)(1 \otimes \psi)(\Delta \otimes 1) = \psi^{-1}(1 \otimes L)(\Delta \otimes 1)$,
where $L := (\mu(1 \otimes \mu)(1^{\otimes}2 \otimes S)(1 \otimes \psi)(\Delta \otimes 1)$,
(25) $d(\mu(v \otimes v) \otimes v) = 1$,
(26) $S^2 = (1 \otimes d)(\psi \otimes 1)(1 \otimes c)$.

The following Theorem is proposed by Habiro.

Theorem 2.3. *(Habiro,* [10]*)*
Let $\mathcal{A}lg$, $\overline{\mathcal{A}lg}$ *as above, and* $\mathcal{C}ob$ *be a category of connected surfaces with one boundary component with cobordisms, as explained in* [11]*. Then there is an isomorphic functor*

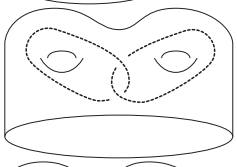
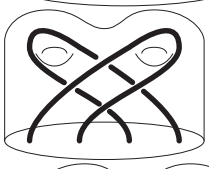
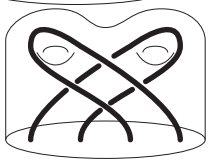
$$\mathcal{F} : \overline{\mathcal{A}lg} \longrightarrow \mathcal{C}ob$$

The functor $\mathcal{F}$ is given by the following. For $k \in \mathcal{O}bj\mathcal{A}lg$,

$$\mathcal{F}(k) = \text{ the genus } k \text{ surface with one puncture.}$$

We denote by k the genus k surface with one puncture as well. The images of the generators of $\mathcal{M}or\mathcal{A}lg$ are the cobordisms given as follows:

- $\mathcal{F}(\eta) =$  $\in \mathrm{Hom}(0,1)$
- $\mathcal{F}(\mu) =$  $\in \mathrm{Hom}(2,1)$
- $\mathcal{F}(\varepsilon) =$  $\in \mathrm{Hom}(1,0)$
- $\Delta =$  $\in \mathrm{Hom}(1,2)$
- $\mathcal{F}(S) =$  $\in \mathrm{Hom}(1,1)$
- $\mathcal{F}(S^{-1}) =$  $\in \mathrm{Hom}(1,1)$

- $\mathcal{F}(c) =$  $\in \mathrm{Hom}(0,2)$
- $\mathcal{F}(v) =$  $\in \mathrm{Hom}(0,1)$
- $\mathcal{F}(v_-) =$  $\in \mathrm{Hom}(0,1)$
- $\mathcal{F}(d) =$  $\in \mathrm{Hom}(2,0)$
- $\mathcal{F}(\psi) =$  $\in \mathrm{Hom}(2,2)$
- $\mathcal{F}(\psi^{-1}) =$  $\in \mathrm{Hom}(2,2)$

The meaning of the right hand side drawings as cobordisms is as given in [11, Theorem 7]: The thick curves are hallow holes drilled through the solid handle bodies from the bottom disks. Each cobordism gives a morphism from the surface S_b to S_t, where S_b is the union of the punctured disk on the bottom with the wall of the drilled holes, and S_t is the remaining part of the boundary of the cobordism together with the boundary of the bottom disk. The dotted loop in $\mathcal{F}(d)$ denotes 0-surgery.

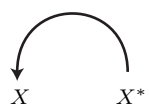
The objective of the paper is the following theorem:

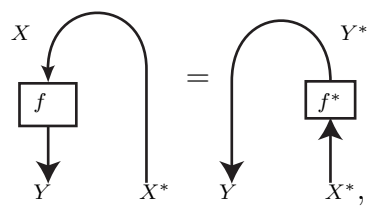
Theorem 2.4. *There is a well-defined functor $\mathcal{T} : \overline{\mathcal{A}lg} \to \mathbf{Vect}^{\mathcal{C}}$, where $\mathcal{C}$ is a small, k-linear, abelian, strict spherical category, and* **Vect** *is a category of vector spaces, and $\mathbf{Vect}^{\mathcal{C}}$ is a category of functors from $\mathcal{C}$ to* **Vect** *with natural transformations.*

Combining with Theorem 2.3, this theorem would imply that for each object in $\mathcal{C}$, $\mathcal{T}$ gives a TQFT functor of puctured surfaces.

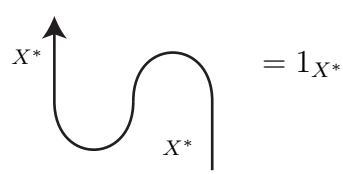
3. The category $\mathcal{S}$

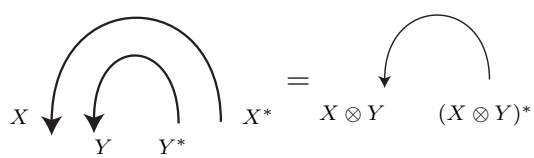
The precise definition of spherical categories is found in [2, Definition 2.1]. Below we review it quickly. A *pivotal category* $\mathcal{C}$ is a category with duals together with a morphism $\varepsilon(X) : \mathbf{1} \to X \otimes X^*$ for each object $X \in \mathcal{C}$, satisfying the relations represented by the following diagrams.

$\varepsilon(X) =$ 

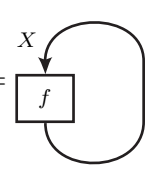


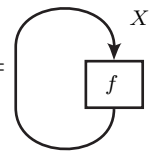
where $f \in \mathcal{M}or(X, Y)$.





We define right (resp. left) trace of an endomorphism $f \in \mathrm{End}(X)$ by

$\mathrm{Tr}_R(f) :=$ 

$\mathrm{Tr}_L(f) :=$ 

A pivotal category $\mathcal{C}$ is said to be *spherical* if for any $f \in \mathrm{End}(X)$, $\mathrm{Tr}_R(f) = \mathrm{Tr}_L(f)$ holds. The dimension $[X]$ of a simple object X is defined by

$$[X] := \mathrm{Tr}_R(1_X) = \mathrm{Tr}_L(1_X).$$

For later convenience we introduce the label ω for curves, which is to be understood in the following sense: a diagram with a component labeled by ω represents a weighted sum of morphisms represented by the diagrams, in which ω is replaced by a simple object. The weight of the summand is given by $[U]$, if ω is replaced by a simple object U. The sum is taken over all the simple objects in $\mathcal{C}$. When there are multiple components labeled ω, the sum is taken for each component. A component labeled by ω is drawn by dotted line, and when there are more than one, the pattern or the thickness of the dotted lines are varied to distinguish them. Sometimes we label such dotted line by a letter to indicate the simple objects. Let k be a field, and let **Vect** denote the category of k-vector spaces. We fix a small, k-linear, abelian, strict spherical category $\mathcal{C}$. Let $\mathbf{Vect}^{\mathcal{C}}$ denote the category whose objects are k-linear functors from $\mathcal{C}$ to **Vect** and whose morphisms are k-linear natural transformations.

3.1. **The functor $U_n\colon \mathcal{C} \to \mathbf{Vect}$.** For $n = \{0, 1, \ldots\}$, define an additive functor $\check{U}_n\colon \mathcal{C} \to \mathbf{Vect}$ by

$$\check{U}_n := \bigoplus_{x_1, y_1, \ldots, x_n, y_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}([x_1, y_1] \otimes \cdots \otimes [x_n, y_n],\ -)$$

on objects, where we set

$$[x, y] := x^* \otimes y^* \otimes x \otimes y \quad \text{for } x, y \in \mathrm{Ob}(\mathcal{C}),$$

and for a morphism $f \in \mathcal{C}(X, Y)$, $\check{U}_n(f) \in \mathrm{Hom}(\check{U}_n(X), \check{U}_n(Y))$ is given by composition $f\circ$.

We simplify the notation as follows. We denote an element in $\mathrm{Ob}(\mathcal{C})^n$ by

$$\vec{x} = (x_1, \ldots, x_n) \in \mathrm{Ob}(\mathcal{C})^n,$$

and for $\vec{x}, \vec{y} \in \mathrm{Ob}(\mathcal{C})^n$, set

$$[\vec{x}, \vec{y}] := [x_1, y_1] \otimes \cdots \otimes [x_n, y_n].$$

For $n = 0$, we have

$$\check{U}_0 := \mathcal{C}(\mathbf{1},\ -).$$

For $z \in \mathrm{Ob}(\mathcal{C})$, let $I_n(z) \subset \check{U}_n(z)$ be the k-submodule generated by

$$h\left(\bigotimes_{i=1}^{n}(1_{x_i^*} \otimes 1_{y_i^*} \otimes f_i \otimes g_i) - \bigotimes_{i=1}^{n}(f_i^* \otimes g_i^* \otimes 1_{x_i'} \otimes 1_{y_i'})\right)$$

for $x_i, y_i, x_i', y_i' \in \mathrm{Ob}(\mathcal{C})$, $f_i \in \mathcal{C}(x_i, x_i')$, $g_i \in \mathcal{C}(y_i, y_i')$ $(i = 1, \ldots, n)$ and $h \in \mathcal{C}(\bigotimes_{i=1}^{n}(x_i^* \otimes y_i^* \otimes x_i' \otimes y_i'), z)$.

Set

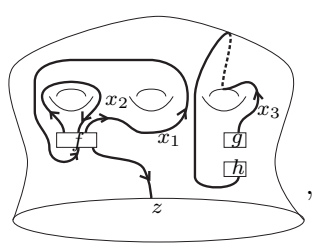
$$U_n(z) = \check{U}_n(z)/I_n(z),$$

which extends to an additive functor

$$U_n\colon\ \mathcal{C} \to \mathbf{Vect}$$

in an obvious way, and the projections $\check{U}_n(z) \to U_n(z)$ form a natural transformation $\check{U}_n \Rightarrow U_n$. In the following we abuse the notation and frequently denote an element in $U_n(z)$ by an element of $\check{U}_n(z)$.

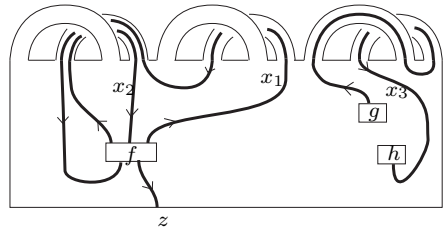
The above definition of $U_n\colon\ \mathcal{C} \to \mathbf{Vect}$ is illustrated as follows. For $z \in \mathrm{Ob}(\mathcal{C})$, $U_n(z)$ can be identified as a "skein space" of "$\mathcal{C}$-decorated graphs" in a surface $\Sigma_{n,0}$ of genus n with one boundary component, with marking by an object z on a point in the boundary, where the graphs consist of oriented edges labeled with objects of $\mathbb{C}$, and vertices are labeled with morphisms in $\mathbb{C}$. An example of a generator of the skein space $U_3(z)$ is given in the picture below:



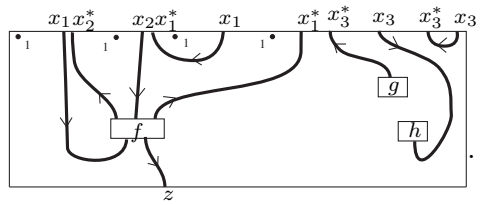
,

where x_i's are objects in $\mathcal{C}$, $f \in \mathcal{M}or(x_2^* \otimes x_2 \otimes x_1^*, x_2^* \otimes z)$, $g \in \mathcal{M}or(x_3^*, \mathbf{1})$, $h \in \mathcal{M}or(\mathbf{1}, x_3^*)$. Note that this is portrayed differently as

below:



Since the arcs in the one-handles are paralel to the handle itself, and their decoration is determined by the decoration of the rest of the graph, one may simply denote the graph by the following picture without the handles:



With this picture, one may easily associate that a decorated graph (with some conditions) in a surface $\Sigma_{n,0}$ corresponds to a morphism in $U_n(z)$ for some $z \in \mathrm{Ob}(\mathcal{C})$. The quotient by the submodule $I_n(z)$ implements that one may move morphisms through the 1-handles. The dots on the top signify that there are empty handles.

3.2. **The category $\mathcal{S}$.** We define an abelian strict monoidal category $\mathcal{S} = \mathcal{S}^{\mathcal{C}}$ with $\mathrm{Ob}(\mathcal{S}) = \{0, 1, \ldots\}$ as follows. For $m, n \in \mathrm{Ob}(\mathcal{S})$, we define $\mathcal{S}(m, n)$ to be the abelian group consisting of all maps

$$\varphi\colon\ \mathrm{Ob}(\mathcal{C})^{2m} \to \coprod_{z \in \mathrm{Ob}(C)} U_n(z)$$

with

$$(\vec{X}, \vec{Y}) \mapsto \varphi_{\vec{X},\vec{Y}} \in U_n([\vec{X}, \vec{Y}]),$$

where $\vec{X}, \vec{Y} \in \mathrm{Ob}(\mathcal{C})^m$, satisfying

$$\left(\bigotimes_{j=1}^{m} (1 \otimes 1 \otimes u_j \otimes v_j)\right) \varphi_{\vec{X},\vec{Y}} = \left(\bigotimes_{j=1}^{m} (u_j^* \otimes v_j^* \otimes 1 \otimes 1)\right) \varphi_{\vec{X}',\vec{Y}'}$$

for $\vec{X}, \vec{Y}, \vec{X}', \vec{Y}' \in \mathrm{Ob}(\mathcal{C})^m$, $u_j \in \mathcal{C}(X_j, X_j')$, $v_j \in \mathcal{C}(Y_j, Y_j')$ $(j = 1, \ldots, m)$. Composition is defined as follows. Suppose that $\varphi \in \mathcal{S}(m, n)$ is such that $\varphi_{\vec{X},\vec{Y}}$ is a morphism

$$\varphi_{\vec{X},\vec{Y}}\colon\ [\vec{X}', \vec{Y}'] \to [\vec{X}, \vec{Y}],$$

and that $\theta \in \mathcal{S}(n,p)$ is such that $\theta_{\vec{X}',\vec{Y}'}$ is a morphism

$$\theta_{\vec{X}',\vec{Y}'}\colon\ [\vec{X}'',\vec{Y}''] \to [\vec{X}',\vec{Y}'],$$

where $\vec{X},\vec{Y} \in \mathrm{Ob}(\mathcal{C})^m$, $\vec{X}',\vec{Y}' \in \mathrm{Ob}(\mathcal{C})^n$ $\vec{X}'',\vec{Y}'' \in \mathrm{Ob}(\mathcal{C})^p$. Then, the composite $\theta\varphi \in \mathcal{S}(m,p)$ is defined so that the morphism $(\theta\varphi)_{\vec{X},\vec{Y}}$ is given by

$$\varphi_{\vec{X},\vec{Y}}\theta_{\vec{X}',\vec{Y}'}\colon\ [\vec{X}'',\vec{Y}''] \to [\vec{X},\vec{Y}].$$

(Recall that we are abusing the notation here: $\varphi_{\vec{X},\vec{Y}}$ etc. are elements in quotient modules $U_n([\vec{X},\vec{Y}]$ etc., but we write as if they are elements in $\check{U}_n([\vec{X},\vec{Y}].)$

Composition of more general morphisms is obtained by linear combination. It follows from the definition that $\theta\varphi \in \mathcal{S}(m,p)$ is well-defined. Clearly, this composition is associative.

Define the morphism $1_m \in \mathcal{S}(m,m)$ so that for $\vec{X},\vec{Y} \in \mathrm{Ob}(\mathcal{C})^m$, the element $(1_m)_{\vec{X},\vec{Y}} \in U_m([\vec{X},\vec{Y}])$ is given by the identity map $1_{[\vec{X},\vec{Y}]}$. Then 1_m is unital with respect to composition. Thus we obtain an abelian category $\mathcal{S}$.

A monoidal category structure is defined as follows. For $m,n \in \mathrm{Ob}(\mathcal{S})$, set $m \otimes n = m+n$. Suppose that

$$\varphi = \{\varphi_{\vec{X},\vec{Y}}\}_{\vec{X},\vec{Y}\in\mathrm{Ob}(\mathcal{C})^m} \in \mathcal{S}(m,n),$$

and

$$\theta = \{\theta_{\vec{X}',\vec{Y}'}\}_{\vec{X}',\vec{Y}'\in\mathrm{Ob}(\mathcal{C})^{m'}} \in \mathcal{S}(m',n'),$$

where $\varphi_{\vec{X},\vec{Y}} \in U_n([\vec{X},\vec{Y}])$ and $\theta_{\vec{X}',\vec{Y}'} \in U_{n'}([\vec{X}',\vec{Y}'])$. Then define $\varphi \otimes \theta \in \mathcal{S}(m+m',n+n')$ by

$$(\varphi\otimes\theta)_{(\vec{X},\vec{X}'),(\vec{Y},\vec{Y}')} = \varphi_{\vec{X},\vec{Y}} \otimes \theta_{\vec{X}',\vec{Y}'} \in U_{m+m'}([\vec{X},\vec{Y}] \otimes [\vec{X}',\vec{Y}']).$$

Note that $[\vec{X},\vec{Y}] \otimes [\vec{X}',\vec{Y}'] = [(\vec{X},\vec{X}'),(\vec{Y},\vec{Y}')]$. Setting $\mathbf{1}_{\mathcal{S}} = 0$, we obtain an abelian strict monoidal category $\mathcal{S}$.

3.3. **The functor $U\colon\ \mathcal{S} \to \mathbf{Vect}^{\mathcal{C}}$.** We define an additive functor $U\colon\ \mathcal{S} \to \mathbf{Vect}^{\mathcal{C}}$ as follows.

For $m \in \mathrm{Ob}(\mathcal{S})$, we set $U(m) = U_m\colon\ \mathcal{C} \to \mathbf{Vect}$. For $\varphi \in \mathcal{S}(m,n)$ and $z \in \mathrm{Ob}(\mathcal{C})$, define a map

$$U(\varphi)_z\colon\ U_m(z) \to U_n(z),$$

similarly to composition in $\mathcal{S}$, as follows. If $\varphi \in \mathcal{S}(m,n)$ is such that $\varphi_{\vec{X},\vec{Y}}$ is given by

$$\check{\varphi}_{\vec{X},\vec{Y}}\colon\ [\vec{X}',\vec{Y}'] \to [\vec{X},\vec{Y}],$$

for $\vec{X},\vec{Y} \in \mathrm{Ob}(\mathcal{C})^m$, $\vec{X}',\vec{Y}' \in \mathrm{Ob}(\mathcal{C})^n$, and if $g \in U_m(z)$ is given by

$$\check{g}\colon\ [\vec{X},\vec{Y}] \to z,$$

then $U(\varphi)_z(g)$ is given by

$$\check{g}\check{\varphi}_{\vec{X},\vec{Y}}\colon\ [\vec{X}',\vec{Y}'] \to z.$$

It is straightforward to check that the $U(\varphi)_z$ forms a natural transformation

$$U(\varphi)\colon\ U_m \Rightarrow U_n,$$

and that we obtain an additive functor

$$U\colon\ \mathcal{S} \to \mathbf{Vect}^{\mathcal{C}}.$$

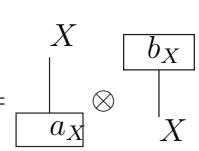
Using this functor, for each object $z \in \mathcal{C}$, we obtain a TQFT functor whenever there is a functor from $\mathcal{C}ob$ to $\mathcal{S}$. In the following section we focus on construction of a functor from $\overline{\mathcal{A}lg}$ to $\mathcal{S}$.

4. The images of the generators of the category $\mathcal{A}lg$.

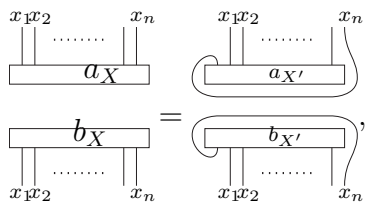
We construct a functor $\mathcal{F} : \overline{\mathcal{A}lg} \to \mathcal{S}$. We define the images of the generators of $\mathcal{A}lg$ under $\mathcal{F}$. First we need to define a certain morphism in $\mathcal{C}$: for $X \in \mathcal{C}$, let the decomposition into the simple modules be given by $X = \oplus_{\lambda,i} X_{\lambda,i}$, where for each λ, $\oplus_i X_{\lambda,i}$ is an isotropic component of X. Let $X_{\mathbf{1},i} \cong \mathbf{1}$. Then we define a morphism

$$c_X = \sum_i a_{X,i} \otimes b_{X,i} \in \mathrm{Hom}(X,\mathbf{1}) \otimes \mathrm{Hom}(\mathbf{1},X)$$

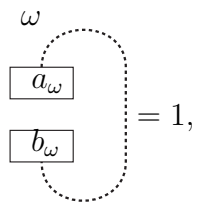
where for each i, $a_{X,i} \in \mathrm{Hom}(X, X_{\mathbf{1},i})$ is a projection, and $b_{X,i} \in \mathrm{Hom}(X_{\mathbf{1},i}, X)$ is an inclusion, such that $a_{X,i} \circ b_{X,i} = \mathrm{id}_{X_{\mathbf{1},i}}$. We express c_X by the following picture:

$c_X =$ 

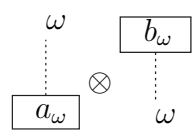
Note that c_X does not depend on the choice of $a_{X,i}$'s. Therefore we observe the relation given by the following picture:



where $X = X_1 \otimes \cdots \otimes X_n$, $X' = X_n \otimes X_1 \otimes \cdots X_{n-1}$. The following also holds:



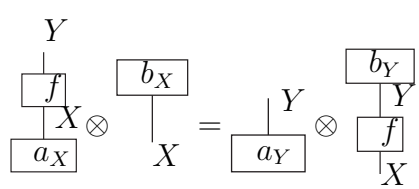
where by



we mean $\sum_{X \in \mathrm{Simple}\mathcal{C}} [X] c_X$, which is in fact the same as $a_{\mathbf{1}} \otimes b_{\mathbf{1}}$. Note that the alignment of the boxes here is irrelevant.

Moreover, the following holds:

Claim 4.1. *Let $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathrm{Hom}(Y, X)$. Then*



Proof.
Let $X = \oplus_{\alpha,i} Z_{\alpha,i}$, $Y = \oplus_{\beta,j} Z_{\beta,j}$ be the decomposition of X and Y into simple objects, where $Z_{\alpha,i} = Z_{\beta,j}$ whenever $\alpha = \beta$, and $Z_{\mathbf{1},i} = \mathbf{1}$. Let $\{a_i\}, \{b_i\}$ be so that $c_X = \sum_i a_i \otimes b_i$, and $\{a'_j\}, \{b'_j\}$ be so that $c_Y = \sum_j a'_i \otimes b'_j$ Suppose $f = 1_{(\gamma,l),(\gamma,s)} \mathrm{pr}_{\gamma,s}$, where $\mathrm{pr}_{\gamma,s}$ is the natural projection from X to $Z_{\gamma,s}$, and $1_{(\gamma,l),(\gamma,s)} \in \mathrm{Hom}(Z_{\gamma,s}, Z_{\gamma,l})$ is the identity. Since a generic morphism in $\mathrm{Hom}(Y, X)$ is a linear combination of morphisms of this form, it suffices to show the claim in this particular case. Then

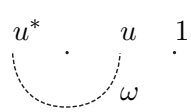
$$\begin{aligned}\mathrm{LHS} &= \sum_i a_i f \otimes b_i = \sum_i a_i (1_{(\gamma,l),(\gamma,s)} \mathrm{pr}_{\gamma,s}) \otimes b_i \\ &= a_l \circ (1_{(\gamma,l),(\gamma,s)} \mathrm{pr}_{\gamma,s}) \otimes b_l \; \delta_{\gamma,\mathbf{1}}\end{aligned}$$

$$\begin{aligned}\mathrm{LHS} &= \sum_j a'_j \otimes f b'_j = \sum_j a'_j \otimes (1_{(\gamma,l),(\gamma,s)}\mathrm{pr}_{\gamma,s}) b'_j \\ &= a'_s \otimes (1_{(\gamma,l),(\gamma,s)}\mathrm{pr}_{\gamma,s}) b'_s \ \delta_{\gamma,\mathbf{1}}\end{aligned}$$

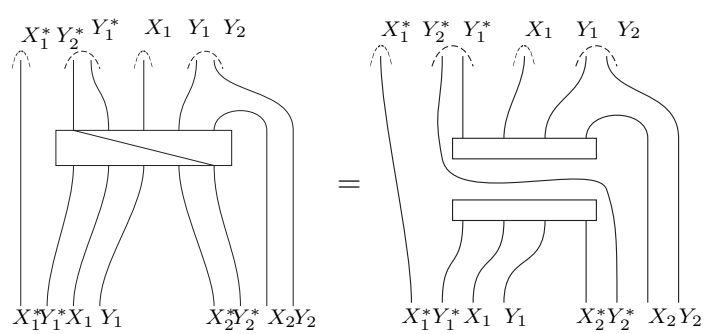
Now, let $a''_s := a_l 1_{(\mathbf{1},l),(\mathbf{1},s)}$, $b''_s := 1_{(\mathbf{1},l),(\mathbf{1},s)} b_l$. Then a''_s's and b''_s's are suitable bases of $\mathrm{Hom}(Y,\mathbf{1})$, $\mathrm{Hom}(\mathbf{1},Y)$ respectively that would give $c_Y = \sum_s a''_s \otimes b''_s$. Thus we may assume that $a'_s = a''_s$, $b'_s = b''_s$, and obtain the desired equality. Q.E.D.

Below we list the images of the generators of $\mathcal{A}lg$ in $\mathcal{S}$, as well as those of some other objects in $\mathcal{A}lg$ that are useful in later computation. For a morphism $\alpha \in \mathcal{A}lg$, we denote $\mathcal{F}(\alpha)$ by $\tilde{\alpha}$, except that the images of identity morphisms in $\mathcal{A}lg(n,n)$ for any n are denoted simply by 1. Recall that a morphism $\varphi \in \mathcal{S}(m,n)$ is determined by $\varphi_{\vec{X},\vec{Y}} \in U_n([\vec{X},\vec{Y}]) \subset \oplus_{\vec{X}',\vec{Y}' \in \mathrm{Ob}(\mathcal{C})^n} \mathcal{C}([\vec{X}',\vec{Y}'],[\vec{X},\vec{Y}])$, where $\vec{X},\vec{Y} \in \mathrm{Ob}(\mathcal{C})^m$. Since the indices are included in the picture, labeling the bottom ends of the curves, we sometimes omit the indices, especially when they are rather long. Small dotted brackets are given sometimes on the top to indicate the "grouping" of the objects. For example, $\tilde{\mu} \in \mathcal{C}([X_1, Y_1 \otimes Y_2], [(X_1,X_2),(Y_1,Y_2)])$.

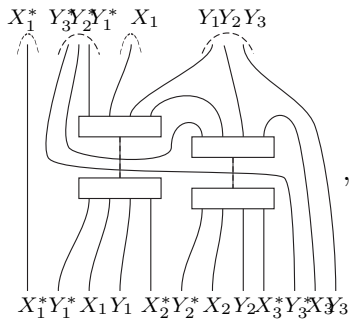
$\tilde{\eta}$ =



$\tilde{\mu}$ =



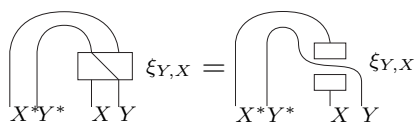
$\tilde{\mu}_3 \quad =$



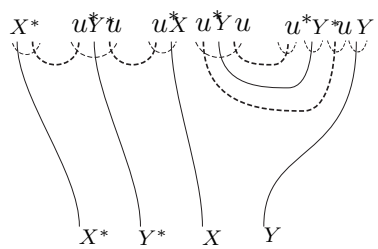
,

where $\mu_3 = \mu(\mu \otimes 1)$, and the dotted vertical lines merely indicate pairing.

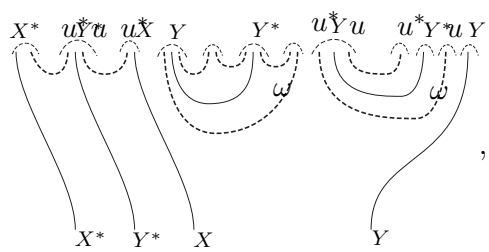
$\tilde{\varepsilon}_{X,Y} \quad =$



$\tilde{\Delta}_{X,Y} \quad =$



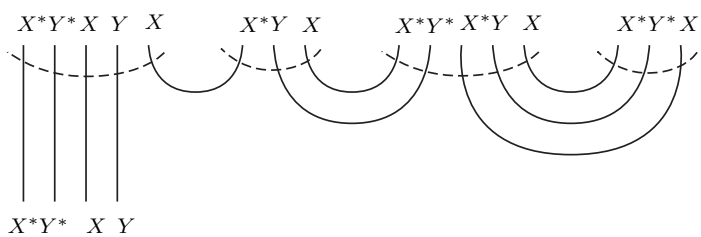
$(\tilde{\Delta}_3)_{X,Y} \quad =$



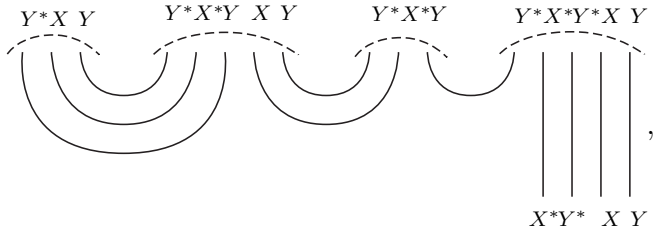
,

where $\Delta_3 = (1 \otimes \Delta)\Delta$,

$\tilde{S}_{X,Y} \quad =$

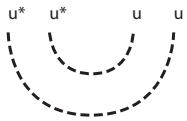


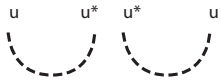
,

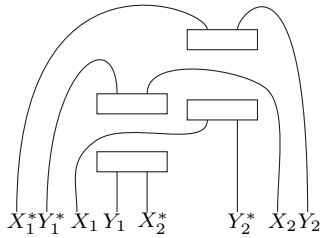
$\tilde{S}^{-1}_{X,Y} =$ 

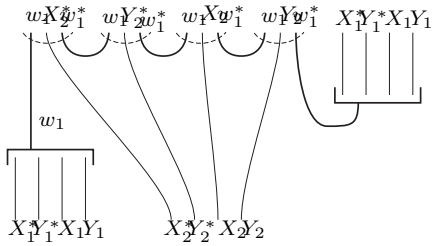
,

$\tilde{c} =$ 

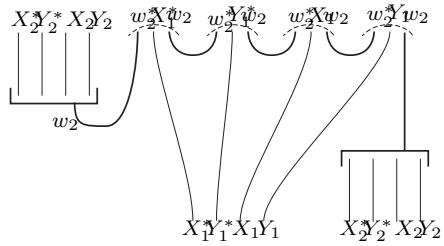
$\tilde{v}^+ =$ 

$\tilde{v}^- =$ 

$\tilde{d}_{X_1,Y_1,X_2,Y_2} =$ 

$\tilde{\psi}_{X_1,Y_1,X_2,Y_2} =$ 

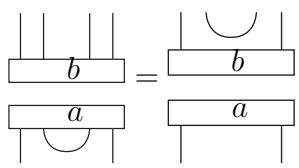
$\tilde{\psi}^{-1}_{X_1,Y_1,X_2,Y_2} =$



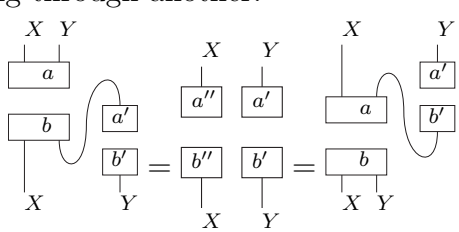
5. Well-definedness of the functor $\mathcal{F}$ on $\overline{\mathcal{A}lg}$

In this section we prove that the images of the generators of $\mathcal{A}lg$ under $\mathcal{F}$ satisfies the relations introduced in Theorem 2.3, thus prove that $\mathcal{F}$ gives a well-defined functor from $\overline{\mathcal{A}lg}$ to $\mathcal{S}$. By Theorem 2.3, this implies that we have a TQFT functor from $\mathcal{C}ob$ to $\mathcal{S}$. Before starting the proof, we list typical moves that give equalities of diagrams.

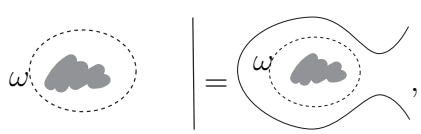
- Push up:



- Pull-down: given by the upside-down picture of above.
- A box going through another:



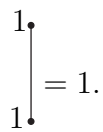
- Handle slide:



(the gray "cloud" indicate that the dotted circle does not necessarily bound a disk.)

- Contraction of the boxes as in the equation 4.

•

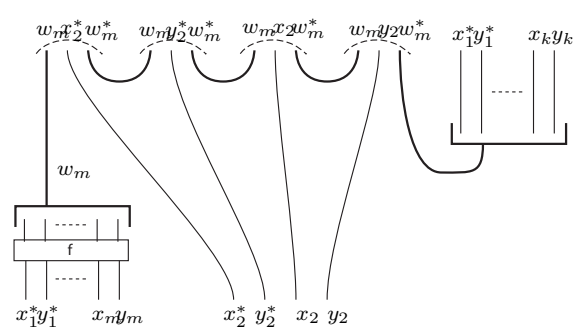


$= 1.$

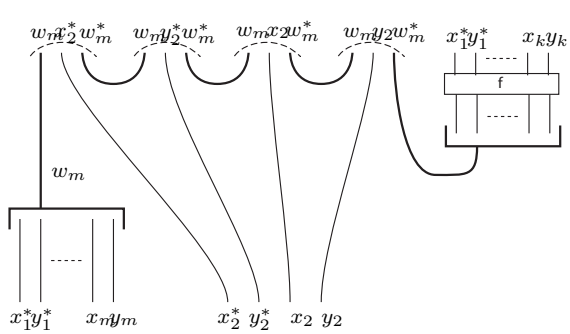
Note that, as explained in the beginning of the section 4, each diagram should be understood as a diagram drawn on a punctured sphere with a proper genus, and one should note that each topological move may involve some movement going through the handles which are not drawn. When there is a complicated deformation of arcs involving push-up's, running through handles, shading is utilized to indicate the region that the arc is to wipe through. Explanation of the deformation of a diagram is given following the resulting diagram.

Relation (1):

$\tilde{\psi}(\tilde{f} \otimes 1) =$



$=$

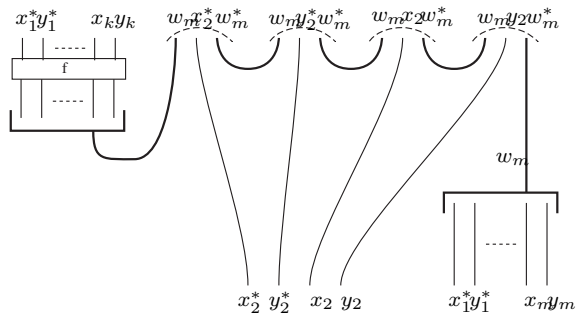


(f runs through)

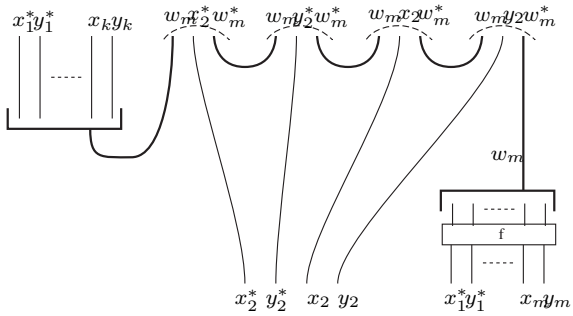
$= (1 \otimes \tilde{f})\tilde{\psi}$

Relation (2):

$(\tilde{f} \otimes 1)\tilde{\psi} =$



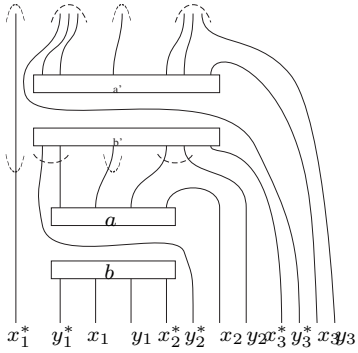
$=$



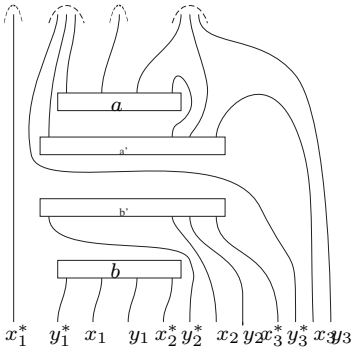
(f runs through)
$= \tilde{\psi}(1 \otimes \tilde{f})$

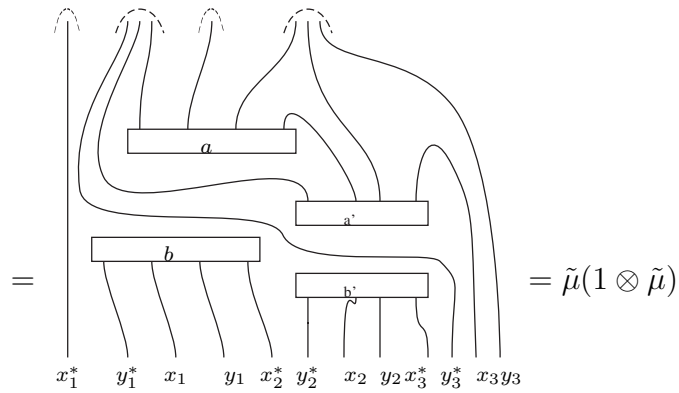
Relation (3):

$\tilde{\mu}(\tilde{\mu} \otimes 1) =$

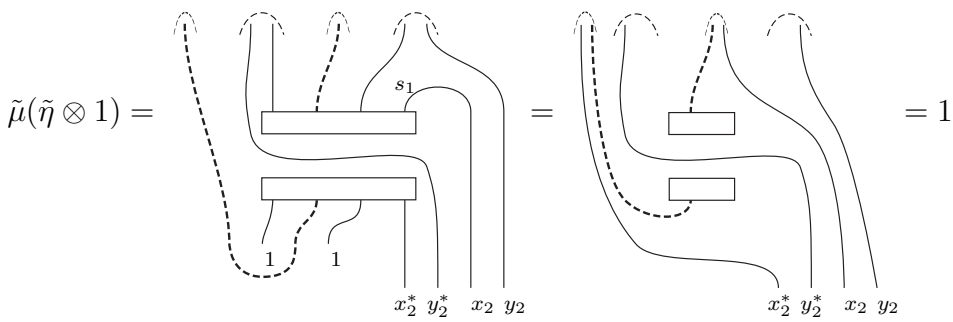


$=$

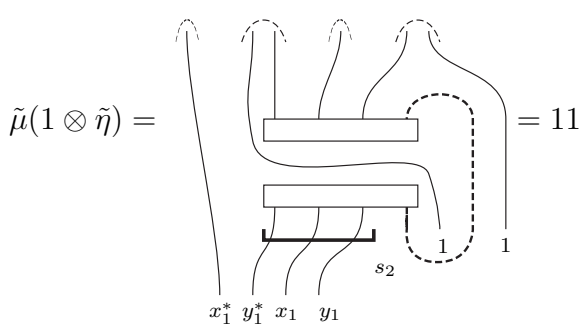




Relation (4):

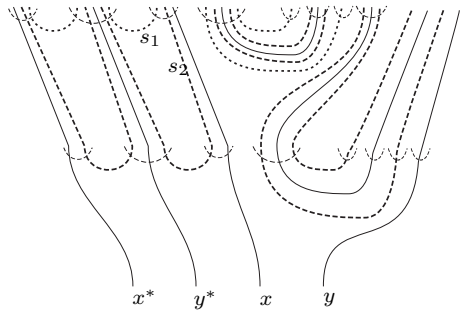


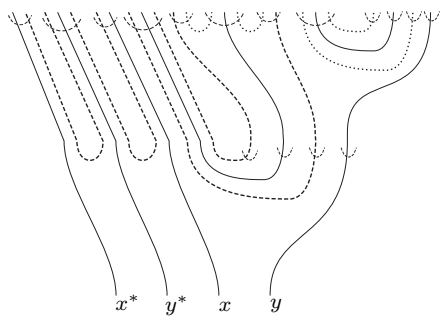
(contract 1, handleslide s_1, contract the boxes).



(contract 1, handleslide s_2, contract the boxes.)

Relation (5):

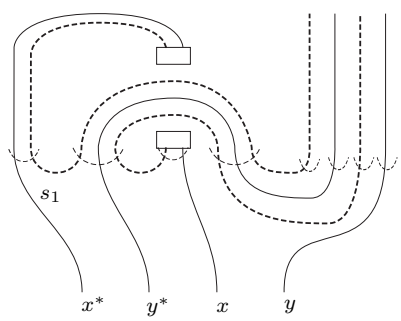
$(\tilde{\Delta} \otimes 1)\tilde{\Delta} =$ 

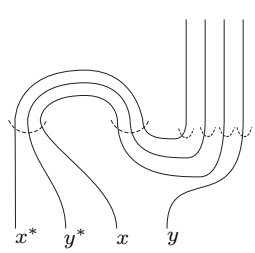
$=$ 

(slide s_2 along s_1)

$= \tilde{\Delta}(1 \otimes \tilde{\Delta})$

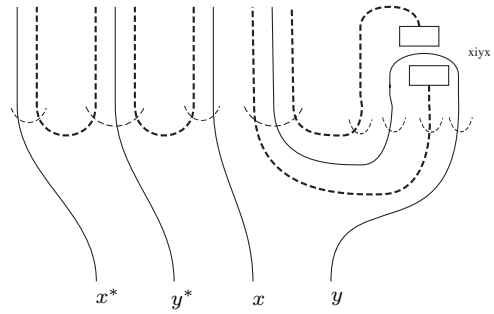
Relation (6):

$(\tilde{\varepsilon} \otimes 1)\Delta =$ 

$=$ 

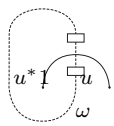
(slide s_1, contract the boxes)

$= 1.$

$(1 \otimes \tilde{\varepsilon})\Delta =$ 

$= 1$ (contract the boxes)

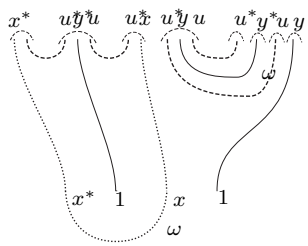
Relation (7):

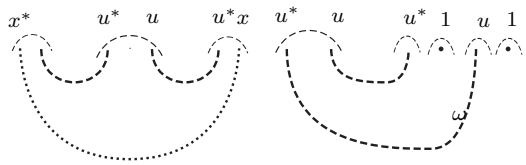
$\tilde{\varepsilon}\tilde{\eta} =$ 

$= \emptyset,$

using Relation (21) and (A).

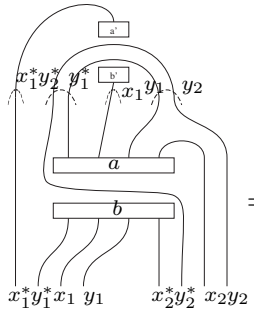
Relation (8):

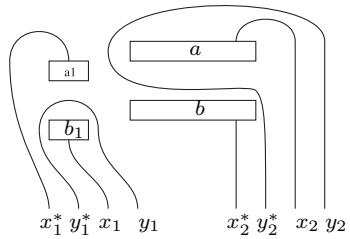
$\tilde{\Delta}\tilde{\eta} =$ 

$=$ 

$= \tilde{\eta} \otimes \tilde{\eta}.$

Relation (9):

$\tilde{\varepsilon}\tilde{\mu} =$ 

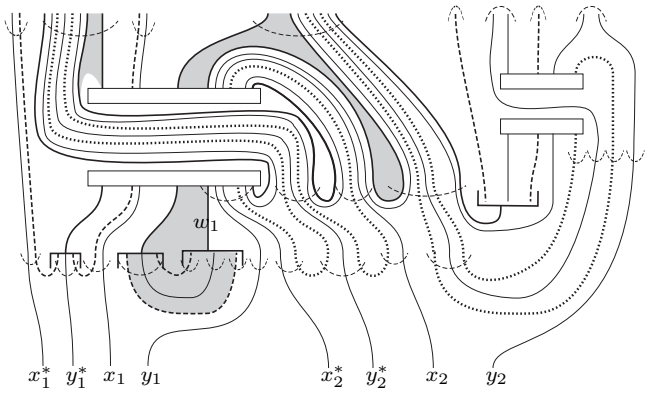
$=$ 

$= \tilde{\varepsilon} \otimes \tilde{\varepsilon}.$

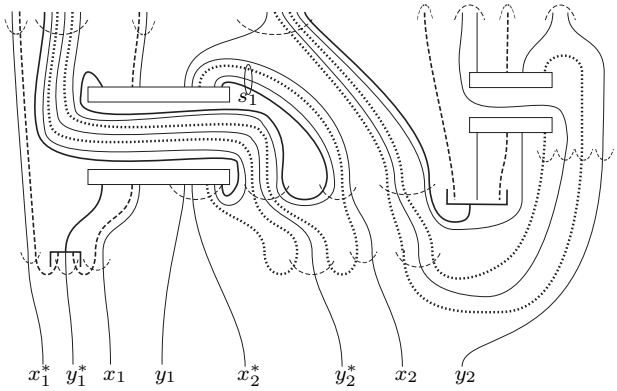
Relation (10):

$(\tilde{\mu} \otimes \tilde{\mu})(1 \otimes \tilde{\psi} \otimes 1)(\tilde{\Delta} \otimes \tilde{\Delta})$

=

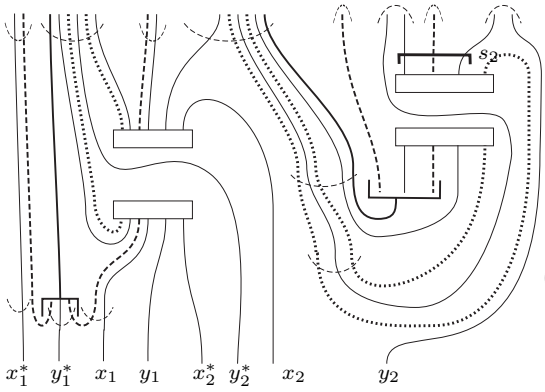


=

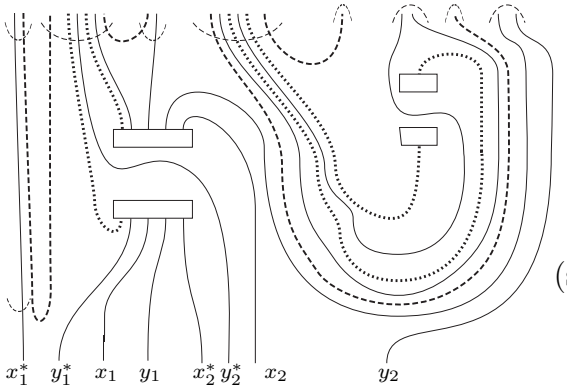


(bring s_1 to the left)

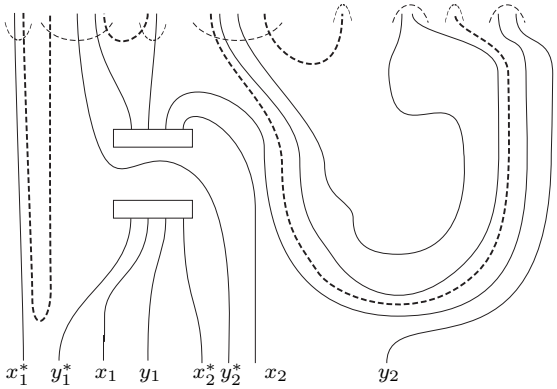
=

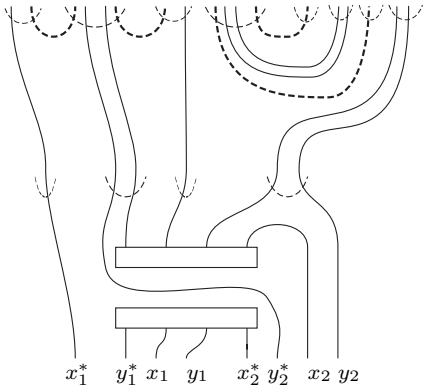


(bring s_1 to the left)

$=$ 

(slide s_2)

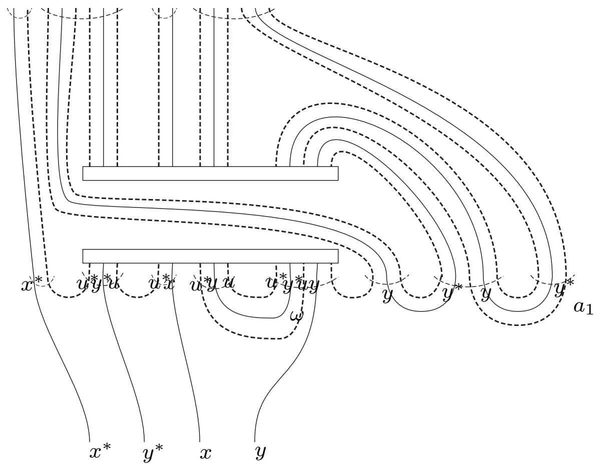
$=$ 

$=$ 

$= \tilde{\Delta}\tilde{\mu}$

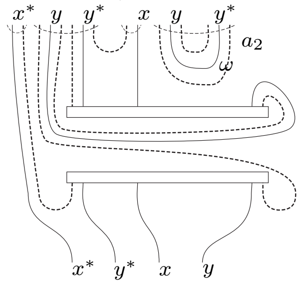
Relation (11):

$\tilde{\mu}(1 \otimes S)\tilde{\Delta} =$

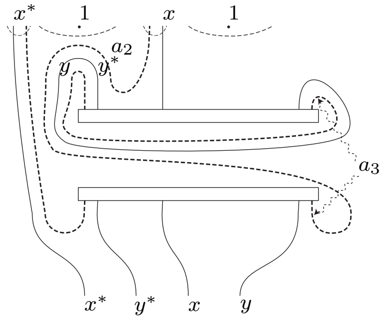


(contract a_1)

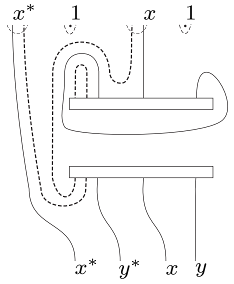
$=$



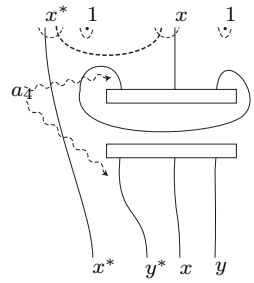
$=$

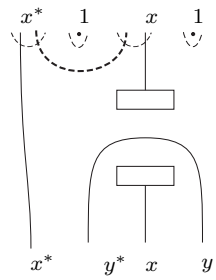


$=$

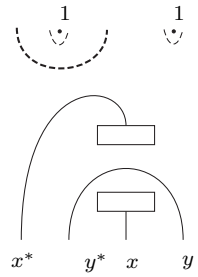


(bring a_3 to the left)

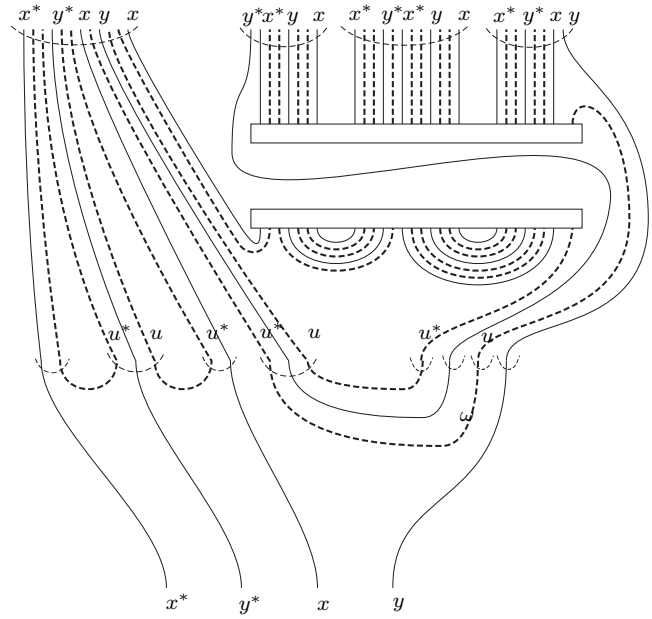
$=$ 

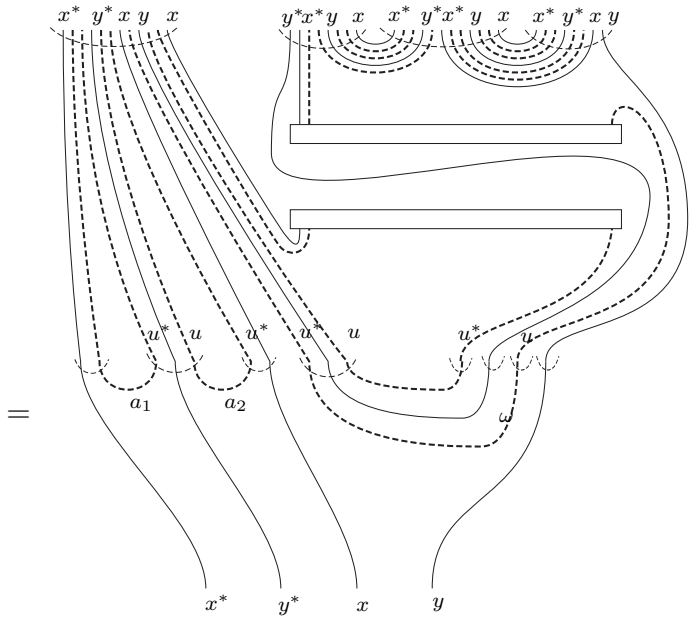
$=$ 

(bring a_4 to the right)

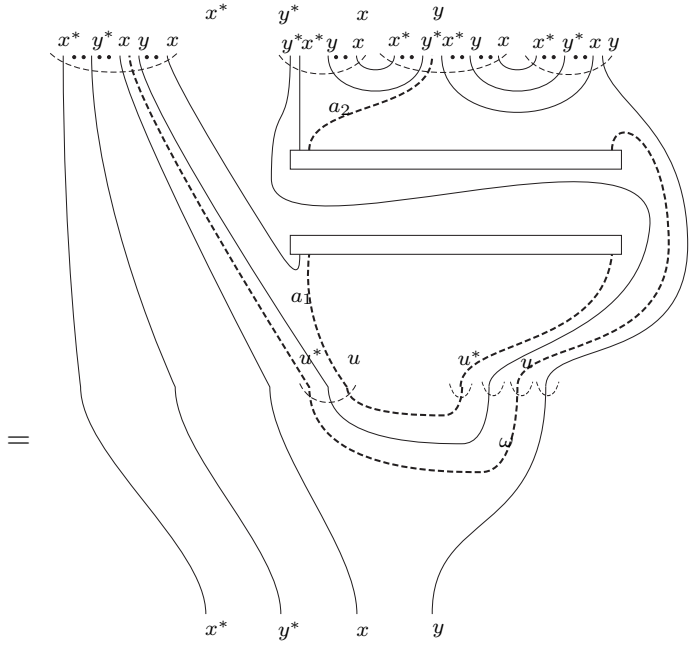
$=$ 

$= \tilde{\eta}\tilde{\varepsilon}$

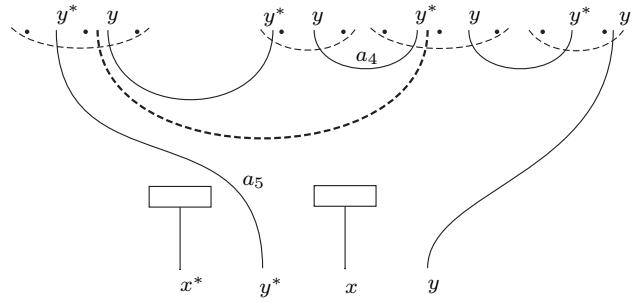
$\tilde{\mu}(\tilde{S} \otimes 1)\tilde{\Delta} =$ 

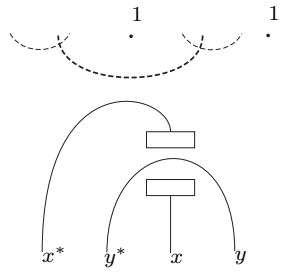

x^* y^* x y x
y^* x^* y x x^* y^* x^* y x x^* y^* x y
u^* u u^* u^* u u^* u
a_1 a_2
ω
x^* y^* x y

=


x^* y^* x y x
y^* x^* y x x^* y^* x^* y x x^* y^* x y
a_2
a_1
u^* u u^* u
ω
x^* y^* x y

=

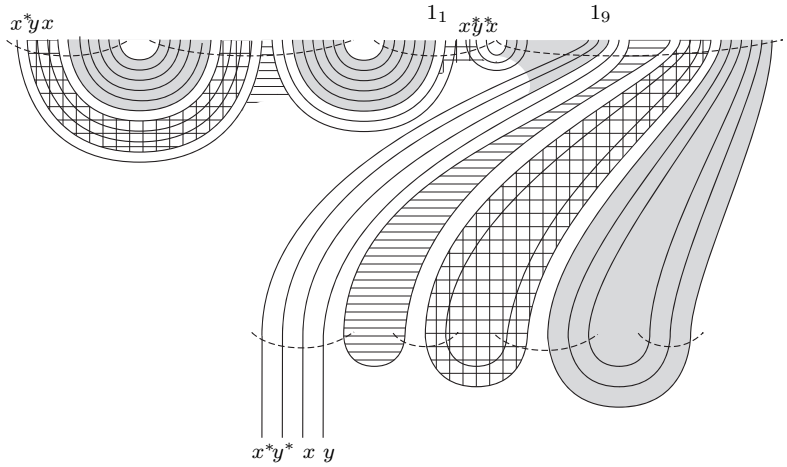
$=$ 

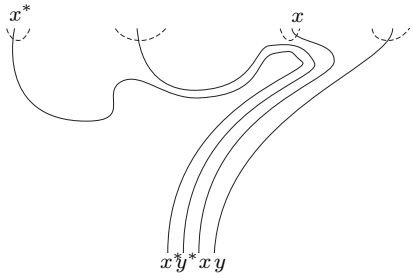
$=$ 

$= \tilde{\eta}\tilde{\varepsilon}.$

(slide a_4 a_5 over ω, pull them)

Relation (12):

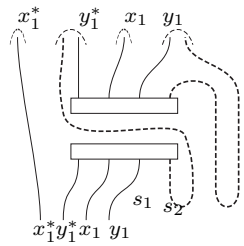
$\tilde{S}^{-1}\tilde{S} =$ 

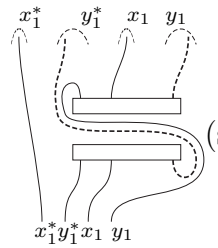
$=$ 

$= 1.$

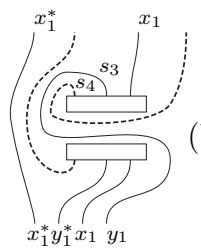
Since the diagrams for S and S^{-1} are totally symmetric, the proof for $SS^{-1} = 1$ is given by the horizontal reflection of the above proof.

Relation (13):

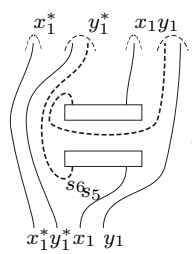
$\tilde{\mu}(1 \otimes \tilde{v}_-) =$ 

$=$ 

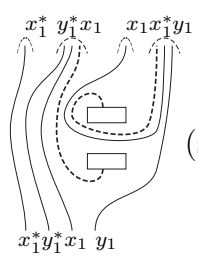
(slide s_1 through s_2)

$=$ 

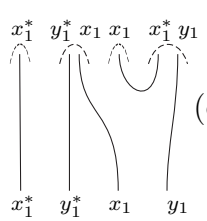
(bring the right ends of the boxes to the left)

$=$ 

(slide s_3 through s_4)

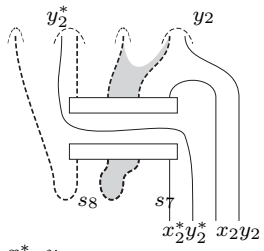
$=$ 

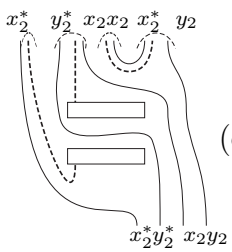
(slide s_5 through s_6)

$=$ 

(contract the boxes) $- -(\sharp)$.

On the other hand

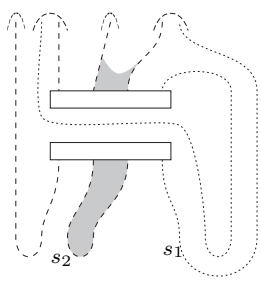
$\tilde{\mu}(\tilde{v}_- \otimes 1) =$ 

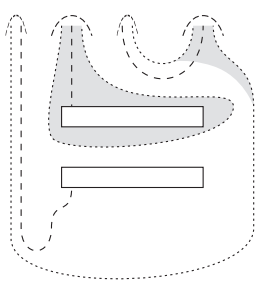
$=$ 

(contract the shade, slide s_7 through s_8)

$= (\sharp)$ (contract the boxes)

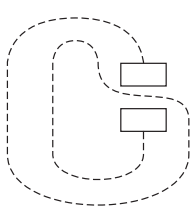
Relation (14):

$\tilde{\mu}(\tilde{v} \otimes \tilde{v}^{-1}) =$ 

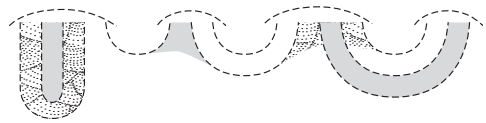
$=$ 
(contract the shade, slide s_1 through s_2)

$= \tilde{\eta}$ (contract the boxes, contract the shade)

Relation (15):

$\tilde{v}\tilde{\varepsilon} =$  $= \emptyset.$

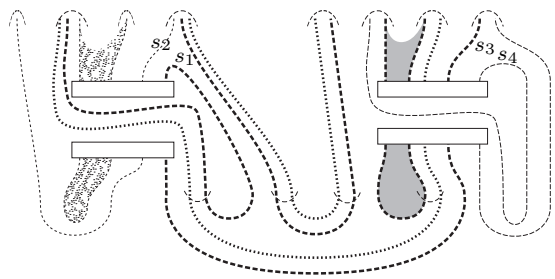
Relation (16):

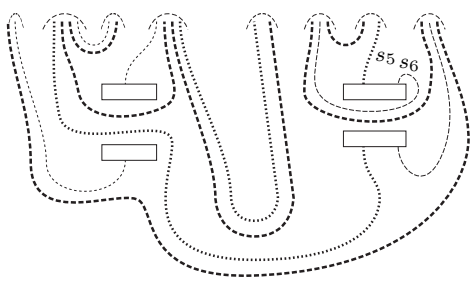
$\tilde{v}\tilde{S} =$ 

$=$  $= \tilde{v}.$

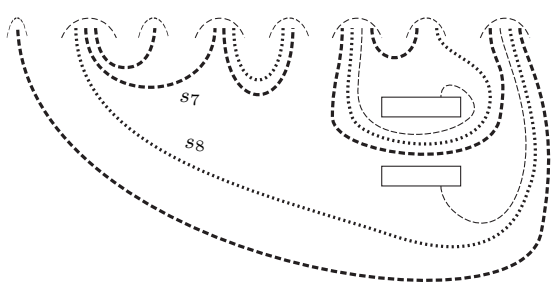
(contract the shaded regions)

Relation (17):

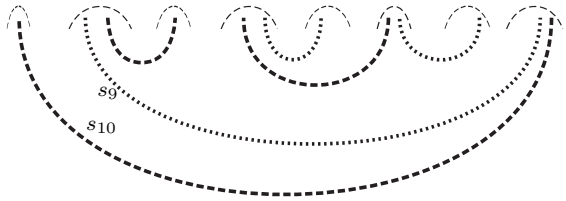
$(\tilde{\mu} \otimes \tilde{\mu})(1 \otimes \tilde{c} \otimes 1)(\tilde{v} \otimes \tilde{v}) =$ 

$=$ 

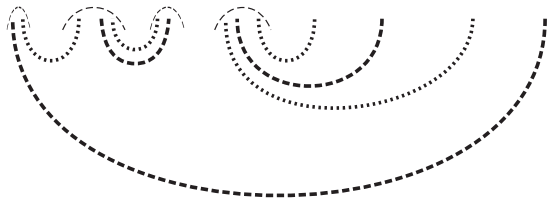
(slide s_1 through s_2, s_3 through s_4)

$=$ 

(slide s_5 through s_6)

$=$ 

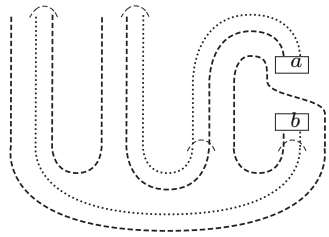
(contract the boxes, slide s_7 through s_8)

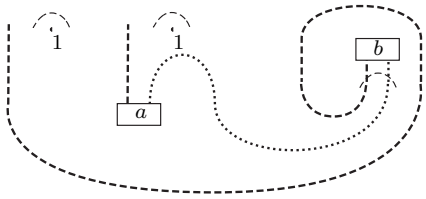
$=$ 

(contract the boxes, slide s_9 through s_{10})

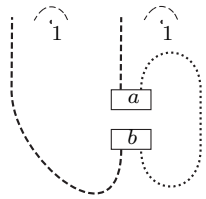
$= \tilde{\Delta}\tilde{v}.$

Relation (18):

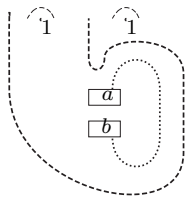
$(1 \otimes \tilde{\varepsilon})\tilde{c} =$ 

$=$ 

(run a)

$=$ 

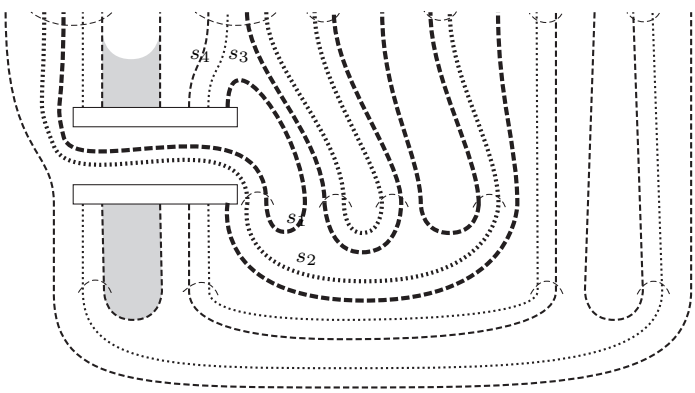
(rotate b)

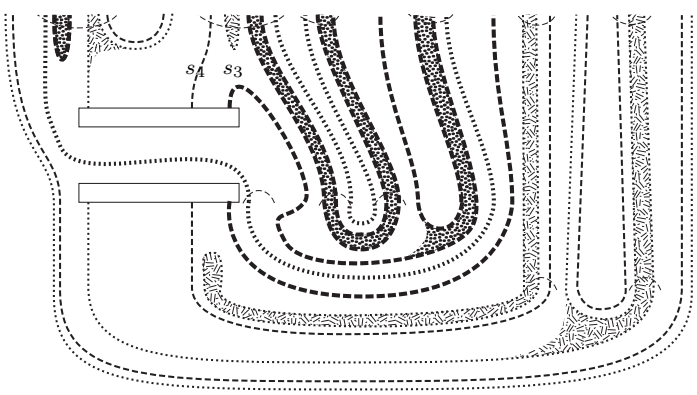
$=$ 

(handle slide) $= \tilde{\eta}$.

Relation (19):

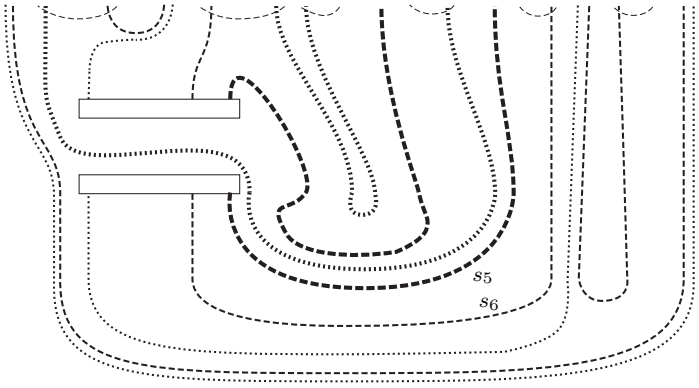
$(\tilde{\mu} \otimes 1^{\otimes 2})(1 \otimes \tilde{c} \otimes 1)c$

$=$ 

$=$ 

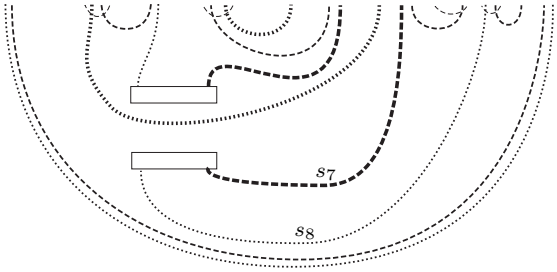
(handleslide s_1 through s_2, s_3 through s_4)

=



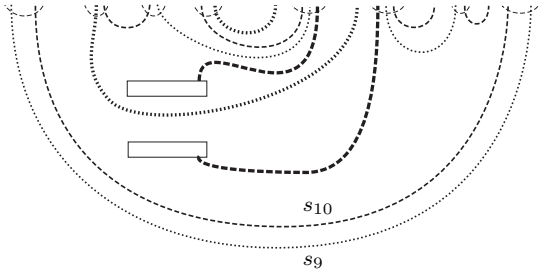
(pull)

=



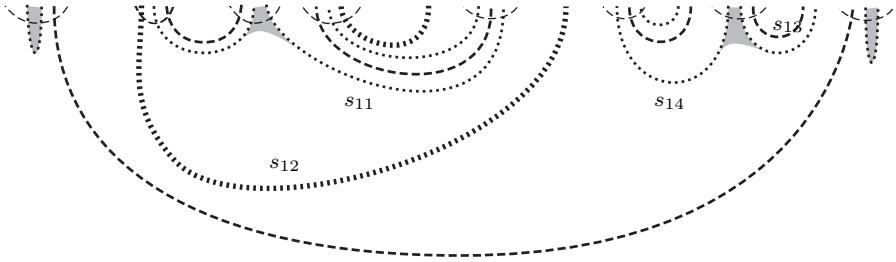
(slide s_6 through s_5, pull)

=

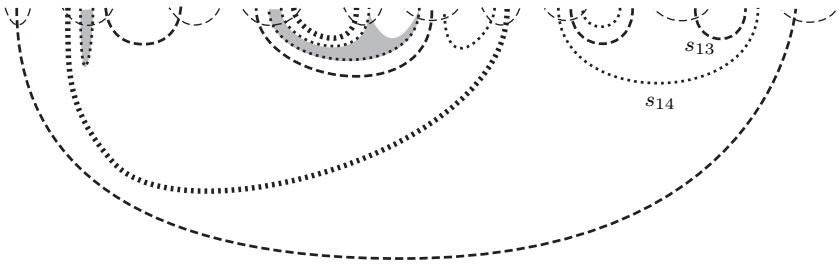


(slide s_8 through s_7, pull)

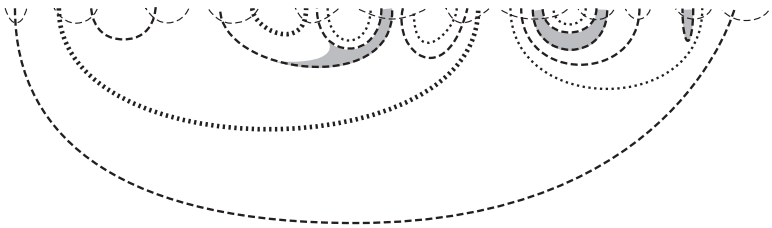
=



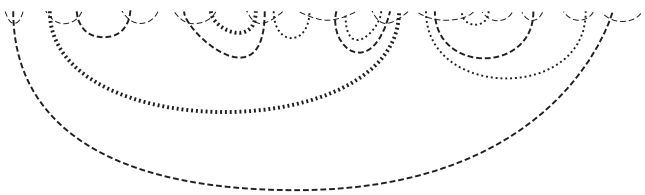
(slide s_9 through s_{10})

= 

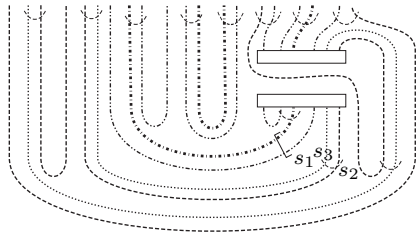
(slide s_{11} through s_{12})

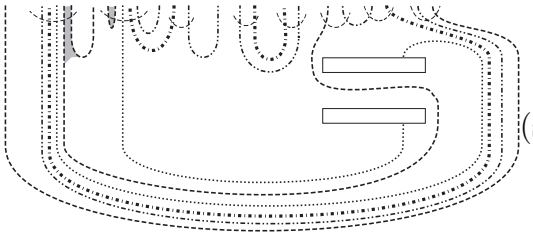
= 

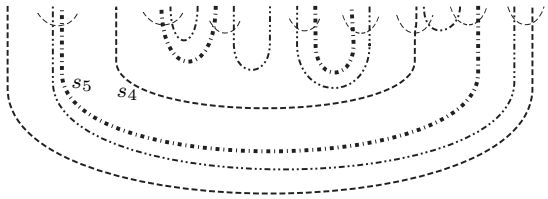
(slide s_{13} through s_{14})

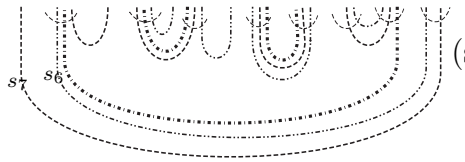
=  $= (1 \otimes \tilde{\Delta})\tilde{c}.$

Relation (20):

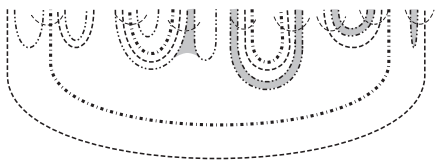
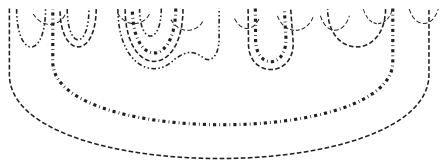
$(1^{\otimes 2} \otimes \tilde{\mu})(1 \otimes \tilde{c} \otimes 1)\tilde{c} =$ 

=  (slide s_1, s_2 through s_3)

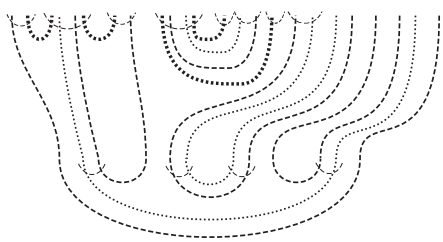
$=$ 

$=$ 

(slide s_4 over s_5)

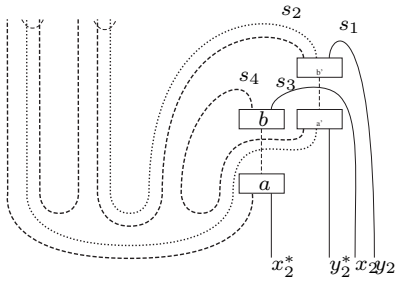
$=$  $=$ 

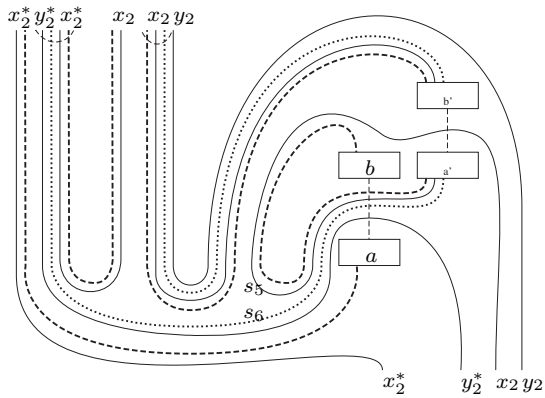
(slide s_6 over s_7)

$=$  $= (\tilde{\Delta} \otimes 1)\tilde{c}$

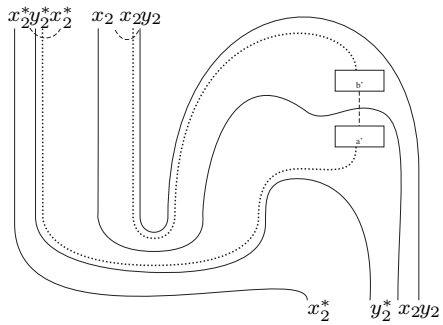
Relation (21):

$(1 \otimes \tilde{d})(\tilde{c} \otimes 1)$

$=$ 

$=$ 

(slide s_1 through s_2, s_3 through s_4)

$=$ 

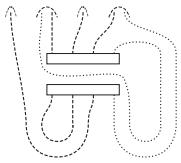
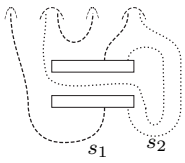
(contract a-b boxes, slide s_5 through s_6)

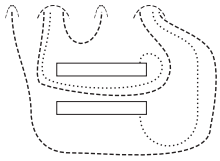
$= 1$. (contract a'-b' boxes.)

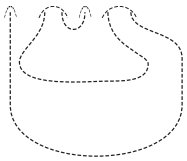
Relation (22):

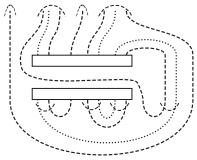
$(\tilde{d} \otimes 1)(1 \otimes \tilde{c}) = 1$ is clear from the proof of Relation (21) and the symmetry of $\tilde{d}$ and $\tilde{c}$.

Relation (23):

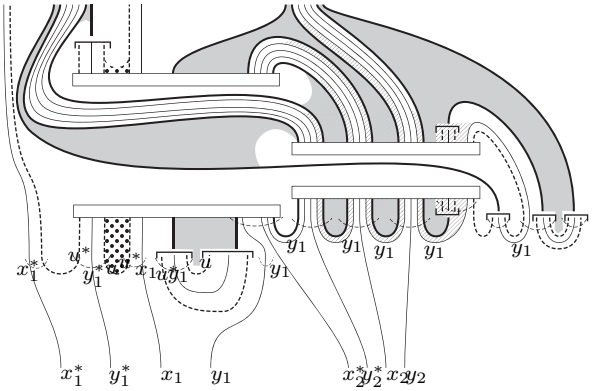
$\tilde{\mu}(\tilde{v} \otimes \tilde{v}) =$  $=$ 

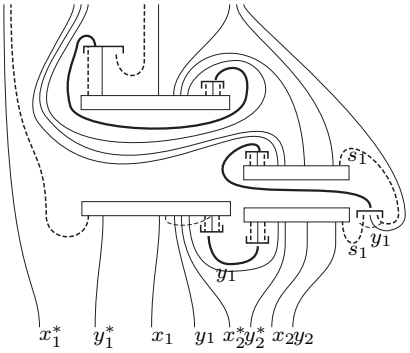
$=$  (slide s_1 through s_2)

$=$  (contract the boxes)

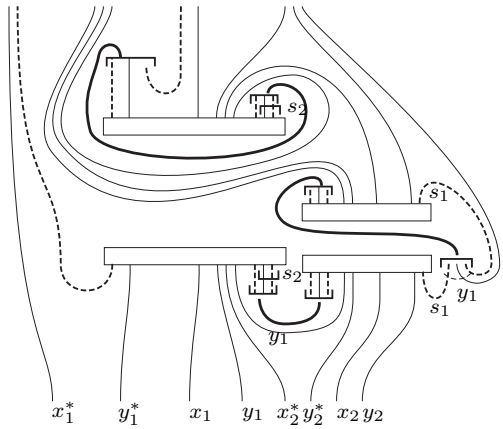
$=$  $= \tilde{\mu}\tilde{c}$

Relation (24): We start with simplifying the diagram $\tilde{L}$.

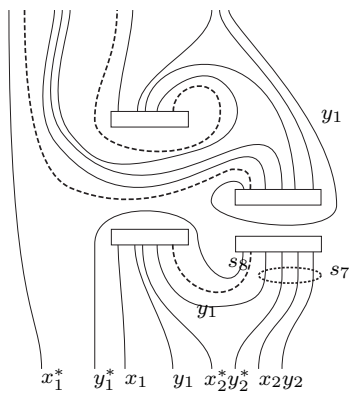
$\tilde{L} =$ 

$=$ 

=

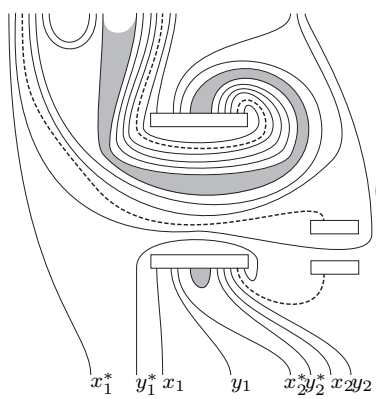


=

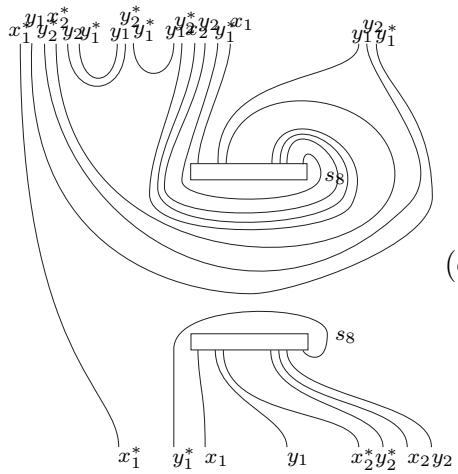


(bring s_1, s_2 to the left of each pair of boxes and pull the strings.)

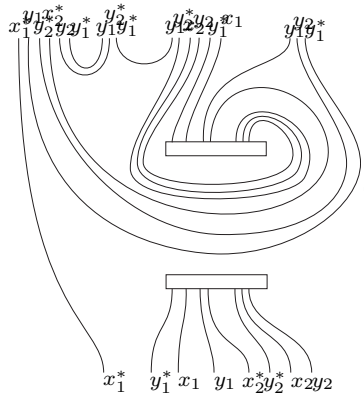
=



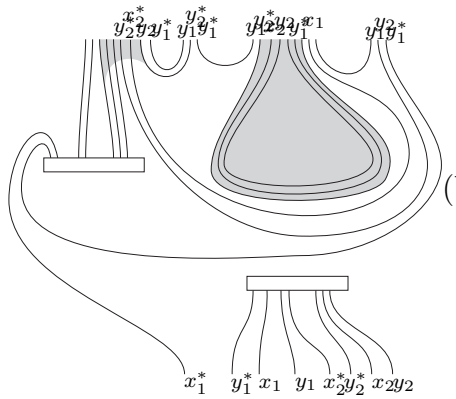
(slide s_7 through s_8.)

$=$ 

(contract the shaded reason)

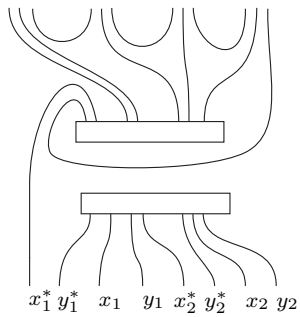
$=$ 

(bring s_8 to the left)

$=$ 

(bring the box through the arm)

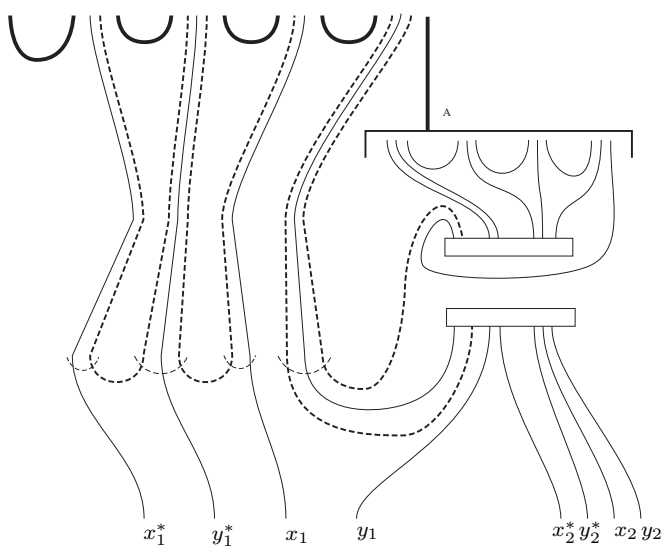
$=$



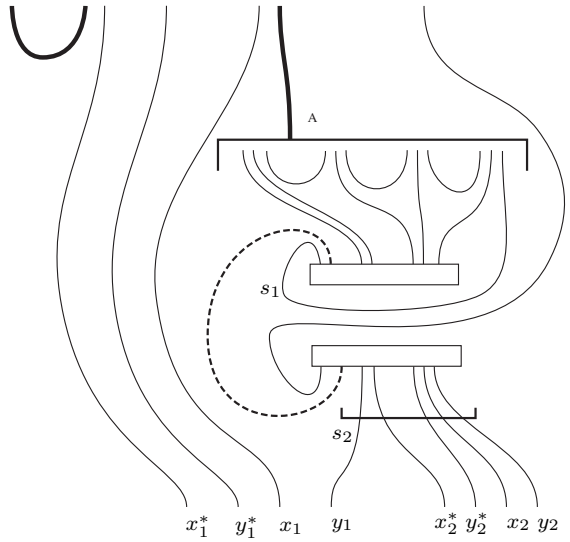
Using the simplification we proceed as follows:

$\tilde{\psi}^{-1}(1 \otimes \tilde{L})(\tilde{\Delta} \otimes 1)$

$=$



=



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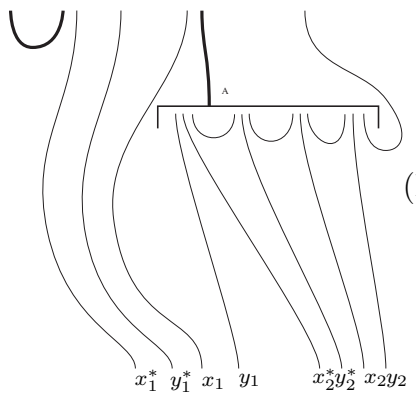
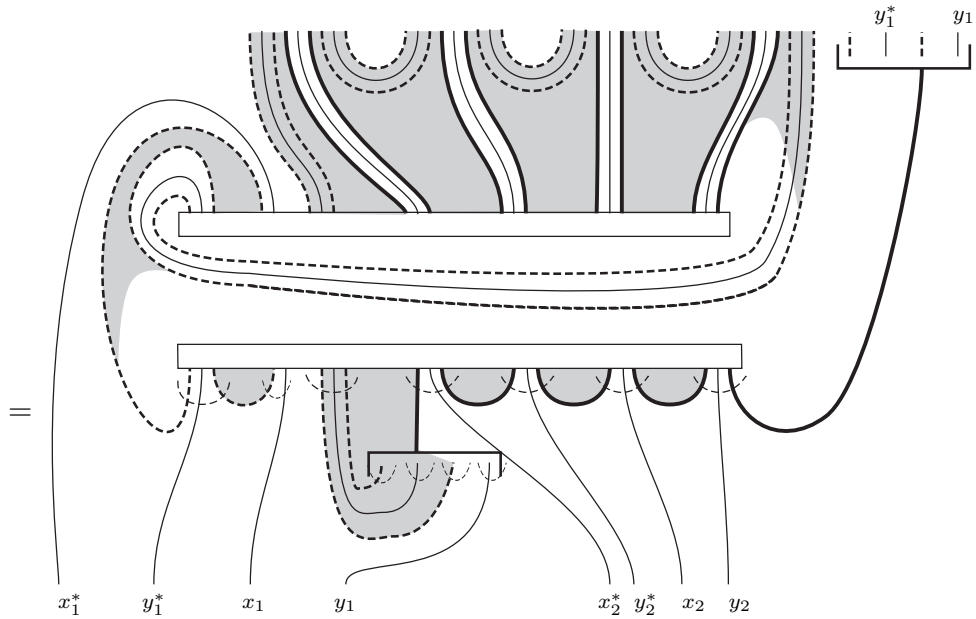
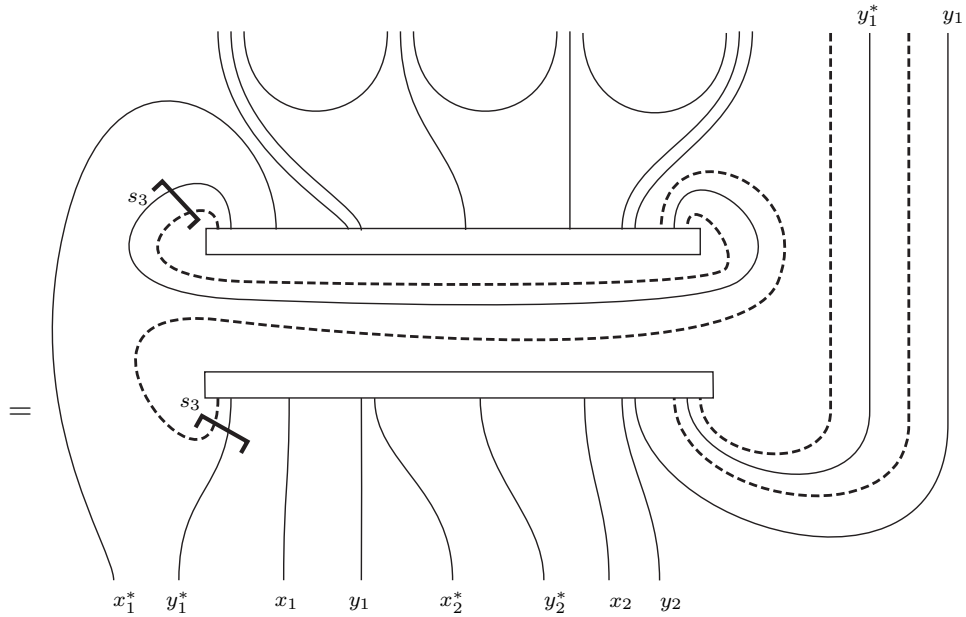
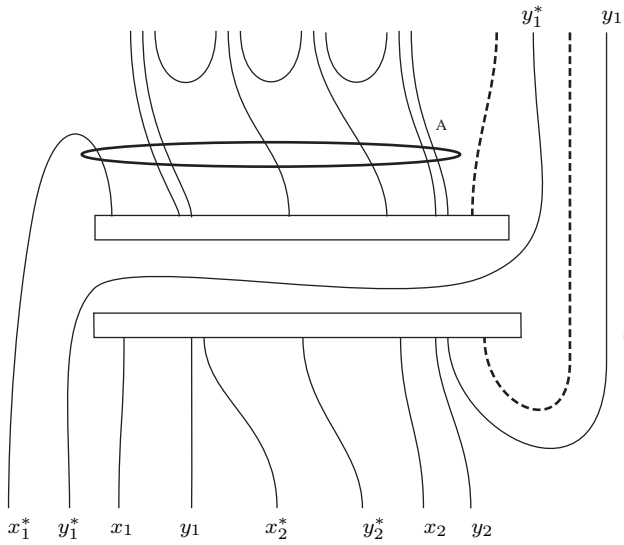


(handleslide s_1, s_2)

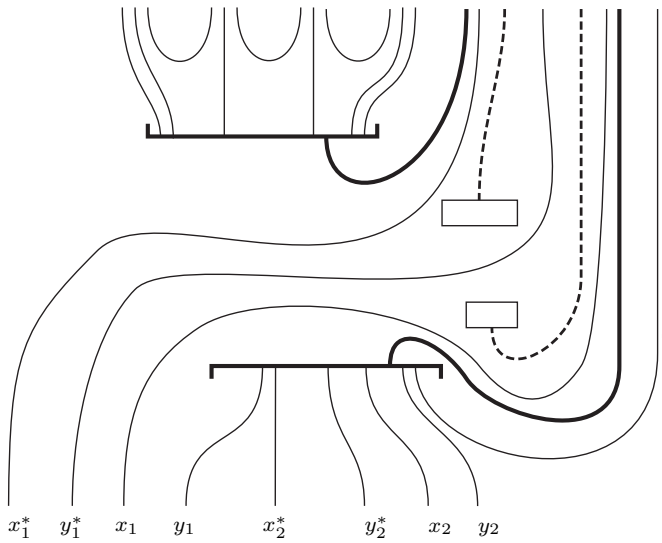
$(\tilde{L} \otimes 1)(1 \otimes \tilde{\psi})(\tilde{\Delta} \otimes 1)$


y_1^*
y_1
=
x_1^*
y_1^*
x_1
y_1
x_2^*
y_2^*
x_2
y_2


y_1^*
y_1
s_3
=
s_3
x_1^*
y_1^*
x_1
y_1
x_2^*
y_2^*
x_2
y_2

$=$ 

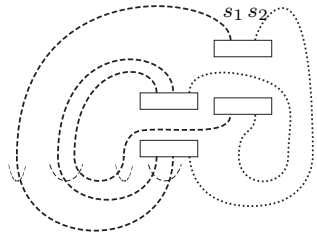
(bring s_3 to the right)

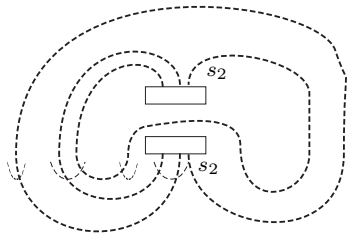
$=$ 

(handleslide A)

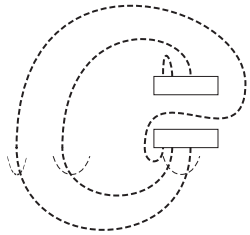
$= \tilde{\psi}^{-1}(1 \otimes \tilde{L})(\tilde{\Delta} \otimes 1)$. (contract the boxes)

Relation (25):

$\tilde{d}(\tilde{\mu}(\tilde{v} \otimes \tilde{v}) \otimes \tilde{v}) =$ 

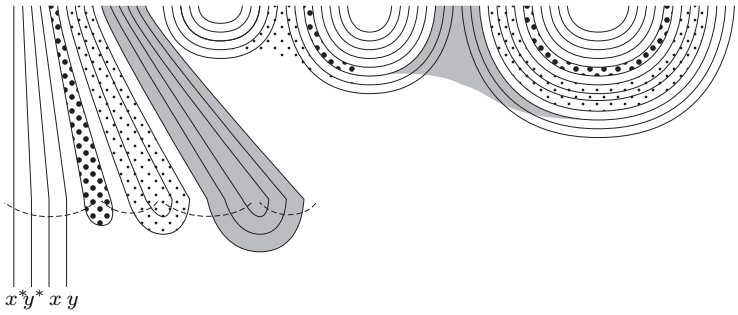
$=$ 

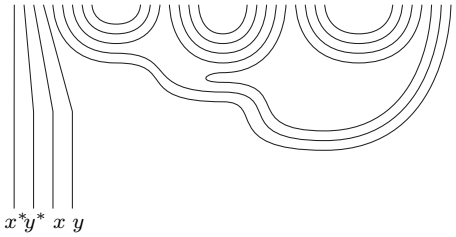
(slide s_1 though s_2, contract the box)

$=$ 
(Bring s_2 to the left)

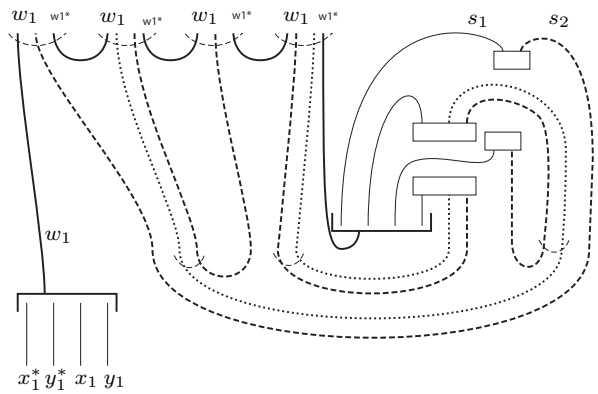
$= 1$

Relation (26):

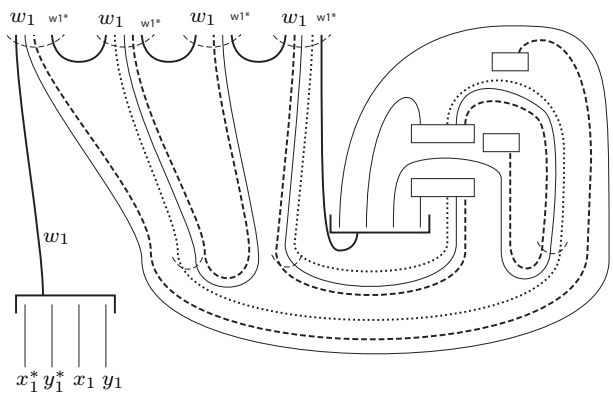
$\tilde{S}^2 =$ 

$=$ 

$(1 \otimes \tilde{d})(\tilde{\psi} \otimes 1)(1 \otimes \tilde{c}) =$

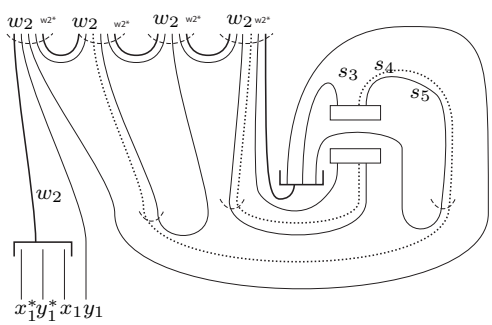


$=$



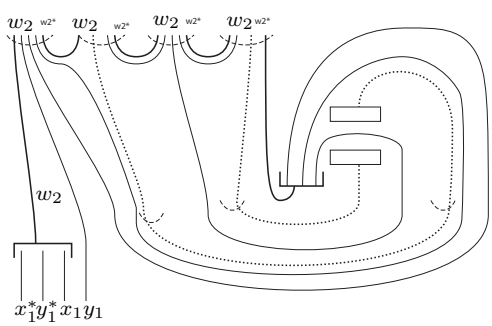
(slide s_1 through s_2).

$=$



(contract the box)

$=$



(slide s_3 and s_5 along s_4)

$= \tilde{S}^2$. (contract the boxes)

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