Math 131, Fall 18
Discussion Section Worksheet 6

Let V, W be vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear map.
Recall:

$$\operatorname{null}(T) = \{v \in V \ : \ T(v) = 0\} \subset V$$

$$\operatorname{range}(T) = \{T(v) \ : \ v \in V\} \subset W$$

We proved that $\operatorname{null}(T)$ is a subspace of V and $\operatorname{range}(T)$ is a subspace of W.
We also showed that T is injective if and only if $\operatorname{null}(T) = \{0\}$.
Our big result (The Fundamental Theorem of Linear Maps) says that if V is finite dimensional, then so is $\operatorname{range}(T)$ and there is an equality:

$$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{range}(T).$$

1. Give an example of a linear map T such that $\dim \operatorname{range}(T) = 3$ and $\dim \operatorname{null}(T) = 2$.

2. Suppose that $T : \mathbb{F}^4 \to \mathbb{F}^2$ is a linear map such that

$$\operatorname{null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 3x_2, \ x_3 = 7x_4\}.$$

   Prove that T is surjective. (Hint: Can you find a basis of $\operatorname{null}(T)$? What is the $\dim \operatorname{null}(T)$? $\dim \operatorname{range}(T)$?)

3. Suppose V and W are finite-dimensional and $\dim V > \dim W$. Show that NO linear map $T : V \to W$ is injective. (Hint: Use the results above and the fact that $\dim\{0\} = 0$.)

4. Suppose V and W are finite-dimensional and $\dim W > \dim V$. Show that NO linear map $T : V \to W$ is surjective. (Hint: Use the results above and the fact that if $\dim \operatorname{range}(T) < \dim W$, then $\operatorname{range}(T) \neq W$.)