Math 131 - HW 5

Read Sections 3.B and 3.C.

1. (a) Find an example of a linear map $S : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\text{range}(S) = \text{null}(S)$.

   (b) Show that there is NO linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $\text{range}(T) = \text{null}(T)$.

2. Suppose that a linear map $T : V \to W$ is injective and that $v_1, \ldots, v_n$ is linearly independent in V. Prove that $T(v_1), \ldots, T(v_n)$ is a linearly independent list in W.

3. Give a proof or counterexample: If $T : V \to W$ is a linear map and $v_1, \ldots, v_n$ spans V, then $T(v_1), \ldots, T(v_n)$ spans $\text{range}(T)$.

4. Suppose that V and W are both finite-dimensional vector spaces. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

5. Suppose that V is finite dimensional and that T is a linear map from V to W. Prove that there is a subspace U in V such that

$$U \cap \text{null}(T) = \{0\} \quad \text{and} \quad \text{range}(T) = \{T(u) \ : \ u \in U\}.$$

   (Hint: for inspiration constructing such a U look back at how we proved the Fundamental Theorem of Linear Maps.)

6. Let $T : V \to W$ be a linear map, where V is finite dimensional. Prove that T is surjective if and only if there exists a linear map $S : W \to V$ such that the composition TS is the identity map on W.

7. (extra credit) Let $T_1, T_2 \in \mathcal{L}(V, W)$ and suppose that W is finite dimensional. Prove that

$$\text{null}(T_1) \subset \text{null}(T_2)$$

   if and only if there exists a linear map $S : W \to W$ such that $ST_1 = T_2$.