Math 131, Fall 18
Discussion Section Worksheet 9

Suppose that V and W are finite dimensional with bases $\beta = (v_1, \ldots, v_n)$ and $\gamma = (w_1, \ldots, w_m)$ respectively. Recall that this gives us a linear map

$$\mathcal{M} : \mathcal{L}(V, W) \to \mathrm{Mat}_{m,n}(\mathbb{F}),$$

defined by $\mathcal{M}(T) = [T]_\beta^\gamma$.

1. We will prove that $\mathcal{M}$ is an isomorphism!
   (a) Prove that $\mathcal{M}$ is injective. (Hint: Suppose that $\mathcal{M}(T)$ is the zero matrix. What does this tell you about $T(v_k)$? What about $T(v)$ for a general vector $v \in V$?)

   (b) Prove that $\mathcal{M}$ is surjective. (Hint: given a matrix $A \in \mathrm{Mat}_{m,n}(\mathbb{F})$, what does it mean for $\mathcal{M}(T)$ to equal A? Can you find such an linear map T?)

2. Let $e_{i,j}$ be the $m \times n$-matrix whose only non-zero entry is a 1 in the i-th row and j-th column. Prove that the set of all $e_{i,j}$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ forms a basis of $\mathrm{Mat}_{m,n}(\mathbb{F})$.

3. Use the previous results to find the dimension of $\mathcal{L}(V, W)$.

4. Suppose X, Y and W are subspaces of a vectors space V. Prove or give a counterexample: If $V = X \oplus W$ and $V = Y \oplus W$, then $X = Y$.