Math 131 - HW 8

Read Section 3.E and 3.F (optional).

1. Suppose $T : V \to W$ is a function. Recall that the graph of T is the subset of $V \times W$ defined as:

$$\text{graph}(T) = \{(v, w) \in V \times W | w = T(v)\}.$$

   Prove that graph(T) is a subspace of $V \times W$ if and only if T is a linear map.

2. (a) Suppose that V is a vector space over $\mathbb{F}$. Prove that there is an isomorphism between $\mathcal{L}(\mathbb{F}, V)$ and V. (Hint: given a linear map $T : \mathbb{F} \to V$, consider the vector $T(1)$...)

   (b) Suppose that $V_1, \ldots, V_m, W$ are vector spaces. Prove that there is an isomorphism between $\mathcal{L}(V_1 \times \ldots \times V_m, W)$ and the vector space $\mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$.

3. (a) Suppose V is a finite dimensional vector space and U, W are subspaces. Prove that if $U \times W$ is isomorphic to $U + W$, then $U \cap W = \{0\}$.

   (b) Find a vector space V and subspaces U, W such that $U \times W$ is isomorphic to $U + W$, but $U \cap W \neq \{0\}$. (Note that V must be infinite dimensional!)

4. Suppose $\phi : V \to \mathbb{F}$ is a linear map and $u \in V$ such that $\phi(u) \neq 0$. Prove that

$$V = \text{null}(\phi) \oplus \{au \mid a \in \mathbb{F}\}.$$