Math 131, Fall 18
Discussion Section Worksheet 8

Today we will discuss what it means for a linear map to be *invertible*.

Definition 1. Let $T : V \to W$ be a linear map. We say that T is *invertible* if there exists a linear map $S : W \to V$ such that the $ST = 1_V$ and $TS = 1_W$.

If such a linear map S exists, then we say that S is an *inverse* of T.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $T(x, y) = (-y, x)$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $S(x, y) = (y, -x)$. Show that S is an inverse of T.

2. Is the zero map $0 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (0, 0)$ invertible? Why or why not?

3. Prove that if a linear map $T : V \to W$ is invertible, then it has a *unique* inverse. (Hint: suppose that S_1 and S_2 are both inverses of T, consider the composition S_1TS_2.)

4. By the previous question, if $T : V \to W$ is invertible, then we can talk about THE inverse of T and we will denote it by $T^{-1} : W \to V$. Today you will prove half of the following theorem (we'll do the rest on Tuesday):

   Theorem 2. *A linear map $T : V \to W$ is invertible if and only if T is both injective and surjective.*

   First we consider the 'forward' direction.

   (a) Prove that if T is invertible, then T is injective. (Hint: Suppose that $u, v \in V$ and $T(u) = T(v)$. Apply T^{-1}... Alternatively, show that $\text{null}(T) = \{0\}$.)

   (b) Prove that if T is invertible, then T is surjective. (Hint: Suppose $w \in W$. Use the fact that $w = T(T^{-1}(w))$.)

5. Use the theorem above to prove that the derivative map $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is not invertible.