Math 131 - Extra Credit

This is an extra credit assignment/ exam review! To get extra credit for it, you must work on the problems on your own (sorry - no collaborating on this one) and turn in your solutions by Thursday at the start of class.

1. Find the matrix representing the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = (-2y - 3z, 5x - 6y)$$

with respect to the standard basis of $\mathbb{R}^3$ and the basis $(1, 1), (1, -1)$ of $\mathbb{R}^2$.

2. Write down a basis of each of the following vector spaces. Write all elements in your basis explicitly with no abbreviations.

   (a) The space $\mathcal{P}_5(\mathbb{R})$ of polynomials in the variable t of degree at most 5 with real coefficients.

   (b) $Mat_{2,3}(\mathbb{R})$.

   (c) $\{(a, b, c, d, e) \in \mathbb{R}^5 \mid a + 4c = 0,\ 3b + d - e = 0\}$

3. Let V and W be vector spaces and $f : V \to W$ be a linear map. Show that if $\text{null}(f) = \{0\}$ then f is injective.

4. Let V and W be vector spaces over $\mathbb{F}$ and let $f : V \to W$ be a linear map. Prove that the range of f,

$$\text{range}(f) := \{y \in W \mid \text{there exists } x \in V \text{ such that } f(x) = y\}$$

is a subspace of W.

5. Let M be a subspace of the finite dimensional vector space V . Use the theorem that any linearly independent set can be extended to a basis to:

   (a) Show that $\dim M \leq \dim V$.

   (b) Show that if $\dim V = \dim M$, then $M = V$.

6. Let $f : V \to W$ be a linear map such that $\dim V = \dim W = n \leq \infty$. Show that if $\dim \text{range}(f) = \dim W$ then f is an isomorphism.

7. Let V be a vector space over $\mathbb{R}$ and let $y, x_1, x_2, x_3 \in V$. Show that if $y \in \text{Span}\{x_1, x_2, x_3\}$ then $\text{Span}\{y, x_1, x_2, x_3\} \subset \text{Span}\{x_1, x_2, x_3\}$.

8. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. Assume that $\{x_1, x_2, ..., x_n\}$ is a basis of W_1 and $\{y_1, y_2, ..., y_m\}$ is a basis of W_2.

   (a) Prove that $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ are linearly independent.

   (b) Prove that $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ span V .